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THE NUMBER OF SUBCONTINUA OF THE REMAINDER OF THE PLANE

ERIC K. VAN DOUWEN

Denote the Euclidean plane by Π, and for a completely regular space X denote its remainder $\beta X - X$ by X^*. We will prove that Π^* has 2^c pairwise nonhomeomorphic subcontinua by finding a family $\mathscr{X}$ of nondegenerate subcontinua each of which has a unique cut point, and then finding 2^c members of $\mathscr{X}$ which are pairwise nonhomeomorphic because their cut points behave differently. It is of interest that the second part uses a method of Frolík originally invented to prove that X^* is not homogeneous for nonpseudocompact X.

Denote the half-line $[0, \infty)$ by $\boldsymbol{H}$. It is well-known that $\boldsymbol{H}^*$ and $(\boldsymbol{R}^n)$, $(2 \leqq n < \omega)$, are continua, [6, 6L]. Evidently, $\boldsymbol{H}^*$ embeds in $(\boldsymbol{R}^n)^*$, $(1 \leqq n < \omega)$, and $(\boldsymbol{R}^m)^*$ embeds into $(\boldsymbol{R}^n)^*$, $(1 \leqq m \leqq n < \omega)$. It was announced in [4] that $\boldsymbol{H}^*$ has at least 5 pairwise nonhomeomorphic nondegenerate (proper) subcontinua. Recently Winslow, [9], proved that $(\boldsymbol{R}^3)^*$, hence $(\boldsymbol{R}^n)^*$ $(3 \leqq n < \omega)$ has 2^c pairwise nonhomeomorphic subcontinua by algebraic means which give no information about $(\boldsymbol{R}^2)^*$. We here show that $\Pi^* = (\boldsymbol{R}^2)^*$, hence $(\boldsymbol{R}^n)^*$, $(2 \leqq n < \omega)$, has 2^c pairwise nonhomeomorpic subcontinua by topological means which give no information about $\boldsymbol{H}^*$. After this paper was written I received Browner's (neé Winslow) [3], where this result also was obtained, with totally different means.

We use ω for the nonnegative integers, and identify Π with the complex plane, so that $\omega \subseteqq \boldsymbol{H} \subseteqq \Pi$. Throughout $^-$ denotes the closure operator in βX, with X being clear from the context.

1. Basic facts about βX.

We here collect basic facts about βX needed in this paper. They are often used without explicit mention.

If X is normal, then $\bar{F} \cap \bar{G} = (F \cap G)^-$ for every two closed $F, G \subseteqq X$.

If A is closed and C^*-embedded in X, in particular if A is closed in X and X is normal, then βA may, and will, be identified with $\bar{A}$ and A^* may, and will, be identified with $\bar{A} \cap X^*$.

Each map $f: X \rightarrow Y$ extends to a map $\beta f: \beta X \rightarrow \beta Y$. If f is a surjection then $\beta f^{\rightarrow} X^* = Y^*$, or, equivalently, $f^{\leftarrow} Y = \beta f^{\leftarrow} Y$, if and only if f is perfect ($\equiv$ closed + compact fibers), [7, 1.5].

Also, if X is normal and A is closed in X, then $(\beta f) \restriction \bar{A} = \beta(f \restriction A)$.

Fact 1.1. *Let $f: X \rightarrow \omega$ be a perfect surjection. Then for all*

$A \subseteq \omega$

$$(f^{\leftarrow}A)^{-} = [\beta f^{\leftarrow}A]^{-} = \beta f^{\leftarrow}A\ .$$

Just observe that $(f^{\leftarrow}A)^{-} \cup (f^{\leftarrow}(\omega - A))^{-} = \beta X$, that $(f^{\leftarrow}A)^{-} \subseteq \beta f^{\leftarrow}\bar{A}$ and $f^{\leftarrow}(\omega - A)^{-} \subseteq \beta f^{\leftarrow}(\omega - A)^{-}$, and that $\beta f^{\leftarrow}\bar{A}$ and $\beta f^{\leftarrow}(\omega - A)^{-}$ are disjoint. □

2. Construction of many subcontinua of Π^*. For $n \in \omega$ let C_n be the circle of radius 1/3 in the upper half plane which touches $\boldsymbol{H}$ in n, i.e.,

$$C_n = \{z \in \Pi : |z - (n + i/3)| = 1/3\}\ .$$

Clearly $Y = \bigcup_n C_n$ is a closed subspace of Π, and $f = \bigcup_n C_n \times \{n\}$ is a well-defined perfect map from Y onto ω. For $p \in \omega$ define

$$C_p = \beta f^{\leftarrow}\{p\}\ .$$

[This does not conflict with our definition of C_n $(n \in \omega)$, for $f^{\leftarrow}\{n\} = \beta f^{\leftarrow}\{n\}$ $(n \in \omega)$ since f is perfect.] Also, for $p \in \omega^*$ define

$$X_p = C_p \cup \boldsymbol{H}^*\ .$$

We first show that C_p touches $\boldsymbol{H}^*$ in p, i.e.,

Fact 2.1. $C_p \cap \boldsymbol{H}^* = \{p\}$, $(p \in \omega^*)$.

Clearly $p \in C_p \cap \boldsymbol{H}^*$ since $p = f(p) \in C_p$ and $p \in \omega^* \subseteq \boldsymbol{H}^*$. Next, for $q \in \beta\omega - \{p\}$ consider $P \subseteq \omega$ such that $\bar{P}$ contains p but not q. Then $[\beta f^{\leftarrow}P]^{-} = \beta f^{\leftarrow}\bar{P}$ by Fact 1.1, hence

$$\begin{aligned} C_p \cap \boldsymbol{H}^* \subseteq [\beta f^{\leftarrow}P]^{-} \cap \boldsymbol{H}^* &= (f^{\leftarrow}P)^{-} \cap \bar{\boldsymbol{H}} \cap \boldsymbol{H}^* \\ &= ((f^{\leftarrow}P) \cap \boldsymbol{H})^{-} \cap \boldsymbol{H}^* = \bar{P} \cap \boldsymbol{H}^*\ . \end{aligned}$$

It follows that $C_p \cap \boldsymbol{H}^* \subseteq \{p\}$. □

The following will be proved in §§ 3 and 4.

Fact 2.2. *C_p is a continuum without cut points,* $(p \in \omega^*)$.

Fact 2.3. *$\boldsymbol{H}^*$ has no cut points.*

[We know already that $\boldsymbol{H}^*$ is a continuum.]
Fact 2.3 also follows from the theorem of Bellamy, [1], and Woods, [10], that $\boldsymbol{H}^*$ is an indecomposable continuum, but we think it is of interest to supply a more direct proof.

COROLLARY 2.3. *X_p is a continuum which has p as unique cut point,* ($p \in \omega^*$).

Fix $p \in \omega^*$. It suffices to show that p is indeed a cut point. To this end we must show that $|\boldsymbol{H}^*| \neq 1 \neq |C_p|$. Now $|\boldsymbol{H}^*| \neq 1$ since $\boldsymbol{H}^* \supsetneqq \omega^*$.

It remains to show that $C_p - \bar{\boldsymbol{H}} \neq \emptyset$. Define $g: \omega \to Y$ by $g = \{\langle n, n + 2i/3\rangle : n \in \omega\}$. Then $f \circ g = \mathrm{id}_\omega$, hence $\beta f \circ \beta g = \mathrm{id}_{\beta\omega}$, hence $\beta g(p) \in \beta g^{\leftarrow}\{p\} = C_p$. But *range* (g) is a closed subset of Π which misses $\boldsymbol{H}$, hence *range* $(\beta g) = (\textit{range}\,(g))^-$ misses $\bar{\boldsymbol{H}}$, hence $\beta g(p) \in C_p - \boldsymbol{H}$. □

We complete this section with pointing out that each X_p is 1-dimensional (in the sense of dim, ind and Ind): Since X_p is a non-degenerate continuum we have $d(X_p) \geqq 1$ for $d \in \{\mathrm{dim}, \mathrm{ind}, \mathrm{Ind}\}$. Since $d(X) \leqq \mathrm{Ind}\, X$ for $d \in \{\mathrm{dim}, \mathrm{ind}\}$ and normal X it remains to show that $\mathrm{Ind}\, X_p \leqq 1$. While there is no general sum theorem for Ind in the class of compact Hausdorff spaces we do have $\mathrm{Ind}\, X_p = \max\{\mathrm{Ind}\, \boldsymbol{H}^*, \mathrm{Ind}\, C_p\}$ since $|\boldsymbol{H}^* \cap C_p| = 1$. But clearly $\max\{\mathrm{Ind}\, \boldsymbol{H}^*, \mathrm{Ind}\, C_p\} \leqq 1$ since Ind is closed monotone and $\mathrm{Ind}\, \beta X = \mathrm{Ind}\, X$ for normal X.

3. Forming Y_p's from Y_n's. Throughout this section let Y be a space which admits a perfect map f onto ω, and for $p \in \beta\omega$ define

$$Y_p = \beta f^{\leftarrow}\{p\} .$$

Note that $Y_n = f^{\leftarrow}\{n\} = \beta f^{\leftarrow}\{n\}$, and that Y is the topological sum of the Y_n's. Hence the Y_p ($p \in \omega^*$) are constructed from the Y_n ($n \in \omega$) the same way we constructed the C_p's from the C_n's in § 2.

There are many properties $\mathscr{P}$ such that if each Y_n ($n \in \omega$) has $\mathscr{P}$ then each Y_p ($p \in \omega^*$) has $\mathscr{P}$. Below we see two examples of this phenomenon.

PROPOSITION 3.1. *If each Y_n ($n \in \omega$) is connected, then so is each Y_p ($p \in \omega^*$).*

Fix $p \in \omega^*$, and let F_0 and F_1 be nonempty disjoint closed subsets of Y_p. We will prove that $F_0 \cup F_1 \neq Y_p$. Since F_0 and F_1 are compact we can find open U_0 and U_1 in βY such that

$$F_i \subseteqq U_i \quad (i \in 2) , \quad \text{and} \quad \bar{U}_0 \cap \bar{U}_1 = \emptyset .$$

Define

$$V_i = \{n \in \omega : Y_n \subseteqq \bar{U}_i\} \quad (i \in 2) , \qquad P = \omega - (V_0 \cup V_1) .$$

We claim that $p \in \bar{P}$: For each $i \in 2$ we have $F_{1-i} \neq \emptyset$, hence $Y_p \nsubseteq \bar{U}_i$, hence $Y_p \nsubseteq (\beta f^{\leftarrow} V_i)^-$; since $(\beta f^{\leftarrow} V_i)^- = \beta f^{\leftarrow} \bar{V}_i$, by Fact 1.1, it follows that $p \notin \bar{V}_i$.

Since each Y_n is connected we can choose $C \subseteq Y$ of the form $\{c_n: n \in P\}$ with $c_n \in Y_n - (\bar{U}_0 \cup U_1)$ $(n \in P)$. Now $\bar{C}$ meets $Y_p = \beta f^{\leftarrow}\{p\}$ since βf is closed, and $P = \beta f^{\leftarrow} C$, and $p \in \bar{P}$. But $\bar{C}$ misses $\bar{U}_i = (Y \cap \bar{U}_i)^-$ since C is closed and misses $Y \cap \bar{U}_i$, and since Y is normal, $(i \in 2)$. It follows that $Y_p - (\bar{U}_0 \cup U_1) \neq \emptyset$, hence $F_0 \cup F_1 \neq Y_p$. □

REMARK 3.2. With some more work one can prove the more general result that $\beta\phi$ is monotone for each monotone perfect surjection ϕ.

This shows that each C_p is a continuum, but does not show yet that no C_p has a cut point. For that result we need the following definition and propositions.

DEFINITION 3.3. A space X is said to have Q if it has a dense subset D such that for every two distinct $x, y \in D$ there are subcontinua K and L of Y with $K \cap L = \{x, y\}$.

PROPOSITION 3.4. *Each space that has Q is connected and has no cut points.* □

PROPOSITION 3.5. *If each Y_n $(n \in \omega)$ has Q, then so has each Y_p $(p \in \omega^*)$.*

Fix $p \in \omega^*$, for each $n \in \omega$ choose $D_n \subseteq Y_n$ which witnesses that Y_n has Q, and define

$$D = \{\beta d(p): d \in \textstyle\prod_n D_n\} .$$

[This definition makes sence since each member of $\prod_n D_n$ is a function $\omega \to Y$.] We show that D witnesses that Y_p has Q in three steps.

Step 1. We show that $D \subseteq Y_p$: For $d \in \prod_n D_n$ we have $f \circ d = \mathrm{id}_\omega$, hence $\beta f \circ \beta d = \mathrm{id}_{\beta\omega}$ by continuity, hence $\beta d(p) \in Y_p = (\beta f)^{\leftarrow}\{p\}$.

Step 2. We show that D is dense: It suffices to prove that $D \cap \bar{U} \neq \emptyset$ for each open U in βY which intersects Y_p. Given such an U, since $\bar{U} = (Y \cap U)^-$ and since βf is continuous, we must have $p \in [\beta f^{\rightarrow}(Y \cap U)]^- = [f^{\rightarrow}(Y \cap U)]^-$. Choose $d \in \prod_n D_n$ such that $d(n) \in U$ for $n \in f^{\rightarrow}(Y \cap U)$. Then $\beta d(q) \in \bar{U}$ for $q \in [f^{\rightarrow}(Y \cap U)]^-$, in particular for $q = p$.

Step 3. For $x, y \in D$ we find subcontinua K, L of Y_p with $K \cap L = \{x, y\}$: Consider $d, e \in \prod_n D_n$ with $x = \beta d(p)$ and $y = \beta e(p)$. For $n \in \omega$ choose subcontinua K_n and L_n of Y_n with $K_n \cap L_n = \{d(n), e(n)\}$. Define

$$K = Y_p \cap (\bigcup_n K_n)^- \quad \text{and} \quad L = Y_p \cap (\bigcup_n L_n)^- .$$

K and L, which obviously are compact, are connected by an obvious generalization of Proposition 3.1, e.g., K is connected since $K = (\beta k)^{\leftarrow}\{p\}$ where $k = f \restriction \bigcup_n K_n$. Also, $K \cap L = A$, where

$$A = \{\beta c(p) \colon c \in \prod_n \{d(n), e(n)\}\} ,$$

so it remains to show that $A \subseteq \{\beta d(p), \beta e(p)\}$ since obviously $A \supseteq \{\beta d(p), \beta e(p)\}$. Indeed, if $c \in \prod_n \{d(n), e(n)\}$ then without loss of generality $p \in \bar{P}$ where $P = \{n \in \omega \colon c(n) = d(n)\}$, and then $\beta c(p) = \beta d(p)$. □

4. **Proving that H^* has no cut points.** It sufficies to prove that if U_0 and U_1 are any two nonempty open subsets of H^* then $|H^* - (U_0 \cup U_1)| = 2^c$. Given such U_i's, choose an open V_i in βH such that

$$\emptyset \neq H^* \cap \bar{V}_i \subseteq U_i \quad (i \in 2) .$$

Then $H \cap \bar{V}_i$ is noncompact since $\bar{V}_i = (H \cap \bar{V}_i)^-$, $(i \in 2)$. It follows that we can find $a, b \colon \omega \to H$ such that

$$n \leq a(n) < b(n) < a(n+1) , \quad \text{and} \quad a(n) \in \bar{V}_0 \quad \text{and}$$
$$b(n) \in \bar{V}_1 , \quad (n \in \omega) .$$

Define $Y \subseteq H$ and $f \colon Y \to \omega$ by

$$Y = \bigcup_n [a(n), b(n)] , \quad \text{and} \quad f = \bigcup_n [a(n), b(n)] \times \{n\} .$$

Then Y is closed in H, hence we may assume $\beta Y = \bar{Y}$, and $Y^* = \bar{Y} \cap H^*$. As f is perfect it follows that

$$\beta f^{\leftarrow}\{p\} \subseteq H^* \quad \text{for} \quad p \in \omega^* .$$

As $f \circ a = f \circ b = \text{id}_\omega$ we have $\{\beta a(p), \beta b(p)\} \subseteq \beta f^{\leftarrow}\{p\}$, $(p \in \omega^*)$. But clearly $\beta a(p) \in \bar{V}_0 \subseteq U_0$ and $\beta b(p) \in \bar{V}_1 \subseteq U_1$. As $\beta f^{\leftarrow}\{n\} = f^{\leftarrow}\{n\} = [a(n), b(n)]$, $(n \in \omega)$, since f is perfect, it now follows from Proposition 3.1 that $\{\beta f^{\leftarrow}\{p\} \colon p \in \omega^*\}$ is a family of $|\omega^*| = 2^c$ pairwise disjoint subcontinua of H^* each of which meets both U_0 and U_1. As U_0 and U_1 are disjoint and open, it follows that $|H^* - (U_0 \cup U_1)| = 2^c$, as required.

We leave generalizations to the reader.

REMARK 3.5. We can use the above to show that there is an infinite connected completely regular space which has no infinite compact subspaces; this answers a question of Bankston (oral communication). Indeed, since H^* has 2^c closed subsets, and since each infinite closed subset of H^* has cardinality 2^c, [6, 9.12], we can find disjoint $X, Y \subseteq H^*$ each of which intersects every infinite closed subset of H^* by an obvious modification of Bernstein's classical construction of totally imperfect subsets of uncountable separable completely metrizable spaces, [8, § 36, I]. Then X has no infinite compact subsets, and is dense in H^* since H^* has no isolated points. So if U_0 and U_1 are nonempty disjoint open sets in X, there are disjoint open V_0 and V_1 in H^* with $X \cap V_i = U_i$, $(i \in 2)$, hence $X - (U_0 \cup U_1) = X \cap (H^* - (V_0 \cup V_1)) \neq \emptyset$ since $H^* - (V_0 \cup V_1)$ is an infinite closed subset of H^*. [Bankston now regrets the fact that he has included my example in [1] without giving proper credit (letter of Oct. 1979).]

5. Finding 2^c distinct X_p's. Frolík [3] has shown that for each space X and each $x \in X$ there is a $\tau(x, X) \subseteq \omega^*$ such that

(1) τ is *topological*, i.e., if $h: X \to Y$ is a homeomorphism onto, then $\tau(h(x), Y) = \tau(x, X)$ for $x \in X$,

(2) τ is monotone in X, i.e., if $x \in X \subseteq Y$ then $\tau(x, X) \subseteq \tau(x, Y)$,

(3) if D is countably infinite closed discrete subset of a completely regular space X which is C-embedded (in particular if X is normal) (so that $\bar{D} \cap X^* = D^*$) then

(a) $\tau(x, D^*) = \tau(x, X^*)$ for $x \in D^*$, and

(b) there is $B \subseteq D^*$ with $|B| = 2^c$ so that $\tau(x, D^*) \neq \tau(y, D^*)$ for every two distinct $x, y \in B$.

[One defines τ by

$$\tau(x, X) = \{p \in \omega^*: \text{there is an embedding } e: \beta\omega \to X \text{ with } e(p) = x\},$$

but we don't need this.]

Applying this with $D = \omega$ we find $B \subseteq \omega^*$ with $|B| = 2^c$ such that $\tau(p, X_p) \neq \tau(q, X_q)$, hence such that X_p and X_q are nonhomeomorphic, for distinct $p, q \in B$, since

$$\tau(p, \omega^*) \subseteq \tau(p, X_p) \subseteq \tau(p, \Pi^*) = \tau(p, \omega^*) \quad \text{for} \quad p \in \omega^* .$$

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OHIO UNIVERSITY
ATHENS, OH 45701