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A NOTE ON REAL ORTHOGONAL MEASURES

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Let X be an open Riemann surface and K a compact subset of X such that $X-K$ has only finitely many connected components. Let $R(K)$ denote the space of meromorphic functions with poles off K. In this note, we investigate the space of real measures supported on ∂K and orthogonal to $R(K)$ and connect it with the first homology group of the interior of K.

1. Introduction and preliminary notations. Let X be a fixed open connected Riemann surface; K a compact subset of X such that $X-K$ has only finitely many connected components. Let $\mathscr{C}(\partial K)$ denote the space of all real valued continuous functions on ∂K; $\mathscr{R}(K)$ denote the space of all meromorphic functions on X with poles outside K; $\overline{\mathrm{Re}\,\mathscr{R}(K)}$ denote the closure of the space of real parts of functions in $\mathscr{R}(K)$ under the sup norm on ∂K. Let $\mathscr{M}(K)$ denote the space of all measures on ∂K that are orthogonal to $\mathscr{R}(K)$ and $m(K)$ denote those measures of $\mathscr{M}(K)$ that are real.

The sole purpose of this note is to establish the following theorems.

THEOREM 1.1. *There exists a natural isomorphism between $m(K)$ and the first cohomology group of $\mathring{K}$ (which we shall denote by Ω hereafter) with real coefficients.*

THEOREM 1.2. *One can select a set of functions depending only on a homology basis of Ω in a natural way so that they form a basis for $\mathscr{C}(\partial K)$ modulo $\overline{\mathrm{Re}\,\mathscr{R}(K)}$.*

When X is the complex plane, Theorem 1.1 has already been established by Ahern and Sarason in [2] and Glicksberg in [5]. Walsh [9] already proved in this case that $\log|z-a_i|$, $1 \leqq i \leqq n$ generate $\mathscr{C}(\partial K)$ modulo $\overline{\mathrm{Re}\,\mathscr{R}(K)}$ where a_i are selected one each from the connected components of $X-K$. He also saw that they need not form a basis as in the case of the crescent moon.

The precise determination of which logarithmic terms are necessary was first given in [2] and later by Glicksberg in [5] by another method. In the case of the plane, we prove these theorems in a separate note without recourse to the techniques of uniform algebras.

2. Topological preliminaries. We need some results that are purely topological and we give proofs where we can not give a good reference.

THEOREM 2.1. *Let U be an open subset of an open Riemann surface Y such that $Y - U$ has only finitely many connected components each of which is noncompact. Then the canonical homomorphism $i: H_1(U) \to H_1(Y)$ is injective where H_1 is the first homology group functor.*

Proof. Let K be a triangulation of Y and $K^{(n)}$ denote the nth barycentric subdivision of K and let $L^{(n)}$ denote the subcomplex made up of all those 2-simplices of $K^{(n)}$ that are contained in U.

Let $i_n: H_1(L^{(n)}) \to H_1(Y)$ be the natural homomorphism. It is enough to prove that i_n is injective for all n since $H_1(U)$ is the direct limit of $H_1(L^{(n)})$. Writing the homology exact sequence

$$H_2(Y, L^{(n)}) \longrightarrow H_1(L^{(n)}) \longrightarrow H_1(Y)$$

we see that it is enough to prove that $H_2(Y, L^{(n)}) = 0$. Since the considerations are the same for all n, we shall drop the superscript n. Let $z = \sum_{i=1}^{k} n_i s_i$ be any two cycle made up of simplices not in L such that $z \in L$. Let $|z|$ denote the set of all points that belong to at least one of the s_i i.e., the so-called support of z. We claim that the topological boundary of $|z|$ is contained in $|L|$ = support of L. Let P be a boundary point of $|z|$ and $P \notin |L|$. But $P \in |\partial s_i|$ for some i. Let a be the 1-simplex of s_i to which P belongs. By hypothesis, $a \notin L$ and since $\partial z \subset L$, this a must get cancelled by another 1-simplex of s_j for some $j \neq i$. Thus if P is not a vertex of s_i, $P \in$ interior of $|z|$. And if P is a vertex of s_i, then star of P must be part of $|z|$. In either case if $P \notin |L|$, $P \in$ interior of $|z|$.

Also the interior of $|z|$ must contain points of $Y - U$ for otherwise $|z|$ would be contained in U and hence $z \subset L$. Hence the interior of $|z|$ must intersect some connected component C of $Y - U$. Since $C \cap |L| = \phi$ and boundary of $|z| \subset |L|$, $C \subset |\mathring{z}|$. But then C is noncompact whereas $|z|$ is compact. A contradiction! □

$$\text{Hence } z = 0 \text{ } ie \text{ } H_2(Y, L) = 0 .$$

LEMMA 2.2. *$H_1(\Omega)$ is finitely generated.*

Proof. We can suitably shrink the ambient Riemann surface X to X_0 so that $K \subset X_0$, $X_0 - K$ has finitely many connected components each of which is noncompact and further $H_1(X_0)$ is a free Abelian group of finite rank.

By the preceding theorem, $H_1(\Omega)$ is a subgroup of $H_1(X_0)$ and hence is a free Abelian group of finite rank.

For complete details regarding barycentric subdivisions, homology groups etc. one can confer [3], Ch. I.

LEMMA 2.3. *Let Y be a connected open Riemann surface and assume $H_1(Y)$ is finitely generated. Then there exists a subregion Ω_0 relatively compact and bounded by simple closed curves $\gamma_1, \gamma_2, \cdots, \gamma_k$ such that every component of $Y - \bar{\Omega}_0$ is an annulus.*

Proof. Canonical form of Y (see [3], p. 94) is (let us say) with p handles and q contours i.e., by cutting out $2p + q$ discs out of the Riemann sphere and then attaching p handles by pairing off $2p$ of the holes, we get a homeomorph of Y.

Thus by taking off q ringed domains one around each hole, we get a subregion Ω_0 such that every connected component of $Y - \bar{\Omega}_0$ is an annulus.

DEFINITION 2.4. Let U be an open subset of a Riemann surface X. A path at x in U is a Jordan arc entirely lying in U except possibly at one endpoint which is x when $x \in \partial U$.

Two paths at x in U are said to be equivalent if and only if given any neighborhood N of x, there exists an arc joining the two paths and lying entirely in $N \cap U$. A point x is said to be a multiple point of U if there exist two inequivalent paths at x in U.

LEMMA 2.5. *Let K be a compact subset of an open connected Riemann surface X such that $X - K$ has only finitely many connected components. Let $\Omega = \mathring{K}$. The set of multiple points of Ω is countable and given any multiple point x of Ω, there exists at most countably many inequivalent paths at x in Ω.*

Proof. Let $x_0 \in \partial\Omega$. Since $X - K$ has only finitely many connected components, there exists a closed parametric disc Δ with center at x_0 such that no connected component of $X - K$ is completely contained in Δ.

Let $\phi: \Delta \to C$ denote the coordinate mapping and C, the image of $\Delta \cap K$ by ϕ. C is compact and the complement of C is connected since any connected component of $X - K$ that intersects Δ would have points on the rim of Δ. Thus any multiple point of Ω contained in the interior of Δ is mapped into a multiple point of $\mathring{C}$ and further any two inequivalent paths at x in Ω are mapped to inequivalent paths at $\phi(x)$ in $\mathring{C}$.

Just for this discussion alone, let us make the convention that capital letters denote paths and small letters their extremeties. Thus xPy shall denote a path P with extremeties x, y and oriented from x to y.

Now let xP_1y_1, xP_2y_2 be two paths at x in $\mathring{C}$ and x a multiple point of $\mathring{C}$. Assume further that these two paths lie in the same connected component U of $\mathring{C}$. We join these two paths by a path y_1Qy_2 completely contained in U. Then $xP_1y_1Qy_2P_2x$ is a Jordan curve completely contained in U but for the point x. Certainly the interior of this curve must be completely contained in $\mathring{C}$ for otherwise it would intersect the complement of C thus trapping a connected component of the complement of C. But complement of C is connected and unbounded leading to a contradiction. Thus $xP_1y_1Qy_2P_2x$ is the boundary of a Jordan domain contained in U. But Jordan domains are locally arc-wise connected even at the boundary (see Goluzin [6], p. 46). Hence xP_1y_1 and xP_2y_2 are equivalent paths at x in $\mathring{C}$.

This proves that two paths are inequivalent if and only if they are contained in different connected components of $\mathring{C}$. Thus the number of inequivalent paths at a point x does not exceed the number of connected components of $\mathring{C}$ and hence they are at most countable.

Now let U_1, U_2 be two connected components of $\mathring{C}$ and let x, u belong to $\partial U_1 \cap \partial U_2$, xP_1y_1, xP_2y_2 be paths at x in U_1 and U_2 respectively and uQ_1z_1, uQ_2z_2 be paths at u in U_1 and U_2 respectively. Let $y_1R_1z_1$, $y_2R_2z_2$ be two paths lying entirely in U_1 and U_2 respectively. Now interior of the Jordan curve $xP_1y_1R_1z_1Q_1uQ_2z_2R_2y_2P_2x$ must trap a component of the complement of $\mathring{C}$ for otherwise it would be completely contained in C and hence in $\mathring{C}$ joining U_1 and U_2 which is impossible. This means that given any multiple point x of $\mathring{C}$, we can associate a pair of coonected components of $\mathring{C}$ where the inequivalent paths to x in C come from and this association is one-to-one. Since the number of connected components of $\mathring{C}$ is at most countable, we obtain that the set of multiple points of $\mathring{C}$ is also at most countable.

Since K can be covered by the interiors of a finite number of parametric discs, the lemma is proved.

LEMMA 2.6. *Let Δ denote the annulus $\delta < |z| < 1$ and $\phi: \Delta \to U$ be a conformal isomorphism and U be a relatively compact subset of a connected open Riemann surface X. Assume $\partial U = C \cup D$ where C and D are both compact and disjoint.*

Let $\phi(|z| = \delta)$ denote the set of all points ζ in X for which there exists a sequence $z_n \in \Delta$, $|z_n| \to \delta$ as $n \to \infty$ and $\phi(z_n) \to \zeta$ as $n \to \infty$. By analogy, we can define $\phi(|z| = 1)$.

Then $\phi(|z|=1)$, $\phi(|z|=\delta)$ are both connected and either $\phi(|z|=1)= C$, $\phi(|z| = \delta) = D$ or $\phi(|z| = \delta) = C$, $\phi(|z| = 1) = D$.

Proof. Evidently $\phi(|z| = \delta)$ is a closed set in X. Assume that $\phi(|z| = \delta)$ is disconnected i.e., $\phi(|z| = \delta) = A_1 \cup A_2$ where A_1, A_2 are mutually disjoint nonempty closed sets in X. Then there exist open sets V_1, V_2 such that $V_i \supset A_i$, $i = 1, 2$ and $V_1 \cap V_2 = \phi$. We claim that $\phi(\delta < |z| < r) \subset V_1 \cup V_2$ for all r sufficiently close to δ. If not, there exists a sequence $r_n \downarrow \delta$ and z_n with $|z_n|=r_n$ and $\phi(z_n) \notin V_1 \cup V_2$.

This is impossible since on the one hand all limit points of $\phi(z_n)$ would belong to $\phi(|z| = \delta)$ and on the other hand should lie outside $V_1 \cup V_2$ which is an open set containing $\phi(|z| = \delta)$.

Since $\phi(\delta < |z| < r)$ is connected, the fact that $\phi(\delta < |z| < r) \subset V_1 \cup V_2$ implies that $\phi(\delta < |z| < r) \subset V_1$ or V_2 which means that $\phi(|z| = \delta) \subset \bar{V}_1$ or $\bar{V}_2$. Since $\bar{V}_1 \cap V_2 = \bar{V}_2 \cap V_1 = \phi$, $\phi(|z| = \delta) \cap V_2 = \phi$ or $\phi(|z| = \delta) \cap V_1 = \phi$. That is impossible. Hence $\phi(|z| = \delta)$ is connected. Similarly $\phi(|z| = 1)$ is connected.

Further any boundary point of U must belong either to $\phi(|z|=\delta)$ or $\phi(|z| = 1)$. Let $\xi_0 \in \partial U$ and $\{\zeta_n\}$ be a sequence of points in U such that $\zeta_n \to \zeta_0$ as $n \to \infty$. Then if $\phi(z_n) = \zeta_n$, $z_n \in \Delta$, any limit point z_0 of $\{z_n\}$ must belong to $\partial\Delta$. For if not, $z_{n_k} \to z_0$ as $k \to \infty$ and $z_0 \in \Delta$ and $\phi(z_{n_k}) = \zeta_{n_k} \to \zeta_0 = \phi(z_0)$ as $k \to \infty$. But $\phi(z_0)$ is an interior point of U and ζ_0 is a boundary point of U. A contradiction. A similar reasoning would prove that $\phi(\partial\Delta) \subset \partial U$. Consequently $\phi(\partial\Delta) = \partial U$.

This proves that ∂U has at most two connected components. By hypothesis ∂U has at least two connected components. Hence C and D must be connected and $\phi(|z| = 1)$, $\phi(|z| = \delta)$ must be disjoint.

Hence $\phi(|z| = 1) = C$ and $(|z| = \delta) = D$ or $\phi(|z| = 1) = D$ and $\phi(|z| = \delta) = C$.

LEMMA 2.7. *Hypothesis and notation same as in the previous lemma. There exists a Borel set $E \subset [0, 2\pi]$ of length 2π such that $\lim_{r\to 1} \phi(re^{i\theta})$, $\lim_{r\to\delta} \phi(re^{i\theta})$ exist for all $\theta \in E$.*

Proof. Narasimhan [8] proved that any open Riemann surface can be imbedded in C^3 as a closed sub-manifold. Hence there exist three holomorphic functions ψ_i, $i = 1, 2, 3$ such that $\psi(\zeta) = (\psi_1(\zeta), \psi_2(\zeta), \psi_3(\zeta))$ from $X \to C^3$ is a one-one holomorphic map.

Since $\bar{U}$ is compact, ψ/U is bounded and hence $\psi_i \circ \phi$ is bounded for $i = 1, 2, 3$. By Fatou's theorem (see [10] pp. 99–100) on radial limits, there exists a Borel set $E \subset [0, 2\pi]$ of length 2π such that $\lim_{r\to 1} \psi_i \circ \phi(re^{i\theta})$, $\lim_{r\to\delta} \psi_i \circ \phi(re^{i\theta})$ exist for all $\theta \in E$, $i = 1, 2, 3$.

Let $\theta \in E$, $r_n \uparrow 1$ and $\phi(r_n e^{i\theta}) \to \zeta_0$ as $n \to \infty$. Then $\lim_{n\to\infty} \psi_i \circ \phi(r_n e^{i\theta}) = \psi_i(\zeta_0) = \lim_{r\to 1} \psi_i \circ \phi(re^{i\theta})$ for $i = 1, 2, 3$. Since ψ is $1-1$, this shows that ζ_0 does not depend on the sequence $\{r_n\}$. Hence $\lim_{r\to 1} \phi(re^{i\theta})$ exists. Similarly $\lim_{r\to\delta} \phi(re^{i\theta})$ exists for all $\theta \in E$.

LEMMA 2.8. *Hypothesis same as in Lemma 2.6. Further assume that $X - \bar{U}$ has only finitely many connected components. Then by discarding a countable subset of E (E as in Lemma 2.7), we can assume that $\theta \to \lim_{r\to 1} \phi(re^{i\theta})$ and $\theta \to \lim_{r\to\delta} \phi(re^{i\theta})$ are both one-one on E.*

Proof. Let $\theta \in E$, P_θ denote the path $\phi(re^{i\theta})$, $1 - \varepsilon < r < 1$, ε a fixed small positive number; $\zeta_\theta = \lim_{r\to 1} \phi(re^{i\theta})$.

Now if $\theta_1 \neq \theta_2$ and $\zeta_{\theta_1} = \zeta_{\theta_2}$, then ζ_{θ_1} is a multiple point and P_{θ_1}, P_{θ_2} are inequivalent (see [6], pp. 38–39). Thus ζ_{θ_1} is a multiple point of U. By Lemma 2.5, the set of multiple points is countable and at any given multiple point, there can be at most countably many inequivalent paths.

Thus given a $\theta_0 \in E$, the set of all $\theta \in E, \theta \neq \theta_0, \zeta_\theta = \zeta_{\theta_0}$ is countable; further the set of all θ_0 for which there exists a $\theta \neq \theta_0$ such that $\zeta_\theta = \zeta_{\theta_0}$ is also countable. Hence by discarding all such θ_0 out of E, we obtain a new Borel set E of length 2π such that $\theta \to \lim_{r\to 1} \phi(re^{i\theta})$ is a $1-1$ map. A similar reasoning applied as $r \to \delta$ would prove the rest of the lemma.

3. Boundary measures and analytic differentials.

DEFINITION 3.1. Let U be an open subset of a connected open Riemann surface X. An increasing sequence $\{U_n\}$ of open sets is said to be a regular exhaustion of U if U_n is a relatively compact subset of U_{n+1} for all n; $\bigcup_{n=1}^{\infty} U_n = U$; ∂U_n consists of finitely many piecewise analytic Jordan curves and $U - \bar{U}_n$ has no relatively compact connected components in U.

REMARK. Existence of regular exhaustions can be proved by triangulations (see [3], pp. 62–63).

DEFINITION 3.2. Let U be an open subset of X. $\mathscr{H}(U)$ denotes the set of all holomorphic 1-forms ω for which there exists a regular exhaustion $\{U_n\}$ of U such that $\int_{\partial U_n} |\omega| \leqq c$ where c is independent of n.

DEFINITION 3.3. Let U be a relatively compact open subset of an open connected Riemann surface X. Let $\omega \in \mathscr{H}(U)$. A finite Borel measure μ on ∂U is called a boundary measure of ω if there exists a regular exhaustion U_n of U such that $\int_{\partial U_n} h\omega \to \int_{\partial U} hd_\mu$ as $n \to \infty$ for any continuous function h on $\bar{U}$ where ∂U_n is positively oriented with respect to U_n.

THEOREM 3.4. (Bishop-Kadama, see [7]). *Let K be a compact subset of X such that $X - K$ has only finitely many connected components. Let $\mathring{K} = \Omega$. Given any $\omega \in \mathscr{H}(\Omega)$, there exists one and only one boundary measure μ_ω of ω.*

The mapping $\omega \to \mu_\omega$ is a linear isomorphism between $\mathscr{H}(\Omega)$ and $\mathscr{M}(K)$ (see § 1 for the definition of $\mathscr{M}(K)$).

DEFINITION 3.5. Let U be an open subset of X. A point $x \in \partial U$ is said to be an accessible boundary point of U if and only if there exists a path at x in U. Acc ∂U shall denote the set of all accessible boundary point of U.

THEOREM 3.6. *Let K be a compact subset of X and $X - K$ have only finitely many connected components. Let $\mathring{K} = \Omega$. Let $\{U_i, i \in I\}$ be the family of all connected components of Ω. By Lemma 2.2, $H_1(\Omega)$ is finitely generated and consequently $H_1(U_i)$ is finitely generated for all $i \in I$. By Lemma 2.3, there exists a relatively compact subregion V_i of U_i bounded by finitely many analytic Jordan curves such that each component of $U_i - \bar{V}_i$ is an annulus. Let $\{\Delta_{ij}, 1 \leqq j \leqq N(i)\}$ denote the set of all connected components of $U_i - \bar{V}_i$. Let $\omega \in \mathscr{H}(\Omega)$.*

Then $\omega/\Delta_{ij} \in \mathscr{H}(\Delta_{ij})$. Let μ_{ij} denote the boundary measure of ω/Δ_{ij} located on $\partial\Delta_{ij} \cap \partial\Omega$. Then μ_{ij}, $\mu_{i'j'}$ are mutually singular for $(i, j) \neq (i', j')$. Further $\sum_{i \in I} \sum_{1 \leqq j \leqq N(i)} \|\mu_{ij}\|$ is finite and $\mu_\omega = \sum_i \sum_j \mu_{ij}$.

Before proceeding to the proof of the Theorem 3.6, we need two lemmas.

LEMMA 3.7. *Let Δ denote the annulus $\{z; \delta < |z| < 1\}$ and $\omega \in \mathscr{H}(\Delta)$. Let $\omega = f(z)dz$ where f is holomorphic in Δ. Then there exists a Borel measurable function f defined on $\partial\Delta$ such that*

$$\lim_{r \to 1-0} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})| d\theta = 0 \quad and \quad \lim_{r \to \delta+0} \int_0^{2\pi} |f(re^{i\theta}) - f(\delta e^{i\theta}) d\theta = 0 \,.$$

Proof. Let μ denote the boundary measure μ_ω of ω.

$$\text{Let } f_1(z) = \int_{|\zeta|=1} \frac{d\mu(\zeta)}{\zeta - z} \text{ and } f_2(z) = \int_{|\zeta|=\delta} \frac{d\mu(\zeta)}{\zeta - z}$$

so that f_1 is holomorphic in $|z| < 1$ and f_2 is holomorphic in $|z| > \delta$ and $f = f_1 + f_2$ in Δ.

Let ν_1, ν_2 be finite complex Borel measures defined by

$$d\nu_1(\zeta) = d\mu(\zeta) - \frac{1}{2\pi i} f_2(\zeta) d\zeta \text{ on } |\zeta| = 1$$

$$d\nu_2(\zeta) = d\mu(\zeta) + \frac{1}{2\pi i} f_1(\zeta) d\zeta \text{ on } |\zeta| = \delta .$$

Then for $\delta < |z| < 1$,

$$\begin{aligned} \int \frac{d\nu_1(\zeta)}{\zeta - z} &= \int_{|\zeta|=1} \int \frac{d\mu(\zeta)}{\zeta - z} - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f_2(\zeta) d\zeta}{\zeta - z} \\ &= f_1(z) - \frac{1}{2\pi i} \int_{|\zeta'|=\delta} \int_{|\zeta|=1} \frac{d\mu(\zeta') d\zeta}{(\zeta' - \zeta)(\zeta - z)} \\ &= f_1(z) \end{aligned}$$

since

$$\int_{|\zeta|=1} \frac{d\zeta}{(\zeta' - \zeta)(\zeta - z)} = 0 \text{ when } |\zeta'| < 1 \text{ and } |z| < 1 .$$

By analytic continuation, we get that

$$\int \frac{d\nu_1(\zeta)}{\zeta - z} \equiv f_1(z) \text{ for } |z| < 1 .$$

Further for $|z| > 1$,

$$\int \frac{d\nu_1(\zeta)}{\zeta - z} = f_1(z) + f_2(z)$$

since

$$\int_{|\zeta|=1} \frac{d\zeta}{(\zeta' - \zeta)(\zeta - z)} = -2\pi i/(\zeta' - z) .$$

Therefore

$$\begin{aligned} \int \frac{d\nu_1(\zeta)}{\zeta - z} &= f_1(z) \text{ for } |z| < 1 \\ &= 0 \text{ for } |z| > 1 . \end{aligned}$$

By F. and M. Riesz theorem ([4], for a very general form), we obtain that

$$\int_0^{2\pi} |f_1(re^{i\theta}) - f_1(r'e^{i\theta})|\, d\theta \longrightarrow 0 \text{ as } r, r' \longrightarrow 1 .$$

Now by a similar reasoning, we find that

$$\begin{aligned}\int \frac{d\nu_2(\zeta)}{\zeta - z} &= f_2(z) \text{ for } |z| > \delta \\ &= 0 \text{ for } |z| < \delta .\end{aligned}$$

Applying an inversion and F and M. Riesz theorem, we obtain that

$$\int_0^{2\pi} |f_2(re^{i\theta}) - f_2(r'e^{i\theta})|\, d\theta \longrightarrow 0 \text{ as } r, r' \longrightarrow \delta .$$

This together with completeness of $L^1([0, 2\pi])$ proves our lemma.

DEFINITION 3.8. Let $\phi: X \to Y$ be a holomorphic map where X and Y are Riemann surfaces. Then for any holomorphic 1-form ω on Y, $\phi^*\omega$ denotes the holomorphic 1-form defined as follows: for any $p \in X$ and a coordinate function ζ in a neighborhood N of $\phi(P)$, $\phi^*\omega = f(\zeta\circ\phi)d\zeta\circ\phi$ where $\omega = f(\zeta)d\zeta$ in a neighborhood of $\zeta\circ\phi(p)$.

DEFINITION 3.9. Let X, Y be two measurable spaces and $\phi: X \to Y$ be a measurable map. For any measure μ on X, $\phi_*\mu$ denotes the measure defined by $(\phi_*\mu)(S) = \mu(\phi^{-1}(S))$ for any measurable subset S of Y.

LEMMA 3.10. *Let Δ_{ij} be as introduced in Theorem* 3.6 *and ϕ: $\Delta \to \Delta_{ij}$ be a conformal isomorphism where $\Delta = \{z; \delta < |z| < 1\}$ and δ depends on i, j.*

Let B denote the set of all points z on $\partial\Delta$ for which $\lim_{r\to 1-0} \phi(rz)$ *or* $\lim_{r\to\delta+0} \phi(rz)$ *exists and let us extend ϕ to B by these limits. Let $\omega \in \mathscr{H}(\Delta_{ij})$. Then $\phi^*\omega \in \mathscr{H}(\Delta)$ and if ν is the boundary measure of $\phi^*\omega$, there exists a Borel subset B_0 of B on which ν is supported and $\phi_*(\nu)$ is the boundary measure of ω.*

Proof. If $\{U_n\}$ is a regular exhaustion of Δ_{ij}, then $\{\phi^{-1}(U_n)\}$ is a regular exhaustion of Δ and further

$$\int_{\partial\phi^{-1}(U_n)} |\phi^*\omega| = \int_{\partial U_n} |\omega| .$$

Consequently by definition, $\phi^*\omega \in \mathscr{H}(\Delta)$. By Lemma 3.7, if $\phi^*\omega =$

$f(z)dz$; we can extend f as a Borel measurable function to Δ such that

$$(1)\qquad \begin{aligned} &\lim_{r\to 1}\int_0^{2\pi}|f(re^{i\theta})-f(e^{i\theta})|d\theta = 0 \quad \text{and} \\ &\lim_{r\to\delta}\int_0^{2\pi}|f(re^{i\theta})-f(\delta e^{i\theta})|d\theta = 0 . \end{aligned}$$

In view of Lemma 2.8 there exists a Borel set $E\subset[0, 2\pi]$ of measure 2π such that $\lim_{r\to 1}\phi(re^{i\theta})$, $\lim_{r\to\delta}\phi(re^{i\theta})$ exist for all $\theta\in E$. Let B_0 denote the set $\{z; z = e^{i\theta} \text{ or } \delta e^{i\theta} \text{ for some } \theta\in E\}$. Obviously B_0 is a Borel set and ϕ can be extended by radial limits to $\Delta\cup B_0$ as a Borel measurable function.

The above considerations imply that if h is any continuous function on $\bar{\Delta}_{ij}$.

$$(2)\qquad \begin{aligned} &\lim_{r\to 1}\int_{|z|=r} h\circ\phi(z)f(z)dz = \lim_{r\to 1}\int_{\phi(|z|=r)} h\omega \quad \text{and} \\ &\lim_{r\to\delta}\int_{|z|=r} h\circ\phi(z)f(z)dz = \lim_{r\to\delta}\int_{\phi(|z|=r)} h\omega \end{aligned}$$

exist and are respectively equal to

$$\int_{B_0\cap|z|=1} h\circ\phi(e^{i\theta})f(e^{i\theta})de^{i\theta} \quad \text{and} \quad \int_{B_0\cap|z|=\delta} h(\delta e^{i\theta})f(\delta e^{i\theta})d\delta e^{i\theta}$$

for any continuous function h on $\partial\Delta$.

Let us define the boundary measure ν on $\partial\Delta$ as follows: $d\nu = f(e^{i\theta})de^{i\theta}$ on $|z| = 1$ and $d\nu = -f(\delta e^{i\theta})d\delta e^{i\theta}$ on $|z| = \delta$. Because of (1), ν is the boundary measure of $\phi^*\omega$ and because of (2),

$$\int_{\partial\Delta} h\phi d\nu = \lim_{n\to\infty}\int_{\partial V_n} h\omega = \int_{\partial_i\Delta_j}\int hd\phi_*\nu$$

where $V_n = \phi(\{z;\ \delta + 1/n < |z| < 1 - 1/n\})$. Since $\{V_n\}$ is a regular exhaustion of Δ_{ij}, by the Theorem 3.4 follows that $\phi_*\nu$ is indeed the boundary measure of ω on Δ_{ij}.

REMARK 3.11. Boundary measure of ω is supported on $\operatorname{acc}\partial\Delta_{ij}$ and any countable set is a null set for this measure.

Proof of Theorem 3.6. By Remark 3.11, it follows that μ_{ij} is supported on a Borel set contained in $\operatorname{acc}\partial\Delta_{ij}\subset\operatorname{acc}\partial U_i$ and any countable set has measure zero.

Now fixing i, $\operatorname{acc}\partial\Delta_{ij}\cap\operatorname{acc}\partial\Delta_{ij'}$ is countable for $j\neq j'$ thanks to Lemma 2.5. Hence μ_{ij}, $\mu_{ij'}$ are mutually singular.

Let us assume $i\neq i'$. The support of μ_{ij} and support of $\mu_{i'j'}$ are respectively contained in $\operatorname{acc}\partial U_i$ and $\operatorname{acc}\partial U_i'$. By Lemma 2.5, $\operatorname{acc}\partial U_i\cap\operatorname{acc}\partial U_{i'}$ is at most countable and by Remark 3.11 follows that μ_{ij}, $\mu_{i'j'}$ are mutually singular.

Let μ_i denote the boundary measure of ω restricted to U_i. We shall now prove that $\mu_i = \sum_{j=1}^{N(i)} \mu_{ij}$. The boundary of Δ_{ij} falls into two parts, a Jordan curve γ_{ij} contained in U_i and $\partial\Omega \cap \partial\Delta_{ij}$ which of course are disjoint closed sets. Thus as in lemma 3.10, ϕ: $\{z; \delta < |z| < 1\} \rightarrow \Delta_{ij}$ is a conformal isomorphism, by lemma 2.6 the limit sets $\phi(|z| = \delta)$ and $\phi(|z| = 1)$ are disjoint and must coincide with γ_{ij} and $\partial\Omega \cap \partial\Delta_{ij}$ is some order. We can assume without loss of generality that $\phi(|z| = 1) = \partial\Omega \cap \partial\Delta_{ij}$. Let γ_{ijn} denote the Jordan curve $\phi(|z| = 1 - 1/n)$ oriented positively with respect to $\phi(\delta < |z| < 1 - 1/n)$. For any fixed n and i, $\{\gamma_{ijn}\}_{1 \leqq j \leqq N(i)}$ bound a domain U_{in} contained in U_i and further for any continuous function h on $\bar{\Omega}$,

$$\lim_{n\rightarrow\infty} \int_{\gamma_{ijn}} h\omega = \int hd\mu_{ij} \text{ because of Lemma 3.10.}$$

Hence

$$\lim_{n\rightarrow\infty} \int_{\partial U_{in}} h\omega = \sum_{j=1}^{N(i)} \int hd\mu_{ij} ,$$

i.e., $\mu_i = \sum_{j=1}^{N(i)} \mu_{ij}$. This also proves that μ_i, $\mu_{i'}$ are mutually singular if $i \neq i'$. Now we shall prove that $\sum_{i\in I} \|\mu_i\| < \infty$.

Since $\omega \in \mathscr{H}(\Omega)$, it follows that there exists a regular exhaustion $\{\Omega_n\}$ of Ω such that

$$\int_{\partial\Omega_n} |\omega| \leqq C \text{ where } C \text{ does not depend on } n .$$

Further for any h continuous on $\bar{\Omega}$, $\int_{\partial\Omega_n} h\omega \rightarrow \int hd\mu_\omega$ as $n \rightarrow \infty$.

Let F be a finite subset of I and let $U_F = \bigcup_{i\in F} U_i$. Now from the above considerations, we obtain that $\int_{\partial(\Omega_n \cap U_F)} |\omega| \leqq C$ for all n and by weak compactness of measures follows that by passing to a subsequence if necessary that $\int_{\partial(\Omega_n \cap U_F)} h\omega \rightarrow \int hd\mu_F$ as $n \rightarrow \infty$ where μ_F is the boundary measure of ω restricted to U_F. Hence $\|\mu_F\| < C$. But since as $n \rightarrow \infty$, $\int_{\partial(\bigcup_{i\in F} U_{in})} h\omega = \sum_{i\in F} \sum_{j=1}^{N(i)} \int_{\gamma_{ijn}} h\omega \rightarrow \sum_{i\in F} \int hd\mu_i$ and $\{\bigcup_{i\in F} U_{in}\}$ is a regular exhaustion of U_F, we see that $\sum_{i\in F} \mu_i$ is also a boundary measure of ω/U_F. By Theorem 3.4, $\sum_{i\in F} \mu_i = \mu_F$.

Consequently $\|\sum_{i\in F} \mu_i\| \leqq C$ for an arbitrary finite subset F of I and now by the fact that μ_i are mutually singular, we obtain that $\sum_{i\in I} \|\mu_i\| \leqq C$.

Now if $\mu' = \sum_{i\in I} \mu_i$, we can prove that any function f meromorphic on X with poles off ∂K, $\int fd\mu' = \int fd\mu_\omega$. It is enough to prove for a function with one pole. If the pole is not in Ω, it is immediate that $\int_{\partial\Omega_n} f\omega = 0$ and $\int_{\partial U_{in}} f\omega = 0$ for all i and n. Hence

$\int f d\mu' = \int f d\mu_\omega = 0$. Now if the pole is in some U_i, then $\int_{\partial\Omega_n} f\omega = \int_{\partial U_{in}} f\omega$ provided the pole is in $\Omega_n \cap U_{in}$. Hence by going to the limits, $\int f d\mu_\omega = \int f d\mu_i$ and of course $\int f d\mu_j = 0$ for $j \neq i$.

Thus $\int f d(\mu' - \mu_\omega) = 0$ for all functions meromorphic with poles off ∂K. By a theorem of Kodama (see [7]), we obtain $\mu' \equiv \mu_\omega$.

Thus $\mu = \sum_{i\in I} \mu_i = \sum_{i\in I} \sum_{j=1}^{N(i)} \mu_{ij}$.

COROLLARY 3.12. *Let $\bar{U}_i = K_i$. Given $\mu_i \in \mathscr{M}(K_i)$ such that $\sum \|\mu_i\| < \infty$, then $\sum \mu_i \in \mathscr{M}(K)$. Further, μ_i are mutually singular. Conversely given any $\mu \in \mathscr{M}(K)$, μ can be uniquely expressed as $\sum \mu_i$ where $\mu_i \in \mathscr{M}(K_i)$ and $\sum \|\mu_i\| < \infty$.*

Proof. By Theorem 3.6 μ_i is supported on a Borel set contained in $\operatorname{acc} \partial U_i$ and any countable set is a null set modulo μ_i. By Lemma 2.5, $\operatorname{acc} \partial U_i \cap \operatorname{acc} \partial U_j$ is a countable set and consequently, μ_i and μ_j are mutually singular.

Since $\int f d\mu_i = 0$ for any f continuous on K_i and analytic in U_i, $\int f d\mu_i = 0$ for any f continuous on K and analytic in Ω. Therefore $\mu_i \in \mathscr{M}(K) \forall i$ and $\sum \mu_i \in \mathscr{M}(K)$.

For the converse, the fact that $\mu = \sum \mu_i$, $\mu_i \in \mathscr{M}(K_i)$ is a consequence of Theorem 3.6. Uniqueness follows from mutual singularity.

COROLLARY 3.13. *Assume that $m(K_i) \approxeq H^1(U_i) \forall i \in I$. Then $m(K) \approxeq H^1(\Omega)$.*

Proof. $H^1(\Omega)$ is finitely generated by Lemma 2.2. Hence $H^1(U_i) = 0$ but for finitely many i. The set of i for which $H'(U_i) \neq 0$, we shall denote by F.

Then $H^1(\Omega) \approxeq \bigoplus_{i\in F} H^1(U_i)$. On the other hand, given any $\mu \in m(K)$ by Corollary 3.12, $\mu = \sum_{i\in I} \mu_i$, $\mu_i \in M(K_i)$, μ_i, μ_j are mutually singular; which implies that μ_i is real for all i, i.e., $\mu_i \in m(K_i)$ for every i and by our assumption above

$$\mu_i = 0 \text{ for } i \notin F .$$

Thus the natural mapping $m(K) \to \bigoplus_{i\in F} m(K_i)$ is an isomorphism.

Thus by our hypothesis,

$$H^1(\Omega) \approxeq m(K) .$$

4. Harmonic 1-forms, real boundary measures.

LEMMA 4.1. *Let ω be a holomorphic 1-form defined on an* *annulus* $D=\{z;\delta<|z|<1\}$. *Assume that* $\exists$ *a real measure* μ *on* $|z|=1$ *such that for any continuous function* h *on* D, $\int_{|z|=r} h\omega \to \int h d\mu$ *as* $r\to 1-0$. *Then* $\iint \omega \Lambda * \omega < \infty$ *and for any* $\mathscr{C}^1$*-function* h *defined on* D, *vanishing in a neighborhood of* $|z|=\delta$ *and* $\iint_D dh\Lambda * dh<\infty$, $\iint_D dh\Lambda \operatorname{Im}\omega=0$.

(For the definition of $*\,\omega$, $\operatorname{Im}\omega$ see Ahlfors-Sario [3] p. 271.)

Proof. Since ω is a holomorphic 1-form, there exists a holomorphic function $g(z)$ on D such that $\omega=g(z)dz$.

Let $\tilde{D}$ denote the annulus $\delta<|z|<1/\delta$, the double of D. Define $\tilde{\omega}$ a holomorphic 1-form on $\tilde{D}$ in the following way. Define $\tilde{\omega}=g(z)dz$ for $|z|<1$ and for $|z|>1$,

$$\tilde{\omega}=-\bar{g}\left(\frac{1}{\bar{z}}\right)\frac{dz}{z^2}. \quad \text{We note that } \tilde{\omega} \text{ is not defined on } |z|=1\,.$$

By hypothesis, we obtain that there exists a constant C such that $\int_{|z|=r}|\omega|<C$ for r such that $(1+\delta)/2\leqq r<1$.

$$\text{i.e., } \int_{|z|=r}|g(z)|\,|dz|\leqq C\,.$$

Thus if g is defined as $g(z)$ on $|z|<1$ and $-\bar{g}(1/\bar{z})1/z^2$ on $|z|>1$, g belongs $L^{1,\mathrm{loc}}(\tilde{D})$. We shall now prove that $\partial\tilde{g}/\partial\bar{z}=0$ in the sense of distributions.

Let h be any C^∞-function with compact support in $\tilde{D}$. Then

$$\iint_{\tilde{D}}\frac{\partial h}{\partial\bar{z}}\tilde{g}(z)d\bar{z}\Lambda dz=\iint_{\tilde{D}}dh\Lambda\tilde{g}(z)dz=\iint_{\tilde{D}}dh\Lambda\tilde{\omega}$$
$$=\lim_{\varepsilon\to 0}\int_{|z|=1-\varepsilon}h\omega-\int_{|z|=1+\varepsilon}-\,h\bar{g}\left(\frac{1}{\bar{z}}\right)\frac{dz}{z^2}$$

(by Stoke's formula applied to the annulii $\delta<|z|<1-\varepsilon$, $1+\varepsilon<|z|<1/\delta$)

$$=\int hd\mu+\lim_{\varepsilon\to 0}\int_{|z|=1+\varepsilon}h\bar{g}\left(\frac{1}{\bar{z}}\right)\frac{dz}{z^2}$$
$$=\int hd\mu-\lim_{\varepsilon\to 0}\int_{|z|=1/1+\varepsilon}h\left(\frac{1}{\bar{z}}\right)\bar{g}(z)d\bar{z}$$
$$=\int hd\mu-\int\overline{hd\mu}=0 \text{ since } \mu \text{ is real}\,.$$

Therefore we obtain that g can be defined suitably on $|z|=1$ so that g is holomorphic in all of $\tilde{D}$. Hence $\iint_{(1+\delta/2)<|z|<(2/1+\delta)}\omega\Lambda*\omega<\infty$ and consequently,

$$\iint \omega \Lambda * \omega < \infty .$$

$(1 + \delta)/2 < |z| < 1$.

Also for any real h, $\mathscr{C}^1$ on $\bar{D}$ and vanishing in a neighborhood of $|z| = \delta$,

$$\iint_D dh \Lambda \omega = \int_{|z|=1} h\omega = \int h d\mu$$

and so

$$\operatorname{Im} \iint_D dh \Lambda \omega = \iint_D dh \Lambda \operatorname{Im} \omega = \iint_D dh \Lambda \operatorname{Im} \omega = \operatorname{Im} \int h d\mu = 0 .$$

Now given any h, $\mathscr{C}^1$ on D and vanishing in a neighborhood of $|z| = \delta$, define $h_\varepsilon(z) = h(z/(1+\varepsilon))$. Then h_ε is $\mathscr{C}^1$ on $\bar{D}$ for every $\varepsilon > 0$ and vanishes in a neighborhood of $|z| = \delta$ and furthermore $\iint dh_\varepsilon \Lambda * dh_\varepsilon < \infty$ and $\iint d(h - h_\varepsilon)\Lambda*(dh - dh_\varepsilon) \to 0$ as $\varepsilon \to 0$.

Hence, since we already know that $\iint_D dh_\varepsilon \Lambda \operatorname{Im} \omega = 0$ for all ε and $\iint \operatorname{Im} \omega \Lambda * \operatorname{Im} \omega < \infty$, we can take the limit under the integral sign and obtain that

$$\iint_D dh \Lambda \operatorname{Im} \omega = 0 . \qquad \square$$

LEMMA 4.2. *Let ω be a holomorphic 1-form on $D = \{z; \delta < |z| < 1\}$ such that $\iint_D \omega \Lambda * \omega < \infty$. Further assume that for any h, $\mathscr{C}^1$ on D and vanishing in a neighborhood of $|z| = \delta$ and $\iint_D dh \Lambda * dh < \infty$, $\iint_D dh \Lambda \operatorname{Im} \omega = 0$.*

Then $\exists$ a real measure μ on $|z| = 1$ such that for any continuous function h on $\bar{D}$, $\int_{|z|=r} h\omega \to \int h d\mu$ as $r \to 1 - 0$.

Proof. Let $\omega = g(z)dz$ for $\delta < |z| < 1$ and $\tilde{\omega}$ be defined as ω on $\delta < |z| < 1$ and

$$= -\bar{g}\left(\frac{1}{\bar{z}}\right)\frac{dz}{z^2} \text{ on } 1 < |z| < \frac{1}{\delta} .$$

By hypothesis, $\iint_{\delta < |z| < 1/\delta} \omega \Lambda * \omega < \infty$. We shall now establish that $\bar{\partial}\tilde{\omega} = 0$ in the sense of distributions.

Let h be any $\mathscr{C}^1$-function with compact support in $\delta < |z| < 1/\delta$. Then

$$\iint_{\delta<|z|<1/\delta}\bar{\partial}h\Lambda\tilde{\omega}=\iint dh\Lambda\tilde{\omega}=\lim_{r\to1-0}\iint_{\delta<|z|<r}dh\Lambda\tilde{\omega}+\iint_{1/r<|z|<1/\delta}dh\Lambda\tilde{\omega}$$

$$=\text{(By Stoke's)}\lim_{r\to1-0}\left(\int_{|z|=r}h\omega-\int_{|z|=1/r}h\omega\right)$$

$$=\lim_{r\to1-0}\left(\int_{|z|=r}h\omega-\int_{|z|=1/r}h(z)\left(-\bar{g}\left(\frac{1}{\bar{z}}\right)\right)\frac{dz}{z^2}\right)$$

$$=\lim_{r\to1-0}\left(\int_{|z|=r}h\omega-\int_{|z|=r}h\left(\frac{1}{\bar{z}}\right)\bar{g}(z)dz\right)$$

$$=\lim_{r\to1-0}\int_{|z|=r}h\omega-\int_{|z|=r}h\left(\frac{1}{\bar{z}}\right)\bar{\omega}$$

$$=\lim_{r\to1-0}\left(\int_{|z|=r}\left(h(z)-h\left(\frac{1}{\bar{z}}\right)\right)\operatorname{Re}\omega+i\int_{|z|=r}\left((h(z)+h\left(\frac{1}{\bar{z}}\right)\operatorname{Im}\omega\right)\right).$$

Since $h(z)+h(1/\bar{z})$ vanishes in a neighborhood of $|z|=\delta$ and $\iint_D dh\Lambda{*}dh<\infty$, we have, by hypothesis,

$$\iint dh\Lambda\operatorname{Im}\omega=0=\iint dh\left(\frac{1}{\bar{z}}\right)\Lambda\operatorname{Im}\omega\ \text{ i.e.,}$$

(By Stoke's)

$$\lim_{r\to1-0}\iint_{\delta<|z|<r}dh\Lambda\operatorname{Im}\omega=\lim_{r\to1-0}\int_{|z|=r}h\operatorname{Im}\omega=0\,.$$

Hence

$$\iint_{\delta<|z|<1/\delta}\bar{\partial}h\Lambda\tilde{\omega}=\lim_{r\to1-0}\int_{|z|=r}\left(h(z)-h\left(\frac{1}{\bar{z}}\right)\right)\operatorname{Re}\omega$$

$$=\lim_{r\to1-0}\iint_{\delta<|z|<r}d\left(h(z)-h\left(\frac{1}{z}\right)\right)\Lambda\operatorname{Re}\omega\ \text{(By Stoke's)}\,.$$

Since $h(z)-h(1/\bar{z})\in H_2(D)$ (here it denotes the Sobolev space) and vanishes on $\partial\ D$, we find that

$h(z)-h(1/\bar{z})\in\mathring{H}_2(D)$ (see Agmon [1], p. 131, Lemma 9.10). But $\iint_D dh\Lambda\operatorname{Re}\omega=0$ for any h that is $\mathscr{C}^1$ and has compact support in D and hence for any h in $\mathring{H}_2^1(D)$.

Therefore $\bar{\partial}\tilde{\omega}=0$. Hence $\tilde{\omega}$ is a holomorphic 1-form on $\delta<|z|<1/\delta$ which implies that $\int_{|z|=r}|g(z)||dz|$ is bounded as $r\to1-0$. That means that ω defines a real boundary measure on $|z|=1$. □

THEOREM 4.3. *Borrowing the notation of Corollary* 3.12, $m(K_i)\cong H^1(U_i)$ *for every* i.

Proof. Let $\Gamma(U_i)$ denote the set of all holomorphic 1-forms ω

such that $\iint_{U_i} \omega \Lambda * \omega < \infty$ and for any $\mathscr{C}^1$-function h on U_i such that $\iint dh \Lambda * dh < \infty$, $\iint eh\Lambda \operatorname{Im} \omega = 0$.

The fact that $H^1(U_i) \approx \Gamma(U_i)$ is well-known and can be found in Ahlfors-Sario [2], p. 284–288. Thus we need only prove that $m(K_i) \approx \Gamma(U_i)$.

Let $\Delta_{ij}(1 \leqq j \leqq N(i))$ be the annulii as introduced in Theorem 3.6. Now if ω is a holomorphic 1-form on U_i whose boundary measure is real, then $\omega | \Delta_{ij} \in \mathscr{H}(\Delta_{ij})$ and further its boundary measure μ_{ij} on $\partial U_i \cap \partial \Delta_{ij}$ is real. We can apply now Lemma 4.1 to $\omega | \Delta_{ij}$ and obtain $\iint_{\Delta_{ij}} \omega \Lambda * \omega < \infty$ and $\iint_{\Delta_{ij}} dh \Lambda \operatorname{Im} \omega = 0$ provided h is a $\mathscr{C}^1$-function vanishing in a neighborhood of $\partial \Delta_{ij} - \partial U_i$. Thus using partition of unity, we obtain that $\iint_{U_i} \omega \Lambda * \omega < \infty$ and $\iint_{U_i} dh \Lambda \operatorname{Im} \omega = 0$ for any h, $\mathscr{C}^1$ on U_i and $\iint dh \Lambda * dh < \infty$.

Now assume that $\omega \in \Gamma(U_i)$. Now $\omega | \Delta_{ij}$ satisfies the following conditions: $\iint_{\Delta_{ij}} \omega \Lambda * \omega < \infty$ and any $\mathscr{C}^1$-function h vanishing in a neighborhood of $\partial \Delta_{ij} - \partial U_i$ and $\iint_{\Delta_{ij}} dh \Lambda * dh < \infty$, $\iint dh \Lambda \operatorname{Im} \omega = 0$. This is easily obtained by defining $h = 0$ on $U_i - \Delta_{ij}$. Now we can apply Lemma 4.2 to obtain that the boundary measure μ_{ij} of ω on $\partial \Delta_{ij} \cap \partial U_i$ is real. Since boundary measure μ_i of ω is $\sum_{j=1}^{N(i)} \mu_{ij}$ by Theorem 3.6, μ_i is real. □

THEOREM 4.4. $m(K) \approx H^1(\Omega)$.

Proof. It is immediate from Corollary 3.13 and Theorem 4.3.

5. A natural basis for $\mathscr{C}(\partial K)/\overline{\operatorname{Re} \mathscr{R}(K)}$ (Theorem 1.2). We may assume without loss of generality that X is a noncompact surface with analytic boundary and K a compact subset of X such that $X - K$ has only finitely many connected components none of which is relatively compact. By Theorem 2.1, the canonical homomorphism $H_1(\mathring{K}) \to H_1(X)$ is injective.

Let $\gamma_i(1 \leqq i \leqq k)$ be a homology basis for $\mathring{K}$ and $\gamma_i(1 \leqq i \leqq k+l)$ be a homology basis for X. Let $\ominus$ denote the space of all harmonic functions h on X such that $\iint dh \Lambda * dh < \infty$.

We contend that given any $\sum a_i \gamma_i \neq 0$, a_i real, there exists $h \in \ominus$ such that

$$\int_{\sum a_i \gamma_i} * dh \neq 0. \quad \text{Assume the contrary .}$$

Then there exists a harmonic differential σ with compact support (see Ahlfors-Sario [3], p. 288) such that

$$\int_{\Sigma a_i \gamma_i} * dh = \iint \sigma \Lambda * dh$$

and so

$$\iint \sigma \Lambda * dh = 0 \forall h \in \ominus ,$$

i.e. $*\sigma$ also has compact support. But $\sigma - i*\sigma$ is a holomorphic 1-form and it can not have compact support unless $\sigma = i*\sigma \equiv 0$ which implies $\sum a_i \gamma_i$ is homologous to zero.

This proves that the mapping $\psi: \ominus \to R^{k+l}$ given by $\psi(h) = \left(\int_{\gamma_1} *dh, \cdots, \int_{\gamma_{k+l}} * dh\right)$ is a surjection. Now let us pick $h_i \in \ominus$ such that $\int_{\gamma_i} *dh_i = 1$ and $\int_{\gamma_j} * dh_i = 0$ for $j \neq i$.

We claim now that $h_1, h_2 \cdot, h_k$ form a basis of $\mathscr{C}(\partial K)$ modulo $\overline{\operatorname{Re} \mathscr{R}(K)}$. Assume $\sum a_i h_i \in \overline{\operatorname{Re} \mathscr{R}(K)}$. Then there exists a function f holomorphic in a neighborhood of K such that $|\sum a_i h_i - \operatorname{Re} f| < \varepsilon$ on ∂K.

Since γ_i lie in $\mathring{K}$ for $1 \leqq i \leqq k$, and $\int_{\gamma_j} |\sum a_i * dk_i - \operatorname{Im} df| < C\varepsilon$ where C depends only on γ_j.

Since $\int_{\gamma_j} df = 0$ and $\int_{\gamma_j} * dh_i = \delta_{ij}$ (Kronecker δ), we obtain that $|a_i| < C\varepsilon$ for $1 \leqq i \leqq k$. Since this is true for all $\varepsilon > 0$, $a_i = 0 \forall i$. Thus $\{h_i\}_{1 \leqq i \leqq k}$ are linearly independent modulo $\overline{\operatorname{Re} \mathscr{R}(K)}$ and because $\dim \mathscr{C}(\partial K)/\overline{\operatorname{Re} \mathscr{R}(K)} = k$, we have that $\{h_i\}_{1 \leqq i \leqq k}$ is a basis for $\mathscr{C}(\partial K)/\overline{\operatorname{Re} \mathscr{R}(K)}$.

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