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SOME BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

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Joseph D. E. Konhauser discussed two polynomial sets $\{Y_n^\alpha(x;k)\}$ and $\{Z_n^\alpha(x;k)\}$, which are biorthogonal with respect to the weight function $x^\alpha e^{-x}$ over the interval $(0,\infty)$, where $\alpha > -1$ and k is a positive integer. The present paper attempts at exploring certain novel approaches to these biorthogonal polynomials in simple derivations of their several interesting properties. Many of the results obtained here are believed to be new; others were proven in the literature by employing markedly different techniques.

1. **Introduction.** Konhauser ([10]; see also [9]) has considered two classes of polynomials $Y_n^\alpha(x;k)$ and $Z_n^\alpha(x;k)$, where $Y_n^\alpha(x;k)$ is a polynomial in x, while $Z_n^\alpha(x;k)$ is a polynomial in x^k, $\alpha > -1$ and $k = 1, 2, 3, \cdots$. For $k = 1$, these polynomials reduce to the Laguerre polynomials $L_n^{(\alpha)}(x)$, and their special cases when $k = 2$ were encountered earlier by Spencer and Fano [19] in certain calculations involving the penetration of gamma rays through matter, and were subsequently discussed by Preiser [16]. Furthermore, we have [10, p. 303]

$$\int_0^\infty x^\alpha e^{-x} Y_i^\alpha(x;k) Z_j^\alpha(x;k)\,dx = \frac{\Gamma(kj+\alpha+1)}{j!}\delta_{ij}\,, \qquad \forall\, i, j \in \{0, 1, 2, \cdots\}\,, \tag{1.1}$$

which exhibits the fact that the polynomial sets $\{Y_n^\alpha(x;k)\}$ and $\{Z_n^\alpha(x;k)\}$ are *biorthogonal* with respect to the weight function $x^\alpha e^{-x}$ over the interval $(0, \infty)$, it being understood that $\alpha > -1$, k is a positive integer, and δ_{ij} is the Kronecker delta.

An explicit expression for the polynomials $Z_n^\alpha(x;k)$ was given by Konhauser in the form [10, p. 304, Eq. (5)]

$$Z_n^\alpha(x;k) = \frac{\Gamma(kn+\alpha+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(kj+\alpha+1)}\,. \tag{1.2}$$

As for the polynomials $Y_n^\alpha(x;k)$, Carlitz [3] subsequently showed that [op. cit., p. 427, Eq. (9)]

$$Y_n^\alpha(x;k) = \frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n\,, \tag{1.3}$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(1.4)\qquad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & , \text{ if } n = 0, \lambda \neq 0 , \\ \lambda(\lambda+1)\cdots(\lambda+n-1) , & \forall\, n \in \{1, 2, 3, \cdots\} . \end{cases}$$

The object of the present paper is to show that several interesting properties of the biorthogonal polynomials $Y_n^\alpha(x; k)$ and $Z_n^\alpha(x; k)$ follow fairly readily from relatively more familiar results by applying explicit expressions (1.2) and (1.3). A number of properties thus the derived are believed to be new, and others were proven in the literature by employing markedly different techniques.

2. The biorthogonal polynomials $Y_n^\alpha(x; k)$. We begin by recalling the polynomials $G_n^{(\alpha)}(x, r, p, k)$ which were introduced by Srivastava and Singhal [24] in an attempt to provide an elegant unification of the various known generalizations of the classical Hermite and Laguerre polynomials. These polynomials are defined by the generalized Rodrigues formula [op. cit., p. 75, Eq. (1.3)]

$$(2.1)\qquad G_n^{(\alpha)}(x, r, p, k) = \frac{x^{-kn-\alpha}\exp(px^r)}{n!}(x^{k+1}D_x)^n\{x^\alpha \exp(-px^r)\} ,$$

where $D_x = d/dx$, and the parameters α, k, p and r are unrestricted, in general. We also have the explicit polynomial expression [24, p. 77, Eq. (2.1)]

$$(2.2)\qquad G_n^{(\alpha)}(x, r, p, k) = \frac{k^n}{n!}\sum_{i=0}^{n}\frac{(px^r)^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{rj+\alpha}{k}\right)_n$$

On comparing (2.2) with Carlitz's result (1.3), we at once get the known relationship [23, p. 315, Eq. (83)]

$$(2.3)\qquad Y_n^\alpha(x; k) = k^{-n}G_n^{(\alpha+1)}(x, 1, 1, k) , \quad \alpha > -1 , \quad k = 1, 2, 3, \cdots ,$$

which evidently enables us to derive the following properties of the Konhauser biorthogonal polynomials $Y_n^\alpha(x; k)$ by suitably specializing those of the Srivastava-Singhal polynomials $G_n^{(\alpha)}(x, r, p, k)$.

I. *Rodrigues' formula.* In (2.1) we set $p = r = 1$, replace α by $\alpha + 1$, and appeal to the relationship (2.3). We thus obtain

$$(2.4)\qquad Y_n^\alpha(x; k) = \frac{x^{-kn-\alpha-1}e^x}{k^n n!}(x^{k+1}D_x)^n\{x^{\alpha+1}e^{-x}\} ,$$

where, by definition, $\alpha > -1$ and k is now restricted to be a positive integer.

Alternatively, we may recall that [15, p. 802, Eq. (2.6)]

$$(2.5) \qquad Y_n^{\alpha}(x; k) = \frac{x^{k-\alpha-1}e^x}{n!} D_s^n\{s^{n-1+(\alpha+1)/k}\exp(-s^{1/k})\}\Big|_{s=x^k},$$

which indeed is equivalent to

$$(2.6) \qquad Y_n^{\alpha}(x; k) = \frac{x^{k-\alpha-1}e^x}{n!}[s^{-n-1}(s^2D_s)^n\{s^{(\alpha+1)/k}\exp(-s^{1/k})\}]_{s=x^k},$$

since

$$(2.7) \qquad (x^2D_x)^n\{g(x)\} = x^{n+1}D_x^n\{x^{n-1}g(x)\}$$

for every non-negative integer n.

Now we set $s = x^k$ and $s^2D_s = k^{-1}x^{k+1}D_x$ in (2.6), and the Rodrigues formula (24.) follows at once.

Incidentally, the Rodrigues type representation (2.4) is due to Calvez et Génin [2, p. A41, Eq. (1)]; it is stated *erroneously* in a recent paper by Patil and Thakare [12, p. 921, Eq. (1.2)].

II. *Recurrence relations.* In view of the relationship (2.3), the known results [24, p. 80, Eq. (4.3), (4.4), (4.5) and (4.6)] readily yield

$$(2.8) \qquad k(n+1)Y_{n+1}^{\alpha}(x; k) = xD_xY_n^{\alpha}(x; k) + (kn + \alpha - x + 1)Y_n^{\alpha}(x; k),$$

$$(2.9) \qquad D_xY_n^{\alpha}(x; k) = Y_n^{\alpha}(x; k)Y_n^{\alpha+1}(x; k),$$

$$(2.10) \qquad (\alpha - k + 1)Y_n^{\alpha}(x; k) = xY_n^{\alpha+1}(x; k) + (n+1)kY_{n+1}^{\alpha-k}(x; k)$$

and

$$(2.11) \qquad k(n+1)Y_{n+1}^{\alpha}(x; k) = (kn + \alpha + 1)Y_n^{\alpha}(x; k) - xY_n^{\alpha+1}(x; k).$$

The recurrence relation (2.8) was given earlier by Konhauser [10, p. 308, Eq. (16)], while (2.9), (2.10) and (2.11) are believed to be new. Notice, however, that by eliminating the term $xY_n^{\alpha+1}(x; k)$ between (2.10) and (2.11) we obtain

$$(2.12) \qquad Y_{n+1}^{\alpha-k}(x; k) = Y_{n+1}^{\alpha}(x; k) - Y_n^{\alpha}(x; k),$$

which is equivalent to the familiar generalization (cf. [10], p. 311) of a well-known recurrence relation for the Laguerre polynomials [18, p. 203, Eq. (8)].

III. *Operational formulas.* Making use of the relationship (2.3), we can specialize the Srivastava-Singhal results [24, p. 85, Eq. (7.5) and (7.6)] to obtain the following operational formulas involving the biorthogonal polynomials $Y_n^{\alpha}(x; k)$:

$$(2.13) \qquad \prod_{j=0}^{n-1}(\delta + \alpha + jk - x + 1) = k^n n! \sum_{j=0}^{n} \frac{(kx^k)^{-j}}{j!} Y_{n-j}^{\alpha}(x; k)(x^{k+1}D_x)^j$$

and

$$Y_n^\alpha(x; k) = \frac{1}{k^n n!} \prod_{j=0}^{n-1} (\delta + \alpha + jk - x + 1) \cdot 1 , \tag{2.14}$$

where $\delta = xD_x$.

IV. *Generating functions.* From the known results [24, p. 78, Eq. (3.2); p. 79, Eq. (3.4) and (3.6)], due to Srivastava and Singhal [24], it readily follows on appealing to (2.3) that

$$\sum_{n=0}^{\infty} Y_n^\alpha(x; k)t^n = (1 - t)^{-(\alpha+1)/k} \exp(x[1 - (1 - t)^{-1/k}]) , \tag{2.15}$$

$$\sum_{n=0}^{\infty} Y_n^{\alpha-kn}(x; k)t^n = (1 + t)^{(\alpha-k+1)/k} \exp(x[1 - (1 + t)^{1/k}]) , \tag{2.16}$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^\alpha(x; k)t^n \\ &\quad = (1 - t)^{-m-(\alpha+1)/k} \exp(x[1 - (1 - t)^{-1/k}]) Y_m^\alpha(x(1 - t)^{-1/k}; k) , \end{aligned} \tag{2.17}$$

where m is a non-negative integer.

Furthermore, by using the definition (2.1) and the aformentioned result [24, p. 79, Eq. (3.4)] it is not difficult to derive the generating function

$$\begin{aligned} &\sum_{n=0}^{\infty} \binom{m+n}{n} G_{m+n}^{(\alpha-k)}(x, r, p, k)t^n \\ &\quad = (1 + kt)^{(\alpha-k)/k]} \exp(px^r[1 - (1 + kt)^{r/k}]) \\ &\quad\quad \times G_m^{(\alpha)}(x(1 + kt)^{1/k}, r, p, k) , \quad k \neq 0 , \end{aligned} \tag{2.18}$$

which, for $p = r = 1$, yields a generalization of (2.16) in the form:

$$\begin{aligned} &\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha-kn}(x; k)t^n \\ &\quad = (1 + t)^{(\alpha-k+1)/k} \exp(x[1 - (1 + t)^{1/k}]) \\ &\quad\quad \times Y_m^\alpha(x(1 + t)^{1/k}; k) , \quad \forall\, m \in \{0, 1, 2, \cdots\} , \end{aligned} \tag{2.19}$$

where, by definition, k is a positive integer.

The generating function (2.15) was derived earlier by Carlitz [3, p. 426, Eq. (8)], while (2.16), (2.17) and (2.19) are due to Calvez et Génin [2]. In fact, (2.15) and (2.17) were also given *independently* by Prabhakar [15, p. 801, Eq. (2.3); p. 803, Eq. (3.3)].

Incidentally, in view of the known generating function [24, p. 78, Eq. (3.2)] and Lagrange's expansion in the form [13, p. 146, Problem 207]:

$$\frac{f(\zeta)}{1-t\phi'(\zeta)} \sum_{n=0}^{\infty} \frac{t^n}{n!} D_x^n\{f(x)[\phi(x)]^n\}\bigg|_{x=0}, \tag{2.20}$$

where

$$\zeta = t\phi(\zeta), \qquad \phi(0) \neq 0, \tag{2.21}$$

it is fairly easy to show that

$$\begin{aligned} &\sum_{n=0}^{\infty} G_n^{(\alpha+\beta n)}([x^r + ny^r]^{1/r}, r, p, k)t^n \\ &\quad = \frac{(1-u)^{-\alpha/k} \exp(px^r[1-(1-u)^{-r/k}])}{1-k^{-1}u(1-u)^{-1}[\beta - rpy^r(1-u)^{-r/k}]}, \quad k \neq 0, \end{aligned} \tag{2.22}$$

or equivalently,

$$\begin{aligned} &\sum_{n=0}^{\infty} G_n^{(\alpha+\beta n)}([x^r + ny^r]^{1/r}, r, p, k)t^n \\ &\quad = \frac{(1+v)^{\alpha/k} \exp(px^r[1-(1+v)^{r/k}])}{1-k^{-1}v[\beta - rpy^r(1+v)^{r/k}]}, \quad k \neq 0, \end{aligned} \tag{2.23}$$

where u and v are functions of t defined by

$$u = kt(1-u)^{-\beta/k} \exp(py^r[1-(1-u)^{-r/k}]), \quad u(0) = 0 \tag{2.24}$$

and

$$v = kt(1+v)^{(\beta+k)/k} \exp(py^r[1-(1+v)^{r/k}]), \quad v(0) = 0. \tag{2.25}$$

In their special cases when $p = r = 1$, (2.22) and (2.23) obviously yield the following generating functions for the Konhauser polynomials $Y_n^\alpha(x; k)$:

$$\sum_{n=0}^{\infty} Y_n^{\alpha+\beta n}(x+ny; k)t^n = \frac{(1-\xi)^{-(\alpha+1)/k} \exp(x[1-(1-\xi)^{-1/k})}{1-k^{-1}\xi(1-\xi)^{-1}[\beta - y(1-\xi)^{-1/k}]}, \tag{2.26}$$

where ξ is a function of t defined by

$$\xi = t(1-\xi)^{-\beta/k} \exp(y[1-(1-\xi)^{-1/k}]), \quad \xi(0) = 0; \tag{2.27}$$

$$\sum_{n=0}^{\infty} Y_n^{\alpha+\beta n}(x+ny; k)t^n = \frac{(1+\eta)^{(\alpha+1)/k} \exp(x[1-(1+\eta)^{1/k}])}{1-k^{-1}\eta[\beta - y(1+\eta)^{1/k}]}, \tag{2.28}$$

where η is a function of t given by

$$\eta = t(1+\eta)^{(\beta+k)/k} \exp(y[1-(1+\eta)^{1/k}]), \quad \eta(0) = 0. \tag{2.29}$$

For $y = 0$, the generating functions (2.26) and (2.28) are essentially equivalent to the Calvez-Génin result [2, p. A41, Eq. (2)]. {Indeed, their reductions to (2.15) when $\beta = y = 0$ and to (2.16) when

$\beta = -k$ and $y = 0$ are immediate.} On the other hand, their special cases when $k = 1$, involving Laguerre polynomials, were given recenlty by Carlitz [4, p. 525, Eq. (5.2) and (5.5)].

From the Srivastava-Singhal result [24, p. 78, Eq. (3.2)] we further have

$$(2.30)\qquad \begin{aligned}&(x^{1-r}D_x)^m\{\exp(-px^r)G_n^{(\alpha)}(x, r, p, k)\}\\ &\qquad = (-rp)^m \exp(-px^r)G_n^{(\alpha+mr)}(x, r, p, k)\ , \quad m \geqq 0\ ,\end{aligned}$$

and

$$(2.31)\qquad G_n^{(\alpha+\beta)}([x^r + y^r]^{1/r}, r, p, k) = \sum_{j=0}^{n} G_j^{(\alpha)}(x, r, p, k)G_{n-j}^{(\beta)}(y, r, p, k)\ ,$$

which, for $p = r = 1$, yield the known results

$$(2.32)\qquad D_x^m\{e^{-x}Y_n^\alpha(x; k)\} = (-1)^m e^{-x}Y_n^{\alpha+m}(x; k)\ , \quad m \geqq 0$$

and

$$(2.33)\qquad Y_n^{\alpha+\beta+1}(x + y; k) = \sum_{j=0}^{n} Y_j^\alpha(x; k)Y_{n-j}^\beta(y; k)\ ,$$

due to Génin et Calvez [8, p. A34, Eq. (6); p. A33, Eq. (2)]. {For (2.33) see also [15, p. 803, Eq. (3.2)].}

Applying (2.30) in conjunction with Taylor's theorem, we obtain yet another new generating function in the form:

$$(2.34)\qquad \sum_{n=0}^{\infty} G_m^{(\alpha+nr)}(x, r, p, k)\frac{t^n}{n!} = e^t G_m^{(\alpha)}([x^r - t/p]^{1/r}, r, p, k)\ , \quad m \geqq 0\ ,$$

which, in view of the relationship (2.3), reduces at once to the Génin-Calvez result [8, p. A34, Eq. (7)]

$$(2.35)\qquad \sum_{n=0}^{\infty} Y_m^{\alpha+n}(x; k)\frac{t^n}{n!} = e^t Y_m^\alpha(x - t; k)\ , \quad m \geqq 0\ .$$

We conclude this part by recoding the following special case of a known result given by Srivastava and Singhal [24, p. 84, Eq. (7.3)];

$$(2.36)\qquad Y_n^\alpha(x; k) = \sum_{j=0}^{n} \binom{j - 1 + (a - \beta)/k}{j} Y_{n-j}^\beta(x; k)\ ,$$

which is due to Prabhakar [15, p. 802, Eq. (3.1)]; for $k = 1$, (2.36) yields a well-known property of the Laguerre polynomials [18, p. 209, Eq. (2)].

Incidentally, the well-known special case $y = 0$ of (2.28) [with β replaced *trivially* by kb], and an *erroneous* version of the Génin-Calvez result (2.35), were rederived in a recent paper by B. K. Karande and K. R. Patil [*Indian J. Pure Appl. Math.* 12 (1981), 222–225; especially p. 224, Eq. (12), and p. 223, Eq. (6)] *without* any reference to the relevant earlier papers [2], [7] and [8].

V. *Mixed multilateral generating functions.* The generating-function relationships (2.17) and (2.19) enable us to apply the results of Srivastava and Lavoie [23], and we are led rather immediately to the following interesting variations of a general bilateral generating function [op. cit., p. 319, Eq. (107)]:

$$(2.37)\qquad \begin{aligned}\sum_{n=0}^{\infty} Y_{m+n}^{\alpha}(x; k)\Lambda_n(y_1, \cdots, y_N; z)t^n \\ = (1-t)^{-(km+\alpha+1)/k} \exp(x[1-(1-t)^{-1/k}]) \\ \times F[x(1-t)^{-1/k}; y_1, \cdots, y_N; zt^q/(1-t)^q]\end{aligned}$$

and

$$(2.38)\qquad \begin{aligned}\sum_{n=0}^{\infty} Y_{m+n}^{\alpha-kn}(x; k)\Lambda_n(y_1, \cdots y_N; z)t^n \\ = (1+t)^{(\alpha-k+1)/k} \exp(x[1-(1+t)^{1/k}]) \\ \times G[x(1+t)^{1/k}; y_1, \cdots, y_N; zt^q/(1+t)^q] ,\end{aligned}$$

where

$$(2.39)\qquad F[x; y_1, \cdots, y_N; z] = \sum_{n=0}^{\infty} c_n Y_{m+qn}^{\alpha}(x; k)\Delta_n(y_1, \cdots, y_N)z^n ,$$

$$(2.40)\qquad G[x; y_1, \cdots, y_N; z] = \sum_{n=0}^{\infty} c_n Y_{m+qn}^{\alpha-kqn}(x;k)\Delta_n(y_1, \cdots, y_N)z^n ,$$

$c_n \neq 0$ are arbitrary complex constants, $m \geqq 0$ and $q \geqq 1$ are integers, and in terms of the non-vanishing functions $\Delta_n(y_1, \cdots, y_N)$ of N variables $y_1, \cdots, y_N$, $N \geqq 1$,

$$(2.41)\qquad \Lambda_n(y_1, \cdots, y_N; z) = \sum_{j=0}^{[n/q]} \binom{m+n}{n-qj} c_j \Delta_j(y_1, \cdots, y_N)z^j .$$

By assigning suitable values to the arbitrary coefficients c_n, it is fairly straightforward to derive, from the general formulas (2.37) and (2.38), a considerably large variety of bilateral generating functions for the polynomials $Y_n^{\alpha}(x; k)$ and $Y_n^{\alpha-kn}(x; k)$, respectively. On the other hand, in every situation in which the multivariable function $\Delta_n(y_1, \cdots, y_N)$ can be expressed as a suitable product of several simpler functions, we shall be led to an interesting class of mixed multilateral generating functions for the Konhauser polynomials considered and, of course, for the Laguerre polynomials when $k = 1$, and for the polynomial systems studied by Spencer and Fano [19] and Preiser [16] when $k = 2$.

VI. *Further finite sums.* The results to be presented here are in addition to the finite summation formulas (2.33) and (2.36) and their general forms involving the Srivastava-Singhal polynomials $G_n^{(\alpha)}(x, r, p, k)$. Indeed, from the known generating functions [24, p. 78, Eq. (3.2); p. 79, Eq. (3.4)] it is readily observed that

$$(2.42)\qquad G_n^{(\alpha)}(x, r, p, k) = \sum_{j=0}^{n-1} (-k)^j \binom{n-1}{j} G_{n-j}^{(\alpha-k+kn)}(x, r, p, k)\ ,$$

$$(2.43)\qquad G_n^{(\alpha)}(x, r, p, k) = \sum_{j=0}^{n-1} k^j \binom{n-1}{j} G_{n-j}^{(\alpha+k-kn+kj)}(x, r, p, k)$$

and

$$(2.44)\qquad G_n^{(\alpha)}(x, r, p, k) = \sum_{j=0}^{n} k^j \binom{(\alpha-\beta)/k}{j} G_{n-j}^{(\beta+kj)}(x, r, p, k)\ ,$$

which, on setting $p = r = 1$ and appealing to (2.3), yield the following new results involving the Konhauser polynomials $Y_n^\alpha(x; k)$:

$$(2.45)\qquad Y_n^\alpha(x; k) = \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} Y_{n-j}^{\alpha-k+kn}(x; k)\ ,$$

$$(2.46)\qquad Y_n^\alpha(x; k) = \sum_{j=0}^{n-1} \binom{n-1}{j} Y_{n-j}^{\alpha+k-kn+kj}(x; k)$$

and

$$(2.47)\qquad Y_n^\alpha(x; k) = \sum_{j=0}^{n} \binom{(\alpha-\beta)/k}{j} Y_{n-j}^{\beta+kj}(x; k)\ ,$$

respectively.

This last formula (2.47) is analogous to the earlier result (2.36).

3. The biorthogonal polynomials $Z_n^\alpha(x; k)$. Since the parameter k in (1.2) is restricted, by definition, to take on positive integer values, by the well-known multiplication theorem for the Γ-function we have

$$(3.1)\qquad \Gamma(kj+\alpha+1) = \Gamma(\alpha+1)\prod_{i=1}^{k}\left(\frac{\alpha+i}{k}\right)_j, \quad j = 0, 1, 2, \cdots ,$$

where $(\lambda)_n$ is given by (1.4). From (1.2) and (3.1) we obtain the hypergeometric representation

$$(3.2)\qquad Z_n^\alpha(x; k) = \frac{(\alpha+1)_{kn}}{n!}\,{}_1F_k[-n; (\alpha+1)/k, \cdots,)\alpha+k)/k; (x/k)^k]\ ,$$

which can alternatively be used to derive the following properties of the biorthogonal polynomials $Z_n^\alpha(x; k)$ by simply specializing those of the generalized hypergeometric function

$$(3.3)\qquad {}_pF_q[\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z] = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^p (\alpha_j)_m}{\prod_{j=1}^q (\beta_j)_m} \frac{z^m}{m!} ,$$

where $\beta_j \neq 0, -1, -2, \cdots, \forall\, j \in \{1, \cdots, q\}$.

I. *Differential equations.* Denoting the first member of the preceding equation (3.3) by F, we have the well-knowu hypergeometric differential equation [18, p. 77, Eq. (2)]

$$(3.4)\qquad \left[\theta \prod_{j=1}^{q} (\theta + \beta_j - 1) - z \prod_{j=1}^{p} (\theta + \alpha_j)\right] F = 0 , \quad p \leqq q + 1 ,$$

where, for convenience, $\theta = zD_z$.

In (3.4) we set $p = 1$, $q = k$, $z = (x/k)^k$, $\theta = k^{-1}\delta$, where $\delta = xD_x$, and apply the hypergeometric representation (3.2). We thus obain a differential equation satisfied by the polynomials $Z_n^\alpha(x; k)$ in the form:

$$(3.5)\qquad \left\{\prod_{j=1}^{k} (\delta + \alpha - k + j)\right\} \delta Z_n^\alpha(x; k) = x^k(\delta - kn) Z_n^\alpha(x; k) .$$

Recalling that (cf., e.g., [26, p. 310, Eq. (19)])

$$(3.6)\qquad f(\delta + \alpha)\{g(x)\} = x^{-\alpha} f(\delta)\{x^\alpha g(x)\} , \quad \delta = xD_x ,$$

it is easily verified that

$$(3.7)\qquad \prod_{j=1}^{k} (\delta + \alpha - k + j)\{g(x)\} = x^{k-\alpha} D_x^k \{x^\alpha g(x)\} ,$$

and the differential equation (3.5) obviously reduces to its equivalent form [10, p. 306, Eq. (10)]

$$(3.8)\qquad D_x^k\{x^{\alpha+1} D_x Z_n^\alpha(x; k)\} = x^\alpha (xD_x - kn) Z_n^\alpha(x; k) .$$

II. *Recurrence relations.* It is well known that (cf., e.g., [11, p. 279, Problem 20])

$$(3.9)\qquad \begin{aligned} &D_z\, {}_pF_q[\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z] \\ &\quad = \frac{\alpha_1 \cdots \alpha_p}{\beta_1 \cdots \beta_q} {}_pF_q[\alpha_1 + 1, \cdots, \alpha_p + 1; \beta_1 + 1, \cdots, \beta_q + 1; z] , \end{aligned}$$

whence, by setting $p = 1$, $q = k$, $z = (x/k)^k$, $D_z = (k/x)^{k-1} D_x$, and applying (3.2), we have

$$(3.10)\qquad D_x Z_n^\alpha(x; k) = -kx^{k-1} Z_{n-1}^{\alpha+k}(x; k) ,$$

or more generally,

$$(3.11)\qquad (x^{1-k}D_x)^m Z_n^\alpha(x;k) = (-k)^m Z_{n-m}^{\alpha+km}(x;k)\ ,\quad n \geqq m \geqq 0\ .$$

Similarly, from the known results ([18, p. 82, Eq. (12), (13) and (15); see also [17]), involving the generalized hypergeometric function (3.3), we readily obtain the following mixed recurrence relations:

$$(3.12)\qquad xD_xZ_n^\alpha(x;k) = knZ_n^\alpha(x;k) - \frac{k\Gamma(kn+\alpha+1)}{\Gamma(k(n-1)+\alpha+1)}Z_{n-1}^\alpha(x;k)\ ,$$

$$(3.13)\qquad xD_xZ_n^\alpha(x;k) = (kn+\alpha)Z_n^{\alpha-1}(x;k) - \alpha Z_n^\alpha(x;k)\ ,$$

$$(3.14)\qquad Z_n^\alpha(x;k) - Z_n^{\alpha-1}(x;k) = \frac{k\Gamma(kn+\alpha)}{\Gamma(k(n-1)+\alpha+1)}Z_{n-1}^\alpha(x;k)\ .$$

It is not difficult to verify that the recurrence relation (3.14) results from (3.12) and (3.13) by eliminating their common term $xD_xZ_n^\alpha(x;k)$. If, however, we eliminate this derivative term in (3.12) or (3.13) by using (3.10) instead, we shall arrive at the recurrence relations

$$(3.15)\qquad x^kZ_n^{\alpha+k}(x;k) = (kn+\alpha+1)_kZ_n^\alpha(x;k) - (n+1)Z_{n+1}^\alpha(x;k)$$

and[1]

$$(3.16)\qquad kx^kZ_n^{\alpha+k}(x;k) = \alpha Z_{n+1}^\alpha(x;k) - (kn+\alpha+k)Z_{n+1}^{\alpha-1}(x;k)\ .$$

Formulas[2] (3.10) and (3.12) were given earlier by Konhauser [10, p. 306, Eq. (8); p. 305, Eq. (6)], (3.14) is due to Génin et Calvez [7, p. A1565, Eq. (5)], while (3.15) was derived by Prabhakar [14, p. 215, Eq. (2.6)] by using a contour integral representation for $Z_n^\alpha(x;k)$.

For a direct proof of (3.15), we observe from (1.2) that

$$\begin{aligned} x^kZ_n^{\alpha+k}&(x;k)\\ &= \frac{\Gamma(k(n+1)+\alpha+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{k(j+1)}}{\Gamma(k(j+1)+\alpha+1)}\\ &= \frac{\Gamma(k(n+1)+\alpha+1)}{n!}\sum_{j=1}^{n+1}(-1)^{j-1}\binom{n}{j-1}\frac{x^{kj}}{\Gamma(kj+\alpha+1)}\ ,\end{aligned}$$

and since

$$-\binom{n}{j-1} = \binom{n}{j} - \binom{n+1}{j}\ ,\quad 0 \leqq j \leqq n+1\ ,$$

it follows that

[1] The pure recurrence relation (3.16) appears *erroneously* in a recent paper by K. R. Patil and N. K. Thakare [*J. Mathematical Phys.* **18** (1977), 1724-1726; especially p. 1725].

[2] It may be of interest to mention here that the known results (3.10) and (3.15) were rederived, using Prabhakar's version [14, p. 214, Eq. (2.2)] of the generating function (3.20) of this paper, by B. Nath [*Kyungpook Math. J.* **14** (1974), 81-82].

$$
\begin{aligned}
&x^k Z_n^{\alpha+k}(x; k) \\
&= \frac{\Gamma(k(n+1)+\alpha+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\alpha+1)} \\
&\quad - \frac{\Gamma(k(n+1)+\alpha+1)}{n!} \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} \frac{x^{kj}}{\Gamma(kj+\alpha+1)} \\
&= (kn+\alpha+1)_k Z_{n+1}^{\alpha}(x; k) - (n+1) Z_n^{\alpha}(x; k) \ ,
\end{aligned}
$$

which precisely is the pure recurrence relation (3.15).

III. *Generating functions.* Chaundy [5] has shown that [op. cit., p. 62, Eq. (25)]

$$
\begin{aligned}
(3.17)\qquad &\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_{p+1}F_q[-n, \alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z] t^n \\
&= (1-t)^{-\lambda} {}_{p+1}F_q[\lambda, \alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; zt/(t-1)] \ , \\
&\quad |t| < 1 \ .
\end{aligned}
$$

If we replace t on both sides of (3.17) by t/λ and take their limits as $\lambda \to \infty$, we shall readily obatin Rainville's result:

$$
\begin{aligned}
(3.18)\qquad &\sum_{n=0}^{\infty} {}_{p+1}F_q[-n, \alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z] \frac{t^n}{n!} \\
&= e^t {}_pF_q[\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; -zt] \ .
\end{aligned}
$$

Both (3.17) and (3.18) are stated by Erdélyi et al. [6, p. 267, Eq. (22) and (25)], and their various generalizations have appeared in the literature (cf., e.g., [20, p. 68, Eq. (3.9) and (3.10)].

By specializing (3.17) and (3.18) in view of the hypergeometric representation (3.2) for $Z_n^{\alpha}(x; k)$, we at once get the generating functions

$$
\begin{aligned}
(3.19)\qquad &\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\alpha+1)_{kn}} Z_n^{\alpha}(x; k) t^n \\
&= (1-t)^{-\lambda} {}_1F_k\left[\lambda; \frac{\alpha+1}{k}, \cdots, \frac{\alpha+k}{k}; \frac{x^k t}{(t-1)k^k}\right], \quad |t| < 1
\end{aligned}
$$

and

$$
\begin{aligned}
(3.20)\qquad &\sum_{n=0}^{\infty} Z_n^{\alpha}(x; k) \frac{t^n}{(\alpha+1)_{kn}} \\
&= e^t {}_0F_k\left[-; \frac{\alpha+1}{k}, \cdots, \frac{\alpha+k}{k}; -\left(\frac{x}{k}\right)^k t\right],
\end{aligned}
$$

respectively.

The generating function (3.19) is due essentially to Génin et Calvez [7, p. A1564, Eq. (3)], while (3.20) was given by Srivastava

[21, p. 490, Eq. (7)]; the latter appears also, with an obvious typographycal error, in a recent paper [12, p. 922]. In fact, both (3.19) and (3.20) were given (in disguised forms) by Prabhakar [14, p. 218, Eq. (4.1); p. 214, Eq. (2.2)]. Notice that the so-called generalized Mittag-Leffler function $E^{\lambda}_{k,\alpha+1}(z)$ and the "Bessel-Maitland" function[3] $\phi(k, \alpha + 1; z)$, occurring in Prabhakar's results just cited, are indeed the familiar hypergeometric functions ${}_1F_k$ and ${}_0F_k$, respectively, k being a positive integer. More precisely, we have, for $k = 1, 2, 3, \cdots$,

$$(3.21)\qquad \begin{aligned} E^{\lambda}_{k,\alpha+1}(z) &= \sum_{m=0}^{\infty} \frac{(\lambda)_m z^m}{m!\,\Gamma(km + \alpha + 1)} \\ &= \frac{1}{\Gamma(\alpha + 1)}\, {}_1F_k\left[\lambda;\, \frac{\alpha+1}{k}, \cdots, \frac{\alpha+k}{k};\, \left(\frac{z}{k}\right)^k\right] \end{aligned}$$

and

$$(3.22)\qquad \begin{aligned} \phi(k, \alpha + 1; z) &= \sum_{m=0}^{\infty} \frac{z^m}{m!\,\Gamma(km + \alpha + 1)} \\ &= \frac{1}{\Gamma(\alpha + 1)}\, {}_0F_k\left[-;\, \frac{\alpha+1}{k}, \cdots, \frac{\alpha+k}{k};\, \left(\frac{z}{k}\right)^k\right], \end{aligned}$$

by appealing to the well-known multiplication theorem for the Γ-function.

Next we consider the double series

$$\begin{aligned} &\sum_{m=0}^{\infty} z^m \sum_{n=0}^{\infty} \binom{m+n}{n} Z^{\alpha}_{m+n}(x; k) \frac{t^n}{(\alpha+1)_{k(m+n)}} \\ &= \sum_{n=0}^{\infty} \frac{Z^{\alpha}_n(x; k)}{(\alpha+1)_{kn}} \sum_{m=0}^{n} \binom{n}{m} t^{n-m} z^m = \sum_{n=0}^{\infty} Z^{\alpha}_n(x; k) \frac{(z+t)^n}{(\alpha+1)_{kn}} \\ &= e^{z+t}\, {}_0F_k\left[-;\, \frac{\alpha+1}{k}, \cdots, \frac{\alpha+k}{k};\, -\left(\frac{x}{k}\right)^k (z+t)\right], \quad \text{by (3.20)}, \\ &= \sum_{n,\nu=0}^{\infty} \frac{(-x^k)^n}{n!\,\nu!\,(\alpha+1)_{kn}} \sum_{m=0}^{n+\nu} \binom{n+\nu}{m} t^{n+\nu-m} z^m \\ &= \sum_{m=0}^{\infty} z^m \sum_{u+\nu \geqq m} \binom{n+\nu}{m} \frac{t^{n-m}}{n!} \frac{(-x^k)^n}{(\alpha+1)_{kn}} \frac{t^{\nu}}{\nu!}, \end{aligned}$$

and on equating the coefficients of z^m, we have the generating relation

$$(3.23)\qquad \begin{aligned} &\sum_{n=0}^{\infty} \binom{m+n}{n} Z^{\alpha}_{m+n}(x; k) \frac{t^n}{(\alpha+1)_{k(m+n)}} \\ &\qquad = \sum_{n=m}^{\infty} \binom{n}{m} \frac{t^{n-m}}{n!} \frac{(-x^k)^n}{(\alpha+1)_{kn}}\, {}_1F_1[n+1; n-m+1; t], \end{aligned}$$

[3] Incidentally, the generalized Bessel function $\phi(\alpha, \beta; z)$ was introduced by E. Maitland Wright [**27**, p. 72, Eq. (I.3)]; see also Erdélyi et al. [**6**, p. 211, Eq. (27)].

which holds true for every non-negative integer m

Alternatively, this last generating relation (3.23) may be derived as a special case of our earlier result [20, p. 68, Theorem 3]. Of course, it is not difficult to develop a *direct* proof of (3.23) without using the generating function (3.20).

For $m = 0$, (3.23) evidently reduces to the familiar generating function (3.20). Its special case when $k = 1$ leads to what is obviously contained in the following limiting form of a known result [22, p. 152, Eq. (19)]:

$$\begin{aligned}(3.24)\qquad \sum_{n=0}^{\infty}\binom{m+n}{n}L_{m+n}^{(\alpha)}(x)\frac{t^n}{(\mu)_n} \\ = \binom{\alpha+m}{m}e^x\psi_2[\alpha+m+1;\mu,\alpha+1;t,-x]\,,\end{aligned}$$

where ψ_2 is a (Humbert's) confluent hypergeometric function of two variables defined by [1, p. 126]

$$(3.25)\qquad \psi_2[a;c,c';x,y] = \sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!}\,.$$

Formula (3.24) follows from the known generating function [22, p. 152, Eq. (19)] by writing t/λ in place of t and then letting $\lambda \to \infty$. Furthermore, if we replace t in (3.24) by μt and let $\mu \to \infty$, we shall arrive at the well-known generating function [18, p. 211, Eq. (9)]

$$\begin{aligned}(3.26)\qquad \sum_{n=0}^{\infty}\binom{m+n}{n}L_{m+n}^{(\alpha)}(x)t^n = (1-t)^{-m-\alpha-1}\exp\left(-\frac{xt}{1-t}\right) \\ \times L_m^{(\alpha)}\left(\frac{x}{1-t}\right)\,,\quad m=0,1,2,\cdots\,,\end{aligned}$$

which follows also from (2.17) when $k = 1$.

IV. *Multilinear generating functions.* By making use of the hypergeometric representation (3.2), a number of new multilinear generating functions for the product

$$(3.27)\qquad Z_{n_1}^{\alpha_1}(y_1;k_1)\cdots Z_{n_r}^{\alpha_r}(y_r;k_r)\,,$$

analogous to the Patil-Thakare result [12, p. 921, Eq. (2.1)], can be derived by suitably specializing a general formula earlier by Srivastava and Singhal [25, p. 1244, Eq. (24)] for a product of several generalized hypergeometric polynomials. We omit the details involved.

V. *Finite summation formulas.* In view of the exponential generating function (3.20), Theorem 1 (p. 64) of Srivastava [20] will apply to the biorthogonal polynomials $Z_n^\alpha(x; k)$, and we thus have

$$Z_n^\alpha(x; k) = \left(\frac{x}{y}\right)^{kn} \sum_{j=0}^{n} \binom{\alpha + kn}{kj} \frac{(kj)!}{j!} \left(\frac{y^k - x^k}{x^k}\right)^j Z_{n-j}^\alpha(y; k) , \tag{3.28}$$

or equivalently,

$$Z_n^\alpha(x; k) = \left(\frac{x}{y}\right)^{kn} \sum_{j=0}^{n} \binom{\alpha + kn}{kn - kj} \frac{(kn - kj)!}{(n - j)!} \left(\frac{y^k - x^k}{x^k}\right)^{n-j} Z_j^\alpha(y; k) . \tag{3.29}$$

The summation formula (3.28) can indeed be derived directly (cf. [21, p. 490, § 4]). It can also be rewritten in the form [op. cit., p. 491, Eq. (12)]:

$$Z_n^\alpha(\mu x; k) = \sum_{j=0}^{n} \binom{kn + \alpha}{kj} \frac{(kj)!}{j!} \mu^{k(n-j)} (1 - \mu^k)^j Z_{n-j}^\alpha(x; k) , \tag{3.30}$$

which obviously provides us with an elegent multiplication formula for the biorthogonal polynomials $Z_n^\alpha(x; k)$.

VI. *Laplace transforms.* Employing the usual notation for Laplace's transform, viz

$$\mathscr{L}\{f(t): s\} = \int_0^\infty e^{-st} f(t) dt , \qquad \mathrm{Re}\,(s - \sigma) > 0 , \tag{3.31}$$

where $f \in L(0, R)$ for every $R > 0$, and $f(t) = O(e^{\sigma t})$, $t \to \infty$, we have

$$\begin{aligned} &\mathscr{L}\{t^\beta Z_n^\alpha(xt; k): s\} \\ &\quad = \frac{(\alpha + 1)_{kn} \Gamma(\beta + 1)}{s^{\beta+1} n!} \\ &\qquad \times {}_{k+1}F_k\left[-n, \frac{\beta+1}{k}, \ldots, \frac{\beta+k}{k}; \frac{\alpha+1}{k}, \ldots, \frac{\alpha+k}{k}; \left(\frac{x}{s}\right)^k\right], \end{aligned} \tag{3.32}$$

provided that $\mathrm{Re}\,(s) > 0$ and $\mathrm{Re}\,(\beta) > -1$.

The Laplace transform formula (3.32) can be derived fairly easily from the hypergeometric representation (3.2) by using readily available tables. In the special case when $\beta = \alpha$, it simplifies at once to the elegant form [14, p. 217, Eq. (3.7)]:

$$\mathscr{L}\{t^\alpha Z_n^\alpha(xt; k): s\} = \frac{\Gamma(kn + \alpha + 1)}{s^{kn+\alpha+1} n!} (s^k - x^k)^n , \tag{3.33}$$

where, as before, $\mathrm{Re}\,(s) > 0$ and (by definition) $\alpha > -1$.

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