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APPROXIMATING COMPACT SETS IN NORMED LINEAR SPACES

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It is shown that in normed linear spaces compact sets can be approximated by compact absolute neighborhood retracts in the following sense: If X is a compact subset of a normed linear space, then for every $\varepsilon > 0$ there exists a compact absolute neighborhood retract that contains X and has the property that each point of the retract is within ε of X. If the choice of ε is sufficiently large, the retract can be chosen to be an absolute retract.

Suppose that X is a compact subset of a Banach space B. Then the closure of the convex hull of X, $\overline{\text{conv}(X)}$, is a compact absolute retract that contains X. Browder [4] has shown that if U is an open subset of B that contains X, then there exists a compact absolute neighborhood retract R^* such that $X \subseteqq R^* \subseteqq U$. Both of these results have proven to be useful in Fixed Point Theory. See, for example, the work of Browder mentioned above and the work of Górniewicz and Granas [9].

Let X be a compact subset of a normed linear space N. The purpose of this paper, Theorem 1, is to show that there exists a compact absolute retract R such that $X \subseteqq R \subseteqq N$. Further, it is shown that if U is an open subset of N that contains X, then there exists a compact absolute neighborhood retract R^* such that $X \subseteqq R^* \subseteqq U$.

1. Preliminaries. Absolute retracts and absolute neighborhood retracts for metric spaces will be denoted by AR and ANR respectively. We use the notation $d(x, E)(d(x, y))$ for the distance from a point x to a set E (to a point y). A continuous function $f: X \to R$ will be called a retraction if $R \subseteqq X$ and $f(x) = x$ for each $x \in R$.

LEMMA 1. *Let $(N, \| \ \|)$ be an infinite dimensional normed linear space, X be a compact subset of N, F be a finite dimensional subspace that is disjoint from X, and ε be greater than 0. Then there exists a finite dimensional subspace E that contains F, is disjoint from X, and for all $x \in X$, $d(x, E) < \varepsilon$.*

Proof Let U_* be an open subset of N. We show that there exists a finite dimensional subspace E_* that contains F, meets U, and is disjoint from X. Let B be the closure of an open set that is contained in U and is disjoint from X. For each $b \in B$, let E_b be the subspace generated by b and F. Suppose that for each such b, $E_b \cap X \neq \emptyset$. Let b_n be an arbitrary sequence in B, and let $x_n \in E_{b_n} \cap X$. Now b_n can be expressed in the form $b_n = v_n + t_n x_n$ where $v_n \in F$ and t_n is a real number. The sequences $\|v_n\|$ and $|t_n|$ are bounded, and the sequence x_n lies in the compact set X. Thus there exist subsequences v_{n_k}, x_{n_k}, t_{n_k}, vectors $v \in F$, $x \in X$ and $t \in R$ such that $b_{n_k} = v_{n_k} + t_{n_k} x_{n_k} \to v + tx$. Since B is closed $v + tx \in B$. This leads us to conclude that B is compact contrary to the fact that B has nonempty interior. Therefore, there exists a subspace E_* satisfying the desired properties.

Now cover X with a finite collection of open sets $U_1, U_2, \cdots, U_n$, each with radius less than $\varepsilon/2$. By applying the result in the above paragraph n times, we are able to construct a finite dimensional subspace E that contains F, is disjoint from X, and meets each of the U_j. Let $x \in X$. There exists a U_j and a $y \in E$ such that $x, y \in U_j$. Then $d(x, E) \leqq d(x, y) < \varepsilon$, and this completes the proof.

DEFINITION 1. [5] Let $(N, \| \ \|)$ be a normed linear space. Then the norm is said to be strictly convex if for all x, y not equal to 0, $\|x + y\| = \|x\| + \|y\|$ implies that $y = px$ for some $p > 0$.

Assume that $(N, \| \ \|)$ is a strictly convex normed linear space and E is a finite dimensional subspace of N. It was observed in [2] that for each $x \in N$ there exists a unique closest point, denoted by $\phi(x)$, in E. That is, $\phi(x) \in E$ and $d(x, \phi(x)) = d(x, E)$. The resulting function $\phi: N \to E$, which is called a metric projection, has the following properties that are easily verified [2, 12].

($\mathscr{P}_1$) ϕ is continuous,

($\mathscr{P}_2$) ϕ is idempotent: $\phi^2 = \phi$,

($\mathscr{P}_3$) ϕ is homogeneous: $\phi(tx) = t\phi(x)$ for all $t \in R$ and $x \in N$, and

($\mathscr{P}_4$) ϕ is quasi additive: $\phi(x + y) = \phi(x) + y$ for all $x \in N$ and $y \in E$.

We establish $\mathscr{P}_1$. Let $x \in X$ and suppose x_n is a sequence that converges to x. Without loss of generality we may assume that $\phi(x_n)$ converges to some point $y \in E$. Then $\|x - y\| = \lim_{n \to \infty} \|x - \phi(x_n)\| = d(x, E)$. So $y = \phi(x)$, and we conclude that ϕ is continuous.

LEMMA 2. *Let N be a strictly convex normed linear space, E be a finite dimensional subspace of N, R be an absolute neighborhood retract in E, $\phi: N \to E$ be the metric projection, and e be greater than* 0. *Then $\phi^{-1}(R) = \{x \in N: \phi(x) \in R\}$ and $\{x \in \phi^{-1}(R): d(x, R) \leqq e\}$ are absolute neighborhood retracts.*

Proof. There exists a neighborhood U_* of R in E and a retraction $r_*: U_* \to R$. Set $U = \phi^{-1}(U_*)$ and define $r: U \to \phi^{-1}(R)$ by $r(x) =$

$x + r_*(\phi(x)) - \phi(x)$. It follows by properties $\mathscr{P}_1$ and $\mathscr{P}_4$ that r is a retraction.

Next set $A = \{x \in \phi^{-1}(R) \colon d(x, R) \leqq e\}$ and define $s \colon \phi^{-1}(R) \to A$ by

$$s(x) = \begin{cases} x \quad \text{if } d(x, R) \leqq e \\ \left[\dfrac{d(x, R) - e}{d(x, R)}\right]\phi(x) + \dfrac{ex}{d(x, R)} \quad \text{if } d(x, R) \geqq e \,. \end{cases}$$

The function s is a retraction. Since a retract of an ANR is an ANR, the proof of the lemma is complete.

2. The approximation theorem. A function $f \colon X \to R$ will be called compact retraction provided f is a retraction and R is compact. If N is a normed linear space, and $x \in N$, then $B_\varepsilon(x) = \{y \in N \colon d(x, y) \leqq \varepsilon\}$ is called an N-ball. In order to simplify the proof of the approximation theorem, we state the following definition.

DEFINITION 2. Let K be a compact subset of a normed linear space N. Then an ε-pair of K in N, denoted by $(N, K, P^*, P, \varepsilon)$, consists of ANR's P^* and P such that $K \subseteqq \operatorname{Int}(P^*)$, $P^* \subseteqq P \subseteqq N$ and if $x \in P^*$, $y \in P$ and $d(x, y) \leqq \varepsilon$, then the segment $[x, y] = \{tx + (1-t)y \colon 0 \leqq t \leqq 1\} \subseteqq P$.

The proof of the approximation theorem is similar in certain respects to [3, p. 108].

THEOREM. *Let $(N, \| \ \|)$ be a normed space and let X be a compact subset of N. Then there exists a compact absolute retract R such that $X \subseteqq R \subseteqq N$. If U is an open subset of N that contains X, then there exists a compact absolute neighborhood retract R^* such that $X \subseteqq R^* \subseteqq U$.*

Proof. A straightforward argument establishes the result when the dimension of N is finite. In that which follows we assume that the dimension of N is infinite.

Let D be a countable dense subset of X. Then the closure of the linear span of D is a separable normed linear space that contains X. Thus, without loss of generality, we may assume that N is separable. Further, we may assume that X does not contain the origin. Every separable normed linear space has an equivalent strictly convex normed [5]. Consequently, we may assume that $\| \ \|$ is strictly convex.

It will be shown that for $n = 1, 2, 3, \cdots$, there exists

(I_n) a finite dimensional subspace $E_n \supseteqq E_{n-1}(E_0 = \varnothing)$ with metric projection $\phi_n \colon N \to E_n$ such that if $x \in X$ then $d(x, E_n) < \varepsilon_n \leqq \varepsilon_{n-1}/18$ $(\varepsilon_0 = 18)$,

(II_n) a $3\varepsilon_n$-pair of $\phi_n(X)$ in E_n, $(E_n, \phi_n(X), P_n^*, P_n, 3\varepsilon_n)$,

(III_n) an ANR $A_n = \{x: x \in \phi_n^{-1}(P_n) \text{ and } d(x, P_n) \leqq 3\varepsilon_n\}$ ($A_0 = N$) such that $X \subseteqq \operatorname{Int} A_n$, $A_n \subseteqq \operatorname{Int}(A_{n-1})$ and $P_{n-1} \cap A_n = \varnothing (P_0 = \varnothing)$, and

(IV_n) a compact retraction $f_n: A_{n-1} \to R_n$ ($R_0 = \varnothing, f_0 = \varnothing$) that satisfies $R_n \cap A_n = P_n$, $R_n \cap R_{n-1} = P_{n-1}$, $f_n(x) = f_{n-1}(x)$ for $x \in bd(A_{n-1})$, $f_n(x) = \phi_n(x)$ for $x \in A_n$, and if $x \in A_{n-1}$ and $d(x, R_n) \leqq 3$, then $d(x, f_n(x)) \leqq 3\varepsilon_{n-1}$.

Let $\varepsilon_1 = 1$. By Lemma 1 there exists a finite dimensional subspace E_1 such that if $x \in X$ then $d(x, E_1) < \varepsilon_1$, and $X \cap E_1 = \varnothing$. Let ϕ_1: $N \to E_1$ be the corresponding metric projection. There exists a finite number of points $p_1^1, \cdots, p_{k_1}^1 \in \phi_1(X)$ and corresponding E_1-balls $B_{\varepsilon_1/2}(p_1^1), \cdots, B_{\varepsilon_1/2}(p_{k_1}^1)$ such that $\phi_1(X) \subseteqq \operatorname{Int} \bigcup_{i=1}^{k_1} B_{\varepsilon_1/2}(p_i^1)$.

Set $$P_1^* = \bigcup_{i=1}^{k_1} B_{\varepsilon_1/2}(p_i^1) \quad \text{and} \quad P_1 = \{x \in E_1: d(x, P_1^*) \leqq 3\varepsilon_1\} .$$

It is easy to see that P_1^* and P_1 are ANR's [3, p. 90] and it follows that $(E_1, \phi_1(X), P_1, P_1^*, 3\varepsilon_1)$ is a $3\varepsilon_1$-pair of $\phi_1(X)$ in E_1. Set $A_1 = \{x: x \in \phi_1^{-1}(P_1) \text{ and } d(x, P_1) \leqq 3\varepsilon_1\}$. Clearly, $X \subseteqq \operatorname{Int} A_1$, $A_1 \subseteqq N = \operatorname{Int}(A_0)$ and $P_0 \cap A_1 = \varnothing \cap A_1 = \varnothing$. Set $R_1 = \operatorname{conv}(P_1)$. There exists a retraction[1] $s: E_1 \to R_1$. We define $f_1: N \to R_1$ by $f_1 = s \circ \phi_1$. Clearly, $R_1 \cap A_1 = P_1$, $R_1 \cap R_0 = \varnothing = P_0$, $f_1(x) = f_0(x)$ for $x \in bd(A_0)$ and $f_1(x) = \phi_1(x)$ for $x \in A_1$. Suppose $x \in A_0$ and $d(x, R_1) \leqq 3$. Then it is easy to see that $d(x, f_1(x)) \leqq 3\varepsilon_0$. Thus, the four conditions are satisfied for the case $n = 1$.

Now assume that for $k = 1, 2, \cdots, n$ the conditions can be satisfied. We show that for $k = n + 1$, there exist appropriate functions and sets that satisfy the conditions.

By condition (III_n) we have $X \subseteqq \operatorname{Int}(A_n) = \{x: x \in \phi_n^{-1}(P_n) \text{ and } d(x, P_n) \leqq 3\varepsilon_n\}$. There exists an open set W_n of N such that $X \subseteqq W_n \subseteqq A_n$, $W_n \cap P_n = \varnothing$, and $\phi_n(W_n) \subseteqq \operatorname{Int}(P_n^*)$. This follows from ($II_n$). Let $\varepsilon_{n+1}^* = d(X, N - W_n)$.[2] Set

$$\varepsilon_{n+1} < \min\{\varepsilon_n/18, \varepsilon_{n+1}^*/8\} .$$

By Lemma 2 there exists a finite dimensional subspace E_{n+1} with metric projection $\phi_{n+1}: N \to E_{n+1}$ such that if $x \in X$ then $d(x, E_{n+1}) < \varepsilon_{n+1}$, $E_n \subseteqq E_{n+1}$, and $X \cap E_{n+1} = \varnothing$. Thus, condition ($I_{n+1}$) is satisfied.

There exists a finite number of points $p_1^{n+1}, p_2^{n+1} \cdots p_{k_{n+1}}^{n+1} \in \phi_{n+1}(X)$ and corresponding E_{n+1}-balls $B_{\varepsilon_{n+1}/2}(p_1^{n+1}), \cdots, B_{\varepsilon_{n+1}/2}(p_{k_{n+1}}^{n+1})$ such that $\phi_{n+1}(X) \subseteqq \operatorname{Int} \bigcup_{i=1}^{k_{n+1}} B_{\varepsilon_{n+1}/2}(p_i^{n+1})$. Set

$$P_{n+1}^* = \bigcup_{i=1}^{k_{n+1}} B_{\varepsilon_{n+1}/2}(p_i^{n+1}) \quad \text{and} \quad P_{n+1} = \{x \in E_{n+1}: d(x, P_{n+1}^*) \leqq 3\varepsilon_{n+1}\} .$$

[1] The retraction is constructed in such a manner that $d(x, s(x)) \leqq 2d(x, R_1)$.

[2] $d(X, N - W_n) = \inf\{d(x, N - W_n): x \in X\}$

It is easy to see that P^*_{n+1} and P_{n+1} are ANR's [3, p. 90], and it follows that $(E_{n+1}, \phi_{n+1}(X), P^*_{n+1}, P_{n+1}, 3\varepsilon_{n+1})$ is a $3\varepsilon_{n+1}$-pair of $\phi_{n+1}(X)$ in E_{n+1}. Thus condition $(\prod_{n+1})$ is satisfied.

Suppose $x \in P_{n+1}$. Then there exists a $B_{\varepsilon_{n+1}/2}(p_i^{n+1})$ and a $y \in B_{\varepsilon_{n+1}/2}(p_i^{n+1})$ such that $d(x, y) \leqq 3\varepsilon_{n+1}$. There exists a $z \in X$ such that $\phi_{n+1}(z) \in B_{\varepsilon_{n+1}/2}(p_i^{n+1})$. Thus $d(x, z) \leqq d(x, y) + d(y, \phi_{n+1}(z)) + d(\phi_{n+1}(z), z) < 5\varepsilon_{n+1}$. We conclude the following:

$$(1) \qquad \text{If } x \in P_{n+1} \text{ then } d(x, X) < 5\varepsilon_{n+1} .$$

Set $A_{n+1} = \{x: x \in \phi^{-1}_{n+1}(P_{n+1}) \text{ and } d(x, P_{n+1}) \leqq 3\varepsilon_{n+1}\}$. By Lemma 2, A_{n+1} is an ANR. We have $\phi_{n+1}(X) \subseteqq \operatorname{Int} P^*_{n+1}$ and if $x \in X$ then $d(x, P_n) < \varepsilon_{n+1}$. Thus, $X \subseteqq \operatorname{Int}(A_{n+1})$. Let $x \in A_{n+1}$. Then $d(x, \phi_{n+1}(x)) \leqq 3\varepsilon_{n+1}$ and by (1) $d(\phi_{n+1}(x), X) < 5\varepsilon_{n+1}$. So $d(x, X) < 8\varepsilon_{n+1} < \varepsilon^*_{n+1}$. Thus, $x \in W_n$ and it follows, from the fact that $A_{n+1} \subseteqq W_n \subseteqq \operatorname{Int} A_n$, that $A_{n+1} \subseteqq \operatorname{Int} A_n$. By construction $P_n \cap A_{n+1} = \emptyset$. Condition (III_{n+1}) is satisfied. We also note that $\phi_n(P_{n+1}) \subseteqq P^*_n$. This follows since $P_{n+1} \subseteqq W_n$.

We set $B_{n+1} = \{x: x \in E_{n+1} \cap \phi_n^{-1}(P^*_n) \text{ and } d(x, P^*_n) \leqq (23/18)\varepsilon_n\}$. Suppose $x \in P_{n+1}$. Then $x \in E_{n+1}$. Also, $d(x, P^*_n) \leqq d(x, X) + \varepsilon_n$. By (1) and the definitions of P^*_n and ε_{n+1}, we have $d(x, P^*_n) \leqq 5\varepsilon_{n+1} + \varepsilon_n \leqq (23/18)\varepsilon_n$. We conclude that $P_{n+1} \subseteqq B_{n+1}$. By Lemma 2 and the fact that E_{n+1} is finite dimensional, we have that B_{n+1} is a compact ANR. Furthermore, it is clear that $B_{n+1} \subseteqq \operatorname{Int}(A_n)$. We defined

$$R^*_{n+1} = P_n \cup B_{n+1} \cup A_{n+1} .$$

It is clear that R^*_{n+1} is a closed subspace of A_n and by [3, p. 90] R_{n+1} is an ANR. So there exists an open subset U^*_{n+1} of R_{n+1} in A_n and a retraction $r_{n+1}: U^*_{n+1} \to R^*_{n+1}$. For each $x \in A_{n+1} \cup B_{n+1}$ there exists a pair of neighborhoods M_x^{n+1}, N_x^{n+1} such that $\operatorname{dia}(M_x^{n+1}) < \varepsilon_{n+1}/2$, $\operatorname{dia} \phi_n(M_x^{n+1}) < \varepsilon_{n+1}$, $N_x^{n+1} \subseteqq M_x^{n+1} \subseteqq U^*_{n+1}$ and $r_{n+1}(N_x^{n+1}) \subseteqq M_x^{n+1}$. Set

$$U_{n+1} = \bigcup \{N_x^{n+1}: x \in A_{n+1} \cup B_{n+1}\} .$$

Now suppose $x \in U_{n+1}$. Then it is easy to see that $\phi_n(\phi_{n+1}(r_{n+1}(x))) \in P_n$. We argue that the segment $[\phi_n(\phi_{n+1}(r_{n+1}(x))), \phi_n(x)] \subseteqq P_n$. Assume $r_{n+1}(x) \in A_{n+1}$. Then there exists an M_y such that $x, r_{n+1}(x) \in M_y$. Since $\operatorname{dia}(M_y) < \varepsilon_{n+1}/2$, $d(x, r_{n+1}(x)) < \varepsilon_{n+1}$. By the definition of A_{n+1} it follows that $d(\phi_{n+1}(r_{n+1}(x)), r_{n+1}(x)) < 3\varepsilon_{n+1}$. By (1) $d(\phi_{n+1}(r_{n+1}(x)), X) < 5\varepsilon_{n+1}$. From condition (I_n), we conclude that if $z \in X$ then $d(z, P_n) < \varepsilon_n$. Combining the above we get

$$d(x, \phi_n(\phi_{n+1}(r_{n+1}(x)))) \leqq d(x, r_{n+1}(x)) + d(r_{n+1}(x), \phi_{n+1}(r_{n+1}(x))) + d(\phi_{n+1}(r_{n+1}(x)), X) + \varepsilon_n < 9\varepsilon_{n+1} + \varepsilon_n .$$

Thus, $d(x, \phi_n(x)) \leqq 9\varepsilon_{n+1} + \varepsilon_n$ and

$$d(\phi_n(\phi_{n+1}(r_{n+1}(x))), \phi_n(x)) \leqq 18\varepsilon_{n+1} + 2\varepsilon_n < 3\varepsilon_n .$$

By (II_n) the segment $[\phi_n(\phi_{n+1}(r_{n+1}(x))), \phi_n(x)] \subseteq P_n$. Suppose $r_{n+1}(x) \in B_{n+1}$. As in the case above, $d(x, r_{n+1}(x)) < \varepsilon_{n+1}$. Note that in this case $r_{n+1}(x) = \phi_{n+1}(r_{n+1}(x))$. By the definition of B_{n+1}, $d(\phi_{n+1}(r_{n+1}(x)), \phi_n(\phi_{n+1}(r_{n+1}(x))) \leqq (23/18)\varepsilon_n$. So

$$\begin{aligned} d(x, \phi_n(\phi_{n+1}(r_{n+1}(x)))) &\leqq d(x, r_{n+1}(x)) + d(\phi_{n+1}(r_{n+1}(x)), \phi_n(\phi_{n+1}(r_{n+1}(x)))) \\ &< \varepsilon_{n+1} + \frac{23}{18}\varepsilon_n . \end{aligned}$$

Thus, $d(x, \phi_n(x)) \leqq \varepsilon_{n+1} + (23/18)\varepsilon_n$ and

$$d(\phi_n(\phi_{n+1}(r_{n+1}(x))), \phi_n(x)) \leqq 2\varepsilon_{n+1} + \frac{23}{9}\varepsilon_n < \frac{24}{9}\varepsilon_n < 3\varepsilon_n .$$

Thus, by (II_n) the segment $[\phi_n(\phi_{n+1}(r_{n+1}(x))), \phi_n(x)]\subseteq P_n$. Finally, suppose $r_{n+1}(x) \in P_n$. As in the cases above, $d(x, r_{n+1}(x)) < \varepsilon_{n+1}$. Since $r_{n+1}(x) \in P_n$, $d(x, E_n) \leqq \varepsilon_{n+1}$. Thus, $d(r_{n+1}(x), \phi_n(x)) < 2\varepsilon_{n+1}$. But in this case $r_{n+1}(x) = \phi_n(\phi_{n+1}(r_{n+1}(x)))$. So $d(\phi_n(\phi_{n+1}(r_{n+1}(x))), \phi_n(x)) < 2\varepsilon_{n+1} < 3\varepsilon_n$. We also conclude in this final case that the segment $[\phi_n(\phi_{n+1}(r_{n+1}(x))), \phi_n(x)] \subseteq P_n$.

Set $R_{n+1} = P_n \cup B_{n+1}$. For each $x \in U_{n+1}$ define $a_{n+1}(x) = d(x, A_{n+1} \cup B_{n+1})$ and $b_{n+1}(x) = d(x, A_n - U_n)$. We define

$$f_{n+1}: A_n \longrightarrow R_{n+1}$$

by

$$f_{n+1}(x) = \begin{cases} \phi_n(x) \text{ if } x \in A_n - U_{n+1} , \\ \dfrac{b_{n+1}(x)(\phi_n(\phi_{n+1}(r_{n+1}(x)))) + (a_{n+1}(x) - b_{n+1}(x))\phi_n(x)}{a_{n+1}(x)} \\ \qquad a_{n+1}(x) \geqq b_{n+1}(x) \\ \dfrac{a_{n+1}(x)(\phi_n(\phi_{n+1}(r_{n+1}(x)))) + (b_{n+1}(x) - a_{n+1}(x))\phi_{n+1}(r_{n+1}(x))}{b_{n+1}(x)} \text{ if} \\ \qquad a_{n+1}(x) \leqq b_{n+1}(x) , \\ \phi_{n+1}(x) \colon x \in A_{n+1} . \end{cases}$$

By $\mathscr{P}_3$ and $\mathscr{P}_4$ we have that if $x \in B_{n+1}$, then the segment $[x, \phi_n(x)]\subseteq B_{n+1}$. It follows that f_{n+1} is a compact retraction from A_n to R_{n+1}, $R_{n+1} \cap A_{n+1} = P_{n+1}$, $R_{n+1} \cap R_n = P_n$, $f_{n+1}(x) = f_n(x)$ for $x \in bd(A_n)$ and $f_{n+1}(x) = \phi_{n+1}(x)$ for $x \in A_{n+1}$. It is easy to see that if $x \in A_n$, then $d(x, R_{n+1}) \leqq 3$ and $d(x, f_{n+1}(x)) \leqq 3\varepsilon_n$.

We have satisfied the conditions for $k = n + 1$; thus, the conditions can be satisfied for all k. Set $R = \bigcup_{n=1}^{\infty} (R_n) \cup X$.

We define $f: N \to R$ by

$$f(x) = \begin{cases} x: & \text{if} \quad x \in X \\ f_n(x): & \text{if} \quad x \in A_{n-1} - A_n . \end{cases}$$

It is clear that f is a continuous function for all $x \notin X$. Now suppose $x \in X$ and let $\varepsilon > 0$. By (I_n), there exists an M such that if $n \geqq M$ then $3\varepsilon_n < \varepsilon/2$. Choose a neighborhood N_x of diameter $< \varepsilon/2$ about x in A_M. Then if $y \in N_x$, $d(f(y), y) < 3\varepsilon_M < \varepsilon/2$ and $d(y, x) < \varepsilon/2$. Thus, $d(f(x), f(y)) < \varepsilon$ and we conclude that f is continuous at x. It is easy to see that R is compact and $f(x) = x$ for each $x \in R$. Thus, $f: N \to R$ is a compact retraction. The space R is the desired AR.

Let U be an open set that contains X. Then there exists an n such that A_n is a closed subset of U. Now A_n is an absolute neighborhood retract for metric spaces. So there exists an open set V of U that contains A_n and a retraction $r: V \to A_n$. Then $f\,|\,A_n \circ r$ is the desired retraction, and $R^* = f(A_n)$ is the desired ANR.

3. Applications. In this section, Theorem 1 will be used to establish a number of results.

The following extension theorem is due to Dugundji and Granas [7].

THEOREM 2. *Let A be a closed subset of a normal space X and let N be a normed linear space. Suppose that $f: A \to N$ is a continuous mapping such that $\overline{f(A)}$ is compact. Then there exists an extension, $F: X \to N$, of f such that $\overline{f(X)}$ is compact.*

Proof. The Dugundji extension theorem [6] assures that f has an extension $F^*: X \to N$. Theorem 1 implies that there exists a compact AR R such that $\overline{f(A)} \subseteqq R$. There exists a retraction $r: N \to R$. The composition $r \circ F^* = F$ is the desired extension.

THEOREM 3. [11] *Let X be an* AR *and let $f: X \to X$ be a continuous function such that $\overline{f(X)}$ is compact. Then f has a fixed point.*

Proof. By the Arens-Eells embedding theorem [1], X can be realized as a closed subset of a normed linear space N.

There exists a retraction $r: N \to X$ from N to X. By Theorem 1 there exists a compact AR R such that $f(X) \subseteqq R$. Set $g = f \circ r\,|\,R$. Since every compact AR has the fixed point property, the function $g: R \to R$ has a fixed point x. Thus, $x = g(x) = f(r(x)) = f(x)$. So f has a fixed point.

The Čech homology groups and the singular homology groups of a compact AR are isomorphic [13, p. 145]. Theorem 1 implies that in the class of compact subsets of an open subset of a normed linear space the compact AR's are cofinal. Thus we have the following theorem.

THEOREM 4[3]. *The Čech homology groups with compact support and the singular homology groups of an open subset of a normed linear space are isomorphic.*

A multi-valued upper semi-continuous mapping $\phi: X \to Y$ is said to be admissible if for each $x \in X$, $\phi(x)$ is compact and acyclic [8, 9]. The following theorem, which is a generalization of Theorem 2, is an important special case of the principal result of [8].

THEOREM 5. *Let X be an* ANR *and let $\phi: X \to X$ be an admissible map such that $\overline{\phi(X)}$ is compact. Then the Lefschetz number of ϕ, $\Lambda\phi$, can be defined, and $\Lambda\phi \neq 0$ implies that there exists an $x \in X$ such that $x \in \phi(x)$.*

Proof. Górniewicz and Granas [9] prove this result for the case that X is a topologically complete ANR. Their argument carries over to the incomplete case if Lemma 9.1 of [9] is replaced by Theorem 1.

The following theorem, which is a special case of [4.4, p. 95, 10] follows from Theorem 1 and Theorem 11 of [4].

THEOREM 6. *Let X be an* AR *and $f: X \to X$ be a continuous and locally compact mapping from X to X. If for some positive integer n, $\overline{f^n(X)}$ is compact, then f has a fixed point.*

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