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A CHARACTERIZATION OF M-IDEALS IN $B(\ell_p)$ FOR $1 < p < \infty$

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For $1 < p < \infty$ the only nontrivial M-ideal in $B(\ell_p)$, the bounded linear operators on ℓ_p, is $K(\ell_p)$, the ideal of compact operators on ℓ_p.

1. Introduction. Certain theorems for $B(H)$ (the bounded linear operators on H a separable Hilbert space) are known to hold for $B(\ell_p)$, $1 < p < \infty$. For example, it is well known that the only nontrivial closed two-sided ideal in $B(\ell_p)$, $1 \leqq p < \infty$ is $K(\ell_p)$, the compact linear operators on ℓ_p. Hennefeld [4] has shown that $K(\ell_p)$ is an M-ideal in $B(\ell_p)$ for $1 < p < \infty$. It is also known that $K(\ell_2)$ is the only nontrivial M-ideal in $B(\ell_2)$. This follows from the fact that in a B^*-algebra, the M-ideals are precisely the closed two-sided ideals [5]. The purpose of this paper is to show that this result also generalizes to $B(\ell_p)$, for $1 < p < \infty$. As this paper is largely based on the work of Smith and Ward [5] it is perhaps not surprising that a result of theirs, namely that every nontrivial M-ideal in $B(\ell_p)$ for $1 < p < \infty$ contains $K(\ell_p)$, has a new proof.

2. Preliminaries. A closed subspace L of a Banach space X is said to be an L-ideal [M-summand] if there exists a closed subspace L' such that $X = L \oplus L'$ and $\|\ell + \ell'\| = \|\ell\| + \|\ell'\|$ [$\|\ell + \ell'\| = \max\{\|\ell\|, \|\ell'\|\}$] for every $\ell \in L$ and $\ell' \in L'$. A closed subspace M of a Banach space X is an M-ideal if $M^\perp$ is an L-ideal in X^*. Note that M-summands are M-ideals, but the latter is a more general concept. [For example, $K(\ell_p)$ is an M-ideal in $B(\ell_p)$ but not an M-summand, as $K(\ell_p)$ is not complemented in $B(\ell_p)$.] For basic properties of M-ideals, L-ideals and M-summands, refer to [1].

The state space S of a banach algebra A with identity e is defined to be $\{\phi \in A^* : \phi(e) = \|\phi\| = 1\}$. An element $h \in A$ is hermitian if $\|e^{i\lambda h}\| = 1$ for all real λ. Equivalently [2] h is hermitian if and only if $\{\phi(h) : h \in S\} \subseteqq \boldsymbol{R}$. A^{**} when endowed with Arens multiplication [3] is a Banach algebra with identity e, and by the weak-star density of A in A^{**}, $h \in A^{**}$ is hermitian if and only if h is real valued on the state space of A.

In [5] it is shown that M-ideals in Banach algebras are necessarily subalgebras. Other results of this paper and [6] needed in the sequel are now summarized:

Let M be an M-ideal in $B(\ell_p)$, $1 < p < \infty$. Then clearly $M^{\perp\perp}$ is an M-summand in $B(\ell_p)^{**}$; that is, $B(\ell_p)^{**} = M^{\perp\perp} \oplus_{c_0} M^\#$. Let

$P\colon B(\ell_p)^{**} \to M^{\perp\perp}$ be the associated M-projection. Let I denote the identity in $B(\ell_p)$, and let $P(I) = z$. Throughout this paper, the following arithmetical facts will be collectively referred to as (*):

$z = z^2$ is hermitian, and commutes with every other hermitian element of $B(\ell_p)^{**}$. $zM^{\perp\perp} \subseteqq M^{\perp\perp}$, $zM^{\#} \subseteqq M^{\#}$, and $zM^{\#}z = 0$. Likewise, $(e - z)M^{\perp\perp} \subseteqq M^{\perp\perp}$, $(e - z)M^{\#} \subseteqq M^{\#}$, and $(e - z)M^{\perp\perp}(e - z) = 0$.

If S is the state space of $B(\ell_p)$, then $S = F_1 \oplus_{\mathrm{conv}} F_2$ where $B(\ell_p)^* = M^{\perp} \oplus_{\ell_1} \tilde{M}$, $F_1 = M^{\perp} \cap S$, and $F_2 = \tilde{M} \cap S$ (i.e., $\phi \in S \to$ there exist unique $\phi_1 \in F_1$, $\phi_2 \in F_2$, and $t \in [0, 1]$ such that $\phi = t\phi_1 + (1 - t)\phi_2$). If z is regarded as a real valued affine function on S, then $z|_{F_1} = 0$ and $z|_{F_2} = 1$.

An important fact used in this paper which follows easily from the definition of the hermitian elements is that in $B(\ell_p)$, any diagonal matrix with real entries in hermitian. [These are in fact precisely the hermitian elements of $B(\ell_p)$ if $1 < p < \infty$, $p \neq 2$ [7].]

In § 3, a matrix $A \in B(\ell_p)$ whose ith row jth column entry is a_{ij} will be denoted $\sum_{i,j\geq 1} a_{ij} e_j \otimes e_i$, where $e_j \otimes e_i$ is the rank-one map that sends e_j to e_i. $((e_i)_{i\geq 1}$ is the canonical basis for ℓ_p.) Note that if $A \in B(\ell_p)$, then $\|A(e_i)\| \leqq \|A\|$ for every i. That is, every column of A is an element of ℓ_p whose norm does not exceed $\|A\|$. By considering the adjoint, we have that every row of A is an element of ℓ_q $[1/p + 1/q = 1]$ whose norm is less than or equal to $\|A\|$. Clearly, $|a_{ij}| \leqq \|A\|$ for every i, j, and if A is a matrix with at most one nonzero entry in each row and column, [for example if A is diagonal] then $\|A\|$ is the ℓ_∞-norm of the sequence of nonzero entries.

3. Results.

Assume all notation in § 2, and assume $M \neq 0$. Recall that I denotes the identity on ℓ_p, where throughout this section $1 < p < \infty$, $p \neq 2$.

LEMMA 1. *If h is hermitian in $B(\ell_p)$ and $h^2 = I$, then for every $m \in M$, $hm \in M$ and $mh \in M$.*

Proof. Considering h as canonically embedded in $B(\ell_p)^{**}$, $h = h_1 + h_2$ where $h_1 \in M^{\perp\perp}$, $h_2 \in M^{\#}$, and $\|h\| = \max\{\|h_1\|, \|h_2\|\}$. Note that h_1 and h_2 are themselves hermitian elements of $B(\ell_p)^{**}$, for if $f_1 \in F_1$ then $f_1(h_1) = 0$ and if $f_2 \in F_2$, $f_2(h_1) = f_2(h) \in \boldsymbol{R}$. So for any $\phi \in S$, $\phi(h_1) \in \boldsymbol{R}$, i.e., h_1 is hermitian. The same reasoning applied to h_2 shows that h_2 is also hermitian. $h^2 = I = h_1^2 + h_1h_2 + h_2h_1 + h_2^2$, however it is easy to see that $h_1h_2 = 0 = h_2h_1$, since by (*) we have that

$$h_1h_2 = zh_1h_2 + (e - z)h_1h_2 = h_1zh_2z + (e - z)h_1(e - z)h_2 = 0\ .$$

Similarly, $h_2h_1 = 0$, hence $I = h_1^2 + h_2^2$.

Now pick $m \in M$, and wlog assume $\|m\| = 1$. We'll show that $hm \in M$. [$mh \in M$ is shown in similar fashion.] There exist $m_1 \in M^{\perp\perp}$ and $m_2 \in M^{\#}$ such that $hm = m_1 + m_2$. Claim: $zm_2 = 0 = m_2 z$. To see this, note that $zhm = zm_1 + zm_2$ where [using (*)] $zhm = zhzm \in M^{\perp\perp}$ and $zm_1 \in M^{\perp\perp}$. Hence $zm_2 \in M^{\perp\perp} \cap M^{\#}$ and so $zm_2 = 0$.

To show $m_2 z = 0$ is a little harder: $hmz = h_1 mz + h_2 mz = m_1 z + m_2 z$ where $h_1 mz \in M^{\perp\perp}$ and $m_1 z \in M^{\perp\perp}$. If we knew that $h_2 mz \in M^{\perp\perp}$, then as before we'd have $m_2 z \in M^{\perp\perp} \cap M^{\#} = 0$ and our claim would be established. So suppose $h_2 mz \notin M^{\perp\perp}$. Then there exists some $f_1 \in S \cap M^{\perp}$ so that $f_1(h_2 mz) \neq 0$. [This happens as the state space spans $B(\ell_p)^*$ and hence F_1 spans $M^{\perp}$.] Choose $\theta \in \boldsymbol{R}$ so that $f_1(e^{i\theta} h_2 mz) = \delta > 0$. Then $e^{i\theta} mz \in M^{\perp\perp}$ has norm at most one, $h_2 \in M^{\#}$ has norm at most one, so $\|h_2(e^{i\theta} mz + h_2)\| \leqq 1$. But $1 \geqq f_1(e^{i\theta} h_2 mz + h_2^2) = \delta + f_1(h_2^2) = \delta + f_1(I) = \delta + 1$, a contradiction which proves the claim. Now $(e - z)hm(e - z) = (e - z)m_1(e - z) + (e - z)m_2(e - z)$. But by (*) we have that $(e-z)hm(e-z) = h(e-z)m(e-z) = 0 = (e-z)m_1(e-z)$, so $0 = (e - z)m_2(e - z) = m_2$, that is, $hm = m_1 \in M^{\perp\perp} \cap B(\ell_p) = M$. □

REMARK. Although stated for $B(\ell_p)$, this lemma is true [by the same proof] for any M-ideal M and norm-1 hermitian h where $h^2 = I$.

COROLLARY. *If h is any diagonal matrix in $B(\ell_p)$, then $hM \subseteqq M$ and $Mh \subseteqq M$.*

Proof. At this point we know that if h is a diagonal matrix with only ± 1's on the diagonal, then $h^2 = I$ and so $hM \subseteqq M$ and $Mh \subseteqq M$. But by averaging two such hermitian elements, we have that if h is any diagonal matrix with only 1's or 0's on the diagonal, then $hM \subseteqq M$ and $Mh \subseteqq M$. Hence the result holds for any finite valued diagonal matrix. But such matrices are dense in the diagonal elements of $B(\ell_p)$, and so as M is closed, $hM \subseteqq M$ and $Mh \subseteqq M$ for any diagonal h. □

COROLLARY. $M \supseteqq K(\ell_p)$.

Proof. By the previous corollary, if E_{ij} denotes the elementary matrix with a 1 in the ith row and jth column and zeros elsewhere, then $E_{ii} M E_{jj} \subseteqq M$ for every $i \geqq 1$ and $j \geqq 1$. As $M \neq 0$ there is an $A = \sum a_{ij} e_j \otimes e_i \in M$ such that for some k and ℓ $a_{k\ell} = 1$. Hence $E_{k\ell} = E_{kk} A E_{\ell\ell} \in M$. Claim: for every $p \geqq 1$, $E_{p\ell} \in M$. If there is any $m = \sum m_{ij} e_j \otimes e_i \in M$ so that $m_{p\ell} \neq 0$, then $E_{p\ell} = (1/m_{p\ell}) E_{pp} m E_{\ell\ell} \in M$. So if every $m = \sum m_{ij} e_j \otimes e_i \in M$ has the property that $m_{p\ell} = 0$, then the norm-1 functional $\rho_2 \in B(\ell_p)^*$ defined by $\rho_2(\sum t_{ij} e_j \otimes e_i) = t_{p\ell}$ is in $M^{\perp}$. Let $\rho_1 \in B(\ell_p)^*$ be defined by $\rho_1(\sum t_{ij} e_j \otimes e_i) = t_{k\ell}$. Then

$\|\rho_1\| = 1$. Claim: $\rho_1 \in \tilde{M}$. To see this, suppose that $\rho_1 = \psi_1 + \psi_2$ where $\psi_1 \in M^\perp$, $\psi_2 \in \tilde{M}$. Then $\|\rho_1\| = \|\psi_1\| + \|\psi_2\|$, and $1 = \|\rho_1\| = \rho_1(E_{k\ell}) = \psi_1(E_{k\ell}) + \psi_2(E_{k\ell}) = \psi_2(E_{k\ell})$, so $\|\psi_2\| = 1 \to \|\psi_1\| = 0$. Hence $2 = \|\rho_1 + \rho_2\|$. Choose $T = \sum t_{ij} e_j \otimes e_i \in B(\ell_p)$ so that $\|T\| = 1$ and $|\rho_1(T) + \rho_2(T)| > 2^{1/q}$ where $1/p + 1/q = 1$. Then $2^{1/q} < |t_{p\ell} + t_{k\ell}| \leqq (|t_{p\ell}|^p + |t_{k\ell}|^p)^{1/p} \cdot 2^{1/q} \leqq \|T_{(e_\ell)}\| \cdot 2^{1/q} \leqq 2^{1/q}$, a contradiction implying that $E_{p\ell} \in M$. A similar argument shows that if $E_{ij} \in M$, then for every $k \geqq 1$, $E_{ik} \in M$. Hence $M \supseteqq \{E_{ij}\colon i, j \geqq 1\}$ which is a basis for $K(\ell_p)$, that is, $M \supseteqq K(\ell_p)$. □

Note that if h is hermitian and $h \in M$ then $hB(\ell_p)h \subseteqq M$. This follows from the simple observation that if $h \in M$, then by (*), $(e - z)h = (e - z)^2 h = (e - z)h(e - z) = 0 = h(e - z)$, since h is hermitian. So $zh = hz = h$, and for any $A \in B(\ell_p)$, $hAh = hzAzh \in M$. From this we see that if $I \in M$, then $M = B(\ell_p)$.

LEMMA 2. *If* $A = \sum a_{ij} e_j \otimes e_i \in M$ *where* $(a_{ii})_{i \geqq 1} \in \ell_\infty \backslash c_0$, *then* $M = B(\ell_p)$.

Proof. wlog there exists an infinite sequence of integers $f(1) < f(2) < \cdots$ so that $A = \sum_i e_{f(i)} \otimes e_{f(i)}$. The reduction to this case illustrates a typical use of Lemma 1 that occurs several times in this paper. This time it will be done in detail:

There exists a $\delta > 0$ and a sequence of positive integers $i_1 < i_2 < \cdots$ so that $\delta < |a_{i_k i_k}| \leqq \|A\|$ for each k. As $hA \in M$ where $h = \sum_{k \geqq 1} (1/|a_{i_k i_k}|) e_{i_k} \otimes e_{i_k}$ we may assume wlog that $a_{i_k i_k} = 1$ for every k. Choose a sequence of positive numbers $(\varepsilon_i)_{i \geqq 1}$ so that $\sum_{i \geqq 1} \varepsilon_i < \infty$. Let $f(1) = i_1$ and choose $\alpha_1 > f(1)$ so that

$$\Big(\sum_{j \geqq \alpha_1} |a_{f(1)j}|^q\Big)^{1/q} < \varepsilon_1 \quad \text{and} \quad \Big(\sum_{i \geqq \alpha_1} |a_{if(1)}|^p\Big)^{1/p} < \varepsilon_2 \,.$$

Choose a k_2 so that $i_{k_2} > \alpha_1$ and set $f(2) = i_{k_2}$. Now find $\alpha_2 > f(2)$ so that $(\sum_{j \geqq \alpha_2} |a_{f(2)j}|^q)^{1/q} < \varepsilon_3$ and $(\sum_{i \geqq \alpha_2} |a_{if(2)}|^p)^{1/p} < \varepsilon_4$, etc. Fix $\varepsilon > 0$. There is an n such that $\sum_{i \geqq n} \varepsilon_i < \varepsilon$. If $h = \sum h_{ij} e_j \otimes e_i$ where

$$h_{ij} = \begin{cases} 1 & \text{if} \quad i = j = f(k) \quad \text{for some} \quad k \\ 0 & \text{otherwise} \end{cases}$$

and K denotes the first $f(n)$ rows and columns of $hAh - \sum_{k \geqq 1} e_{f(k)} \otimes e_{f(k)}$, then K represents a compact operator on ℓ_p, and by choice of K $\|hAh - \sum_{k \geqq 1} e_{f(k)} \otimes e_{f(k)} - K\| < \varepsilon$. As $\varepsilon > 0$ is arbitrary and $hAh - K \in M$ we have that

$$\sum_k e_{f(k)} \otimes e_{f(k)} \in M \,.$$

If $f(N)^c$ is finite, then there exists a compact K so that $A + K = I \in M \to M = B(\ell_p)$. So assume $f(N)^c$ is infinite and let g enumerate $f(N)^c$.

Claim. $B = \sum_i e_{g(i)} \otimes e_{f(i)} \in M$.

Note that proving this claim is sufficient to finish the lemma, since the same argument can be modified to show that

$$C = \sum_i e_{f(i)} \otimes e_{g(i)} \in M\ , \quad \text{hence again} \quad I = A + CB \in M\ .$$

We first show that $d(B, M)$ is zero or one.

Now if $h = \sum_{i \in I} e_i \otimes e_i$ where I is any subset of positive integers, then $d(h, M)$ is either zero or one for any M-ideal M, for if there is a $\delta > 0$ and $m \in M$ such that $\|h - m\| = \delta$, then by the first corollary to Lemma 1, $(h - m)^2 = h - (hm + mh - m^2) \to d(h, M) \leqq \delta^2$.

Let P be the permutation matrix which as an operator on ℓ_p interchanges, for every i, $e_{f(i)}$ with $e_{g(i)}$. Then $AP = B$. It is easily checked that $M_P = \{mP: m \in M\}$ is an M-ideal isometric to M. Indeed the isometry $T: B(\ell_p) \to B(\ell_p)$ given by $T(N) = NP$ induces an isometry [call it T again] on $B(\ell_p)^*$ by $\langle N, T\varphi\rangle = \langle NP, \varphi\rangle$. Then $T(M) = M_P$, $T(M^\perp) = M_P^\perp$ and $B(\ell_p)^* = T(M^\perp) \oplus_{\ell_1} T(\tilde{M})$. Therefore $d(B, M) = d(A, M_P) = 1$ or 0.

Now assuming that $B \notin M$, there is a $\varphi \in M^\perp$ so that $\|\varphi\| = 1 = \varphi(B)$. Define $\varphi_A \in B(\ell_p)^*$ by $\varphi_A(N) = \varphi(NB)$. Then $AB = B \to \varphi_A(A) = 1 = \|\varphi_A\|$. But then $\varphi_A \in \tilde{M}$ since $A \in M$. [This calculation occurs in the corollary above stating that $M \supseteqq K(\ell_p)$.] Thus $\|\varphi_A + \varphi\| = 2$. But there is an $\varepsilon > 0$ such that for any norm-1 $N \in B(\ell_p)$, we have that $|\varphi_A(N) + \varphi(N)| \leqq \|\varphi\| \cdot \|N\| \cdot \|B + I\| < 2 - \varepsilon$, a contradiction implying that $B \in M$. □

LEMMA 3. *If $B = \sum b_{ij} e_j \otimes e_i \in M$ where B contains a sequence of entries $(b_{i_k j_k})_{k \geqq 1} \in \ell_\infty \backslash c_0$, then $M = B(\ell_p)$.*

Proof. As in the proof of Lemma 2, we may assume wlog that there exist infinite sequences $f(1) < f(2) < \cdots$ and $g(1) < g(2) < \cdots$ such that $f(i) \neq g(j)$ for all i and j, and so that $\sum_i e_{g(i)} \otimes e_{f(i)} \in M$. Call this matrix B, and let $A = \sum_i e_{f(i)} \otimes e_{f(i)}$. If P and M_P are as in Lemma 2, then $0 = d(B, M) = d(A, M_P) \to$ [by Lemma 2] $M_P = B(\ell_p) \to M = B(\ell_p)$. □

If $T = \sum t_{ij} e_j \otimes e_i \in M$ and T is not compact, then it is not necessarily the case that there is a subsequence of entries $(t_{i_k j_k})_{k \geqq 1} \in \ell_\infty \backslash c_0$. But what is true [and will be shown in the proof of the next theorem] is that T has infinitely many square blocks each of whose norm is larger than some fixed $\varepsilon > 0$. So what essentially remains to be done is to generalize preceding arguments from 1 by 1 blocks to square blocks of arbitrary dimension.

THEOREM. *Suppose* $T = \sum t_{ij} e_j \otimes e_i$ *is not compact. Then* $T \in M \to M = B(\ell_p)$.

Proof. wlog $\|T\| = 1$. The argument of Lemma 2 modifies to show that wlog T is a direct sum of diagonal square blocks $\bar{T}_i$ where $\|\bar{T}_i\| = 1$. Although this is well known, it is included for the sake of completeness. We can do this in more generality as follows:

Suppose $T = \sum t_{ij} e_j^* \otimes e_i \in B(X)$ where X is a reflexive space with 1 unconditional basis $(e_i)_{i \geq 1}$ [so $(e_i^*)_{i \geq 1}$ is a basis for X^*]. Suppose T is in an M-ideal $M \subseteq B(X)$. Since T is not compact, there is a $\delta > 0$ and a sequence $(z_i)_{i \geq 1} \subseteq X$ such that $\|z_i\| = 1$ and $\|T(z_i)\| > 2\delta$ for every i, and $z_i \to 0$ in the weak topology. Let $x_1 = z_1$ where $x_1 = \sum_{k \geq 1} x_k^1 e_k$. Then there exist $p_1 \geq 1$ and $p_1' \geq 1$ so that $\|T(\sum_{k=1}^{p_1} x_k^1 e_k)\| > \delta$, and if $T(\sum_{k=1}^{p_1} x_k^1 e_k) = \sum_{k \geq 1} y_k^1 e_k$, then also $\|\sum_{k=1}^{p_1'} y_k^1 e_k\| > \delta$. Define $m_1 = 0$, let $n_1 = \max\{p_1, p_1'\}$ and let $\bar{T}_1 = \sum_{i,j=m_1+1}^{m_1+n_1} t_{ij} e_j^* \otimes e_i$. Then $\delta < \|\bar{T}_1\| \leq 1$. Choose a sequence $(\varepsilon_i)_{i \geq 1}$ of positive numbers so that $\sum_{i \geq 1} \varepsilon_i < \infty$. Now $\sum_{i=1}^{\infty} \sum_{j=1}^{n_1} t_{ij} e_j^* \otimes e_i$ represents a compact operator [its adjoint is finite rank] and so there exists $\beta_1 > n_1$ such that $\|\sum_{i=\beta_1}^{\infty} \sum_{j=1}^{n_1} t_{ij} e_j^* \otimes e_i\| < \varepsilon_1$ [if $(P_n)_{n \geq 1}$ are the natural basis projections defined by $P_n(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{n} a_i e_i$, then $(\bar{T}_1 P_{n_1} - P_n \bar{T}_1 P_{n_1})(x) \to 0$ for every $x \in X$, and as $\bar{T}_1$ is compact this convergence is uniform on the unit ball, hence $\|\bar{T}_1 P_{n_1} - P_n \bar{T}_1 P_{n_1}\| \to 0$ as $n \to \infty$]. As $\sum_{i=1}^{n_1} \sum_{j \geq 1} t_{ij} e_j^* \otimes e_i$ is finite rank [hence compact] similar reasoning shows that there is an $\alpha_1 > n_1$ so that $\|\sum_{i=1}^{n_1} \sum_{j=\alpha_1}^{\infty} t_{ij} e_j^* \otimes e_i\| < \varepsilon_2$. Define $m_2 = \max\{\alpha_1, \beta_1\}$. Since $z_i \to 0$ weakly, we can use a standard gliding hump argument to find a $k_2 > 1$ such that $x_2 = z_{k_2}$ has the property that if $x_2 = \sum_{k \geq 1} x_k^2 e_k$ then there exists a $p_2 \geq 1$ and $p_2' \geq 1$ such that $\|T(\sum_{k=m_2+1}^{m_2+p_2} x_k^2 e_k)\| > \delta$, and if $T(\sum_{k=m_2+1}^{m_2+p_2} x_k^2 e_k) = \sum_{k \geq 1} y_k^2 e_k$, then also $\|\sum_{k=m_2+1}^{m_2+p_2'} y_k^2 e_k\| > \delta$. Let $n_2 = \max\{p_2, p_2'\}$ and let $\bar{T}_2 = \sum_{i,j=m_2+1}^{m_2+n_2} t_{ij} e_j^* \otimes e_i$. Then $\delta < \|\bar{T}_2\| \leq 1$. Again find $\beta_2 > m_2 + n_2$ and $\alpha_2 > m_2 + n_2$ so that

$$\left\| \sum_{i=\beta_2}^{\infty} \sum_{j=m_2+1}^{m_2+n_2} t_{ij} e_j^* \otimes e_i \right\| < \varepsilon_3 \quad \text{and} \quad \left\| \sum_{i=m_2+1}^{m_2+n_2} \sum_{j=\alpha_2}^{\infty} t_{ij} e_j^* \otimes e_i \right\| < \varepsilon_4 .$$

Let $m_3 = \max\{\alpha_2, \beta_2\}$ and repeat the process on $\sum_{i,j \geq m_3+1} t_{ij} e_j^* \otimes e_i$. Let $h = \sum h_{ij} e_j^* \otimes e_i$ be the hermitian element defined by

$$h_{ij} = \begin{cases} 1 & \text{if there is a } k \text{ so that } m_k + 1 \leq i = j \leq m_k + n_k \\ 0 & \text{otherwise .} \end{cases}$$

Then $hTh\in M$. [Although the corollary to Lemma 1 need not hold here, what the proof of the corollary actually shows is that M is closed under multiplication by real diagonal matrices.] To see that $T'=\sum_i \bar{T}_i\in M$, choose $\varepsilon>0$. There is an ℓ so that $\sum_{i\geqq\ell}\varepsilon_i<\varepsilon$. Let K denote the compact operator represented by the first $m_\ell+n_\ell$ rows and columns of $hTh-T'$. Then by the choice of ℓ, $\|hTh-T'-K\|<\varepsilon$ and as M is closed we have that $T'\in M$. If $h'=\sum h'_{ij}e_j^*\otimes e_i$ is defined by

$$h'_{ij}=\begin{cases}\dfrac{1}{\|\bar{T}_k\|} & \text{if}\quad m_k+1\leqq i=j\leqq m_k+n_k\\ 0 & \text{otherwise,}\end{cases}$$

then $\|h'\|\leqq 1/\delta$, $h'T'\in M$, and $h'T'$ is a direct sum of diagonal square blocks each having norm 1. Returning now to $B(\ell_p)$, we see that we may assume that if T is not compact and $T\in M$, then wlog $T=\sum_i \bar{T}_i$ where each $\bar{T}_k=\sum_{i,j=m_k+1}^{m_k+n_k}t_{ij}e_j\otimes e_i$, $\|\bar{T}_i\|=1$, and $m_k+n_k+1<m_{k+1}$. Since $\|\bar{T}_k\|=1$, there exist $x_k=(x_1^k,\cdots,x_{n_k}^k)\in\ell_p^{n_k}$, $y_k=(y_1^k,\cdots,y_{n_k}^k)$ and $z_k=(z_1^k,\cdots,z_{n_k}^k)\in\ell_q^{n_k}$ all of norm-1 such that $\langle\bar{T}_k(x_k),y_k\rangle=1=\langle z_k,x_k\rangle$ for all k. Define norm-1 matrices A, X, Y, and Z in $B(\ell_p)$ by

$$A=\sum_{k\geqq1}e_{m_k+1}\otimes e_{m_k+1}\,,\qquad X=\sum_{k\leqq1}X_k\,,\qquad Y=\sum_{k\geqq1}Y_k\,,\qquad\text{and}$$
$$Z=\sum_{k\geqq1}Z_k$$

where

$$X_k=\sum_{j\leqq n_k}x_j^ke_{m_k+1}\otimes e_{m_k+j}\,,\qquad Y_k=\sum_{j\leqq n_k}y_j^ke_{m_k+j}\otimes e_{m_k+1}\,,\qquad\text{and}$$
$$Z_k=\sum_{j\leq n_k}z_j^ke_{m_k+j}\otimes e_{m_k+1}\,.$$

Then $ZX=YTX=A$. *Claim*: If $X\in M$, then $M=B(\ell_p)$. For if not, choose $\varphi\in c_0^\perp$ so that $\|\varphi\|=1=\varphi(1,1,\cdots)$. Define $\gamma\in B(\ell_p)^*$ by $\gamma(N)=\varphi[(n_{m_k+n_k+1,m_k+1})_{k\geqq1}]$ where $N=\sum n_{ij}e_j\otimes e_i$. We may assume that $\gamma\in M^\perp$, or else M contains an element with a sequence of entries in $\ell_\infty\backslash c_0$, hence $M=B(\ell_p)$. If $X\in M$, then the functional γ_1 defined by $\gamma_1(N)=\varphi[((ZN)_{m_k+1,m_k+1})_{k\geqq1}]$ is in $\tilde{M}$, as $\gamma_1(X)=1$ and as has been noted before, any functional attaining its norm at a norm-1 element of M is in $\tilde{M}$. Therefore $2=\|\gamma+\gamma_1\|$. However for any $N\in B(\ell_p)$ of norm-1, we have that

$$\begin{aligned}|\gamma(N)+\gamma_1(N)|&=|\varphi[(n_{m_k+n_k+1,m_k+1}+\sum_{j\leqq n_k}z_j^kn_{m_k+j,m_k+1})_{k\geqq1}]|\\&\leqq\|(z_1^k,z_2^k,\cdots,z_{n_k}^k,1)\|_q=2^{1/q}\,,\end{aligned}$$

a contradiction implying that $M=B(\ell_p)$. What this argument in fact shows is that if M contains any element with the same form as X then $M=B(\ell_p)$. In particular the functional φ_2 defined by

$\varphi_2(N) = \varphi[((YN)_{m_k+1, m_k+n_k+1})_{k\geq 1}]$ is in $M^{\perp}$. [For if there is an $m = \sum m_{ij}e_j \otimes e_i \in M$ such that $\varphi_2(m) \neq 0$, then there exists $\varepsilon > 0$ such that $\|\bar{m}_k\| > \varepsilon$ for infinitely many k where $\bar{m}_k = \sum_{j\leq n_k} m_{m_k+j, m_k+n_k+1} e_{m_k+n_k+1} \otimes e_{m_k+j}$. Reasoning as in Lemma 2 we may pass to a subsequence if necessary to get $\sum_{\ell\geq 1} \bar{m}_{k_\ell} \in M$, which up to normalization of the blocks $\bar{m}_{k_\ell}$ has the same form as X.] Finally define $\varphi_1 \in B(\ell_p)^*$ by $\varphi_1(N) = \varphi[((YNX)_{m_k+1, m_k+1})_{k\geq 1}]$. As $\varphi_1(T) = 1$, $\varphi_1 \in \tilde{M}$, and so $2 = \|\varphi_1 + \varphi_2\|$. But for any norm-1 $N \in B(\ell_p)$, we have that

$$\begin{aligned}|\varphi_1(N) + \varphi_2(N)| &\leqq \sup_k \Big|\sum_{j\leqq n_k} (YN)_{m_k+1, m_k+j} x_j^k + (YN)_{m_k+1, m_k+n_k+1}\Big| \\ &\leqq \sup_k \|(x_1^k, \cdots, x_{n_k}^k, 1)\|_p = 2^{1/p}\end{aligned}$$

a contradiction showing that if $T \in M$ then $M = B(\ell_p)$. □

The properties of ℓ_p used to prove this theorem are the existence of a symmetric basis and of certain convexity conditions in the space and its dual.

J. Hennefeld recently announced the following result [AMS Notices Volume 25, Number 6, 760-B8].

THEOREM. *The only* 1-*symmetric spaces* X *for which* $K(X)$ *is an* M-*ideal in* $B(X)$ *are* c_0 *and* ℓ_p, $1 < p < \infty$.

Hence combining these theorems we have that if X is not c_0 or ℓ_p, $1 < p < \infty$, has a symmetric basis in X and X^* and satisfies the required convexity conditions, then there are no nontrivial M-ideals in $B(X)$.

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