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EXCEPTIONAL VALUES OF DIFFERENTIAL POLYNOMIALS

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Let f be a meromorphic non-rational function on C and $Q[f]$, $P[f]$ differential polynomials in f. Assuming that neither of them vanishes identically, functions of the form $f^nQ[f] + P[f]$, $n \in N$, are shown not to have zero as a Picard or Borel exceptional value for sufficiently large n. Examples show that the estimates given for n are optimal.

1. Introduction and results. In the present paper we concern ourselves with the value-distribution of differential polynomials. We make use or results from value-distribution theory and we use the common notations $m(r, f)$, $N(r, f)$, $T(r, f)$, $\bar{N}(r, f)$, $S(r, f)$ and so on. (cf., e.g., [3], [8]).

There has been quite a bit of investigation (cf. [2], [12]-[14]) of Picard values of certain expressions in a meromorphic function f such as f^nf' or $f^n + f'$. Our article extends some of the previous results, especially those of W. K. Hayman [4] and L. R. Sons [9]. Let f be a meromorphic function—in this paper always in the sense of meromorphic in the whole plane—and let $n_0, n_1, \cdots, n_k$ be nonnegative entire numbers. We call

$$M[f] = f^{n_0}(f')^{n_1} \cdots (f^{(k)})^{n_k} \tag{1}$$

a *monomial* in f (cf. L. R. Sons [9]), $\gamma_M: = n_0 + n_1 + \cdots + n_k$ its *degree* and $\Gamma_M: = n_0 + 2n_1 + \cdots + (1 + k)n_k$ its *weight*. Further, let $M_1[f], \cdots, M_\ell[f]$ denote monomials in f and $a_1, \cdots, a_\ell$ meromorphic functions satisfying $T(r, a_j) = S(r, f)$, $1 \leqq j \leqq \ell$, then

$$P[f] = a_1M_1[f] + \cdots + a_\ell M_\ell[f] \tag{2}$$

is called a *differential polynomial* in f of *degree* $\gamma_P := \max_{j=1}^{\ell} \gamma_{M_j}$ and *weight* $\Gamma_P := \max_{j=1}^{\ell} \Gamma_{M_j}$ with *coefficients* a_j.

Using these definitions we can state the following results:

THEOREM 1. *Let f be a nonrational meromorphic function and let $Q[f]$, $P[f]$ be differential polynomials in f satisfying $Q[f](z) \not\equiv 0$, $P[f](z) \not\equiv 0$. Then zero is neither a Picard nor a Borel exceptional value of*

$$\Psi = f^nQ[f] + P[f]$$

for any $n \in N$ with $n \geqq 3 + \Gamma_P$ and in particular

$$\limsup_{r\to\infty} \frac{\bar{N}(r, 1/\Psi)}{T(r, \Psi)} > 0 \, .$$

As an immediate consequence we get

COROLLARY 1. *Let f be a nonrational meromorphic function and*

$$\Psi = af^{n_0} \cdots (f^{(k)})^{n_k}$$

a differential polynomial in f, $a \not\equiv 0$. Barring zero, Ψ has no finite Picard or Borel exceptional values if only $n_0 \geqq 3$ holds. And again

$$\limsup_{r\to\infty} \frac{\bar{N}(r, 1/(\Psi - c))}{T(r, \Psi)} > 0$$

holds for $c \in \boldsymbol{C}\backslash\{0\}$.

REMARK. L. R. Sons proved similar results in [9] for the case $a \equiv 1$ and $n_0 \geqq 2$, however under the additional assumptions $n_k \geqq 1$ and $2^k(n_0 + \sum_{i=0}^{k} (1 + i)n_i) < (2^k + n_0 - 1)(\sum_{i=0}^{k} (1 + i)n_i)$.

Theorem 1 can be sharpened by considering entire functions only.

THEOREM 2. *Let f be a transcendental entire function and let $Q[f]$, $P[f]$ be differential polynomials in f, both not identically vanishing. Then*

$$\Psi = f^n Q[f] + P[f]$$

does not assume zero as a Picard or Borel exceptional value for any $n \in \boldsymbol{N}$, $n \geqq 2 + \gamma_P$; and here also

$$\limsup_{r\to\infty} \frac{\bar{N}(r, 1/\Psi)}{T(r, \Psi)} > 0$$

holds for these n.

REMARK. Assuming f to be entire Corollary 1 holds already for $n_0 \geqq 2$.

We conclude by giving two examples which show that the estimates given for n are optimal in the sense that they cannot be improved. First consider a nonconstant solution of the Riccati differential equation $w' = -2(w - 1)(w + 1)$ which is a transcendental meromorphic function satisfying $w^4 + w' \neq 1$ (cf., e.g., [10], [11]); this settles Theorem 1.

Regarding Theorem 2 we choose an entire transcendental solution of the linear differential equation $w^{(j)} = -2ac(w - c)$, $j \in \boldsymbol{N}$, where a and c are nonzero constants. Then we have $w^{(j)} + aw^2 \neq ac^2$ what is all we wanted to show.

2. Some lemmas. We prove a few auxiliary results. The following notations help to simplify our presentation. By $\lambda(f)$ and $\rho(f)$ we shall always denote the upper and lower order of growth of a meromorphic function f; for a differential polynomial $Q[f]$ in f we write $Q'[f]$ instead of $(d/dz)Q[f]$. (Note that for an arbitrary monomial $M[f]$ in f, $M'[f]$ can always be represented as a differential polynomial in f, each of whose monomials have the same degree as $M[f]$. Those differential polynomials are often called *homogeneous*).

Finally we shall say, following W. K. Hayman [4], that a certain property $\mathscr{P} = \mathscr{P}(r)$, $r \in D \subseteqq \boldsymbol{R}$, holds "nearly everywhere" (n.e.) in D, if there is a subset $A \subseteqq D$ of finite linear measure such that $\mathscr{P}(r)$ holds for all $r \in D \backslash A$.

LEMMA 1. *Let f be a nonconstant meromorphic function. If $Q[f]$ is a differential polynomial in f with arbitrary meromorphic coefficients q_j, $1 \leqq j \leqq n$ then*

(i) $m(r, Q[f]) \leqq \gamma_Q m(r, f) + \sum_{j=1}^{n} m(r, q_j) + S(r, f)$

and

(ii) $N(r, Q[f]) \leqq \Gamma_Q N(r, f) + \sum_{j=1}^{n} N(r, q_j) + O(1)$.

Proof. Starting with $Q[f] = \sum_{j=1}^{n} q_j M_j[f]$ (cf. (2)) we can represent $Q[f]$ as $Q[f] = \sum_{j=1}^{n} q_j^* f^{m_j}$ with $m_j := \gamma_{M_j}$ and with meromorphic functions q_j^* satisfying $m(r, q_j^*) \leqq m(r, q_j) + S(r, f)$, $j = 1, \cdots, n$. This settles (i). Further, in an arbitrary $z_0 \in \boldsymbol{C}$ let $Q[f]$, f, q_j and $M_j[f]$ have poles of order μ, ν, μ_j and ν_j respectively (as usual a meromorphic function f has poles of order zero in points $z \in \boldsymbol{C}$ with $f(z) \neq \infty$). It follows immediately, that $\mu \leqq \max\{\nu_1 + \mu_1, \cdots, \nu_n + \mu_n\}$ and because of $\nu_j \leqq \Gamma_{M_j} \cdot \nu \leqq \Gamma_Q \cdot \nu$, $1 \leqq j \leqq n$, we have

$$\mu \leqq \Gamma_Q \cdot \nu + \sum_{j=1}^{n} \mu_j \, . \tag{3}$$

Hence $n(r, Q[f]) \leqq \Gamma_Q n(r, f) + \sum_{j=1}^{n} n(r, q_j)$ and therefore (ii) holds.

Now we use Lemma 1 to improve a result of Clunie (cf. [1], Lemmas 1 and 2).

LEMMA 2. *Let f be a nonconstant meromorphic function. And let $Q^*[f]$ and $Q[f]$ denote differential polynomials in f with arbitrary meromorphic coefficients $q_1^*, \cdots, q_n^*$ and $q_1, \cdots, q_\ell$ respectively; further, let P be a nonconstant polynomial of degree p. Then from*

$$P(f)Q^*[f] \equiv Q[f]$$

we can infer the following:

(i) *if* $\gamma_Q \leqq p$, *then*

$$m(r, Q^*[f]) \leqq \sum_{j=1}^{n} m(r, q_j^*) + \sum_{j=1}^{\ell} m(r, q_j) + S(r, f)$$

(ii) *if* $\Gamma_Q \leqq p$ *we have in addition*

$$N(r, Q^*[f]) \leqq \sum_{j=1}^{n} N(r, q_j^*) + \sum_{j=1}^{\ell} N(r, q_j) + O(1) .$$

Proof. For a proof of the first proposition see Clunie [1]. (ii) Let $n_f(r, Q^*[f])$ denote the number of those poles of $Q^*[f]$ in $|z| \leqq r$ that are also poles of f with the poles of $Q^*[f]$ being counted according to their order. Set $n^f(r, Q^*[f]) := n(r, Q^*[f]) - n_f(r, Q^*[f])$ and define $N_f(r, Q^*[f])$, $N^f(r, Q^*[f])$ correspondingly. We obtain immediately

$$N^f(r, Q^*[f]) \leqq \sum_{j=1}^{n} N(r, q_j^*) + O(1) . \tag{4}$$

Now we choose a point $z_0 \in \boldsymbol{C}$ where $Q^*[f]$ and f have poles of order μ and ν respectively; denoting by $\nu_1, \cdots, \nu_\ell$ the orders of the poles of $q_1, \cdots, q_\ell$ in z_0 and considering (3) we get

$$p\cdot\nu + \mu \leqq \Gamma_Q\cdot\nu + \max\{\nu_1, \cdots, \nu_\ell\}$$

and $\Gamma_Q \leqq p$ yields

$$n_f(r, Q^*[f]) \leqq \sum_{j=1}^{\ell} n(r, q_j) .$$

Adding (4) this proves (ii).

We conclude by proving a lemma that will enable us to compare the orders of growth of a differential polynomial in f with those of f.

LEMMA 3. *Let* $T_1(r)$, $T_2(r)$ *be real valued, nonnegative and nondecreasing functions defined for* $r > r_0 > 0$ *and satisfying* $T_1(r) = O(T_2(r))$, $r \to \infty$, n.e., *then we have*

(i) $\limsup_{r\to\infty} \overset{+}{\log} T_1(r)/\log r \leqq \limsup_{r\to\infty} \overset{+}{\log} T_2(r)/\log r$

and

(ii) $\liminf_{r\to\infty} \overset{+}{\log} T_1(r)/\log r \leqq \liminf_{r\to\infty} \overset{+}{\log} T_2(r)/\log r$.

This implies in particular that for meromorphic functions f_1 *and* f_2 *with* $T(r, f_1) = O(T(r, f_2))$, $r \to \infty$, n.e., *the inequalities* $\lambda(f_1) \leqq \lambda(f_2)$ *and* $\rho(f_1) \leqq \rho(f_2)$ *hold.*

Proof. (i) Assume without loss of generality that

$$\lambda := \limsup_{r\to\infty} \frac{\overset{+}{\log} T_2(r)}{\log r} < \infty .$$

For arbitrary $\varepsilon > 0$ there exist $R > \max\{r_0, 1\}$, $K > 0$ and $D \subseteqq [R, \infty)$ such that $T_2(r) \leqq r^{\lambda+\varepsilon}$ for $r \geqq R$, $T_1(r) \leqq KT_2(r)$ for $r \in [R, \infty)\backslash D$ and $m := \operatorname{mes}(D) < \infty$. Here m denotes the Lebesgue-measure of D. Now for $r > R + m$ and $r \in D$ one can find $r_1, r_2 \notin D$, $R \leqq r_1 < r < r_2$ and $r_2 - r_1 \leqq m + 1$ such that $T_1(r) \leqq KT_2(r_2) \leqq Kr_2^{\lambda+\varepsilon} \leqq K(r_2/r_1)^{\lambda+\varepsilon} r^{\lambda+\varepsilon} \leqq Cr^{\lambda+\varepsilon}$ with $C: = K(m+2)^{\lambda+\varepsilon}$, i.e., $T_1(r) \leqq Cr^{\lambda+\varepsilon}$ for all $r > R + m$. Hence we obtain

$$\limsup_{r\to\infty} \frac{\overset{+}{\log} T_1(r)}{\log r} \leqq \lambda + \varepsilon \quad \text{for arbitrary} \quad \varepsilon > 0 ;$$

We conclude that (i) holds.

(ii) Assume the contrary and carry on as above.

3. The proofs of Theorems 1 and 2. With the assumptions of Theorem 1 let

$$\Psi = f^n Q[f] + P[f] .$$

By means of Lemmas 1 and 2 we see that Ψ connot be constant and setting $v = \Psi'/\Psi$ we get

$$f^{n-1} H = vP[f] - P'[f] \tag{5}$$

where

$$H = nf'Q[f] + fQ'[f] - vfQ[f] . \tag{6}$$

Now Lemmas 1 and 2 show that $H \not\equiv 0$. Otherwise $\Psi'/\Psi = P'[f]/P[f]$, i.e. $\Psi = KP[f]$ for a suitable $K \in \boldsymbol{C}$ leading to $f^n Q[f] + (1 - K)P[f] \equiv 0$. However, since $\Gamma_P \leqq n - 3$ by assumption this implies $T(r, Q[f]) = S(r, f)$ by use of Lemma 2 and therefore $T(r, f^n) \leqq T(r, P[f]) + S(r, f)$ since $Q[f] \not\equiv 0$, again by assumption. Now Lemma 1 leads to $nT(r, f) \leqq \Gamma_P T(r, f) + S(r, f)$ which is impossible.

Further we infer from $S(r, \Psi) \leqq S(r, f)$

$$vP[f] - P'[f] = T[f] \quad \text{with} \quad \gamma_T \leqq \gamma_P \tag{7}$$

where all coefficients t of the differential polynomial $T[f]$ satisfy $m(r, t) = S(r, f)$.

Therefore we can invoke Lemma 2 and (5) leads to

$$m(r, H) = S(r, f) . \tag{8}$$

It remains to be shown

$$N(r, H) \leqq \bar{N}\left(r, \frac{1}{\Psi}\right) + S(r, f) . \tag{9}$$

First choose $z_0 \in C$ such that $H(z_0) = \infty$.

If $f(z_0) = \infty$ with order ν we get

$$\mu \leqq \Gamma_P \cdot \nu + \max\{\nu_1, \cdots, \nu_n\} + 1 - (n-1)\cdot \nu \leqq \max\{\nu_1, \cdots, \nu_n\}$$

where $\nu_1, \cdots, \nu_n$ and μ denote the orders of the poles of the coefficients $p_1, \cdots, p_n$ of $P[f]$ and H in z_0 respectively (remember that $n \geqq 3 + \Gamma_P$ by assumption).

Using the notations of Lemma 2 we can write this as

$$N_f(r, H) \leqq \sum_{j=1}^{n} N(r, p_j) + S(r, f) = S(r, f) . \tag{10}$$

Further, let $q_1, \cdots, q_\ell$ be the coefficients of Q. Then we can conclude

$$N^f(r, H) \leqq 2 \sum_{j=1}^{\ell} N(r, q_j) + N^f(r, v) + S(r, f)$$

and because of

$$N^f(r, v) \leqq \bar{N}\left(r, \frac{1}{\Psi}\right) + \sum_{j=1}^{\ell} N(r, q_j) + \sum_{j=1}^{n} N(r, p_j) + S(r, f)$$

we finally arrive at

$$N^f(r, H) \leqq \bar{N}\left(r, \frac{1}{\Psi}\right) + S(r, f) . \tag{11}$$

Now (10) and (11) together prove that (9) is valid.

Noting that $H \not\equiv 0$ one infers from (3), (8) and (9) using

$$T(r, f^{n-1}) \leqq T(r, vP[f] - P'[f]) + T(r, H) + S(r, f)$$

and

$$N(r, vP[f] - P'[f]) \leqq \Gamma_P N(r, f) + \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{\Psi}\right) + S(r, f)$$

the inequality

$$T(r, f^{n-1}) \leqq \Gamma_P T(r, f) + \bar{N}(r, f) + 2\bar{N}\left(r, \frac{1}{\Psi}\right) + S(r, f) .$$

Here use was made of Lemma 1(i). Keeping in mind however that $\Gamma_P \leqq n - 3$ we get

$$T(r, f) = O\left(\bar{N}\left(r, \frac{1}{\Psi}\right)\right) , \quad r \longrightarrow \infty , \quad \text{n.e.} \tag{12}$$

The rest is easy.

First one clearly sees that the assumption $\bar{N}(r, 1/\Psi) = S(r, f)$ leads to a contradiction, hence zero cannot be a Picard exceptional value of Ψ and we have

$$\limsup_{r\to\infty} \frac{\bar{N}(r, 1/\Psi)}{T(r, \Psi)} > 0 .$$

Applying Lemma 3 to equation (12) we get

$$\lambda(f) \leqq \limsup_{r\to\infty} \frac{\overset{+}{\log} \bar{N}(r, 1/\Psi)}{\log r} =: \lambda ,$$

and observing $\lambda \leqq \lambda(\Psi) \leqq \lambda(f)$ we see, that zero cannot be a Borel exceptional value of Ψ either. This completes the proof of Theorem 1.

REMARK. Using (12) and Lemma 3 we obtain $\lambda(f) = \lambda(\Psi)$ and $\rho(f) = \rho(\Psi)$ under the stated assumptions.

The proof of Theorem 2 is now easily accomplished. Assume $N(r, f) = S(r, f)$ then due to

$$T(r, P[f]) \leqq (n-2)T(r, f) + S(r, f) \quad \text{and} \quad N(r, Q[f]) = S(r, f)$$

(cf. Lemmas 1 and 2, (5) and (6)) one gets just as in the proof of Theorem 1

$$\Psi \not\equiv c , \quad H \not\equiv 0 , \qquad T(r, H) \leqq \bar{N}\left(r, \frac{1}{\Psi}\right) + S(r, f) \tag{13}$$

where analogous notation is used. And from

$$f^{n-1}H = \frac{\Psi'}{\Psi}P[f] - P'[f]$$

we infer that

$$(n-1)T(r, f) \leqq (n-2)T(r, f) + 2\bar{N}\left(r, \frac{1}{\Psi}\right) + S(r, f)$$

and therefore

$$T(r, f) = O\left(\bar{N}\left(r, \frac{1}{\Psi}\right)\right) , \quad r \longrightarrow \infty , \quad \text{n.e.} ,$$

holds again.

The statements of Theorem 2 are now obvious.

REMARK. As above, Ψ and f have again the same upper and lower orders of growth.

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REFERENCES

1. J. Clunie, *On integral and meromorphic functions*, J. London Math. Soc., **37**, (1962), 17–27.
2. ———, *On a result of Hayman*, J. London Math. Soc., **47** (1967), 389–392.
3. W. K. Hayman, *Meromorphic Functions*, Oxford, Clarendon Press 1975.
4. ———, *Picard values of meromorphic functions and their derivatives*, Ann. of Math., II. Ser. **70** (1959), 9–42.
5. E. Mues, *Über die Nullstellen homogener Differential polynome*, manuscripta math. **23** (1978), 325–341.
6. ———, *Über ein Problem von Hayman*, Math. Z., **164** (1979), 239–259.
7. ———, *Zur Wertverteilung von Differentialpolynomen*, Arch. Math. **32** (1979), 55–67.
8. R. Nevanlinna, *Eindeutige Analytische Funktionen*, Berlin, Heidelberg, New York: Springer 1974.
9. L. R. Sons, *Deficiencies of monomials*, Math. Z. **111** (1969), 53–68.
10. H. Wittich, *Einige Eigenschaften der Lösungen von* $w' = a(z) + b(z)w + c(z)w^2$, Arch. Math., **5** (1954), 226–232.
11. ———, *Neuere Untersuchungen über eindeutige analytische Funktionen*, Berlin, Göttingen, Heidelberg: Springer 1955.
12. C. C. Yang, *Applications of the Tumura-Clunie Theorem*, Trans. Amer. Math. Soc., **151** (1970), 659–662.
13. ———, *On deficiencies of differential polynomials*, Math. Z., **116** (1970), 197–204.
14. ———, *On deficiencies of differential polynomials II*, Math. Z., **125** (1972), 107–112.

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