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EQUIVALENT NILPOTENCIES IN CERTAIN GENERALIZED RIGHT ALTERNATIVE RINGS

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A nonassociative ring is called generalized right alternative if it satisfies the identity $(wx, y, z) + (w, x, [y, z]) = w(x, y, z) + (w, y, z)x$. Generalized right alternative rings which also satisfy $([w, x], y, z) + (w, x, yz) = y(w, x, z) + (w, x, y)z$ or $(x, y, z) + (y, z, x) + (z, x, y) = 0$ are known as generalized alternative or generalized $(-1, 1)$ rings, respectively. For both these varieties it is proved that either left or right nilpotence implies nilpotence. However, characteristic $\neq 2$ is required for generalized $(-1, 1)$ rings in the case of right nilpotence.

1. Introduction. Using the standard notation $(x, y, z) = (xy)z - x(yz)$ for the associator and $[x, y] = xy - yx$ for the commutator, a nonassociative ring which satisfies the identity

$$(wx, y, z) + (w, x, [y, z]) = w(x, y, z) + (w, y, z)x \tag{1}$$

is called generalized right alternative. Such rings which also satisfy

$$([w, x], y, z) + (w, x, yz) = y(w, x, z) + (w, x, y)z \tag{2}$$

are known as generalized alternative, and those that satisfy (1) and

$$(x, y, z) + (y, z, x) + (z, x, y) = 0 \tag{3}$$

are called generalized $(-1, 1)$. The studies of these three varieties were each initiated by E. Kleinfeld [2-4], with the strongest result on the structure of generalized right alternative rings per se due to Hentzel and Cattaneo [1].

Let A be any nonassociative ring. If for some positive integer n every product of n elements from A is zero, no matter how the elements are associated, then A is said to be nilpotent. In this case the least such integer n is referred to as the index of nilpotency of A. Setting $A^{[0]} = A$ and defining inductively $A^{[k]} = AA^{[k-1]}$, then less restrictively A is called left nilpotent of index n if $A^{[n]} = (0)$ and n is the least such integer. Right nilpotence is defined analogously. In addition, if we let $A^{(0)} = A$ and define inductively $A^{(k)} = (A^{(k-1)})^2$, then A is called solvable of index n if $A^{(n)} = (0)$ and $A^{(n-1)} \neq (0)$. It is immediate that nilpotency implies left and right nilpotency, and that left or right nilpotency implies solvability.

As the name generalized alternative suggests, identities (1) and (2) are both satisfied by any alternative ring. In [10] Zhevlakov proved that a left or right nilpotent alternative ring must be nilpotent. In this paper we shall extend his result to the variety of generalized alternative rings.

By definition any $(-1, 1)$ ring satisfies (3), and provided characteristic $\neq 2$ such rings likewise satisfy (1). (That this restriction on characteristic is necessary can be seen using example 2 from [5].) Thus a $(-1, 1)$ ring with characteristic $\neq 2$ is generalized $(-1, 1)$. In [8] there is a proof due to Slin'ko that for $(-1, 1)$ rings left nilpotency implies nilpotency. We shall extend this result to generalized $(-1, 1)$ rings. Also, Pchelincev [6] and Dorofeev each proved that a right nilpotent $(-1, 1)$ ring with characteristic $\neq 2$ is nilpotent. This result extends as well to the variety of generalized $(-1, 1)$ rings. However, as demonstrated by an example, characteristic $\neq 2$ is required there too.

In [7] Pokrass proved that for flexible generalized right alternative rings left or right nilpotency is equivalent to nilpotency. Our approach parallels that used by Pokrass, but the argument applied here is more general in that it does not utilize any sort of result concerning the product of ideals in either of the varieties considered.

2. Main section. Let A be a nonassociative ring. For $a \in A$, L_a and R_a denote the operators of left and right multiplication by a, respectively. The notation S_a is used when the operator S can be either L or R.

We begin by writing (1) in expanded form. After some cancelling of terms, we have

$$\begin{aligned} 0 &= [(wx)y]z - (wx)(zy) + w[x(zy)] \\ &\quad - w[(xy)z] - [(wy)z]x + [w(yz)]x \,. \end{aligned} \tag{1'}$$

Then taking in turn w, x, y, and z as the argument, (1′) in operator form gives

$$\begin{aligned} 0 &= R_x R_y R_z - R_x R_{zy} + R_{x(zy)} \\ &\quad - R_{(xy)z} - R_y R_z R_x + R_{yz} R_x \,, \end{aligned} \tag{4}$$

$$\begin{aligned} 0 &= L_w R_y R_z - L_w R_{zy} + R_{zy} L_w \\ &\quad - R_y R_z L_w - L_{(wy)z} + L_{w(yz)} \,, \end{aligned} \tag{5}$$

$$\begin{aligned} 0 &= L_{wx} R_z - L_z L_{wx} + L_z L_x L_w \\ &\quad - L_x R_z L_w - L_w R_z R_x + R_z L_w R_x \,, \end{aligned} \tag{6}$$

$$\begin{aligned} 0 &= L_{(wx)y} - R_y L_{wx} + R_y L_x L_w \\ &\quad - L_{xy} L_w - L_{wy} R_x + L_y L_w R_x \,. \end{aligned} \tag{7}$$

Also. expanding (3) and taking x as the argument gives

$$(8) \qquad 0 = R_y R_z - R_{yz} + L_{yz} - L_z L_y + L_z R_y - R_y L_z .$$

Since identities (4)-(7) hold in any generalized right alternative ring, they are valid for both generalized alternative and generalized $(-1, 1)$ rings. Identity (8), of course, is only valid for generalized $(-1, 1)$ rings.

We note, too, that if the ring A is generalized alternative, then due to the symmetry between identities (1) and (2) the opposite ring of A is likewise generalized alternative. In particular, this means if such an A satisfies some relation involving multiplication operators, then A also satisfies the opposite relation where L's and R's are interchanged.

Letting B denote the ideal A^2, we first prove

LEMMA 1. *If A is a generalized right alternative ring, then $(B^k A)B \subseteqq B^{k+1}$ for $k \geqq 1$.*

Proof. To show $(B^k A)B \subseteqq B^{k+1}$ we induct on k. For $k = 1$, $(BA)B \subseteqq B^2$ since B is an ideal. Now assume $(B^i A)B \subseteqq B^{i+1}$ for $1 \leqq i < k$ and consider $(B^k A)B$ where $k \geqq 2$. Let $1 \leqq i \leqq k - 1$. Then from (1) we obtain $[(B^i B^{k-i})A]B \subseteqq (B^i B^{k-i})(AB) + (B^i B^{k-i}, A, B) \subseteqq (B^i B^{k-i})(AB) + (B^i, B^{k-i}, [A, B]) + B^i(B^{k-i}, A, B) + (B^i, A, B)B^{k-i} \subseteqq B^{k+1}$, using B is an ideal and the induction assumption. Since $B^k = \sum_{i=1}^{k-1} B^i B^{k-i}$, this proves $(B^k A)B \subseteqq B^{k+1}$ and completes our induction.

In Lemmas 2-7 the ring A can be either generalized alternative or generalized $(-1, 1)$.

LEMMA 2. *Let $T = S_{x_1} S_{x_2} S_{x_3} S_{x_4}$ where for each $1 \leqq i < 4$ either $x_i \in B$ or $x_{i+1} \in B$. Then $(B^k)T \subseteqq (B^{k+1}) \sum S \cdots S$.*

Proof. If $x_1 \in B$ the result is obvious. Thus we assume $x_1 = a \notin B$, so that by assumption $x_2 = b \in B$. Depending on the R's and L's in $S_{x_1} S_{x_2}$ there are now four possible cases. First suppose $S_{x_1} S_{x_2} = R_a R_b$. Then by Lemma 1 $(B^k)T = (B^k) R_a R_b S_{x_3} S_{x_4} \subseteqq (B^{k+1}) S_{x_3} S_{x_4}$ as required.

We next suppose that $S_{x_1} S_{x_2} = L_a R_b$, and begin by assuming $x_3 = b' \in B$. In this case, using (6) with $x = a$, $z = b$, $w = b'$ we have

$$L_a R_b L_{b'} = L_{b'a} R_b - L_b L_{b'a} + L_b L_a L_{b'} - L_{b'} R_b R_a + R_b L_{b'} R_a ;$$

or using (6) with $w = a$, $z = b$, $x = b'$ we have

$$L_a R_b R_{b'} = L_{ab'} R_b - L_b L_{ab'} + L_b L_{b'} L_a - L_{b'} R_b L_a + R_b L_a R_{b'} .$$

Hence in either situation $(B^k)T = (B^k) L_a R_b S_{b'} S_{x_4} \subseteqq (B^{k+1}) \sum S \cdots S$ as required. On the other hand, if $x_3 = a' \notin B$ then $x_4 = b' \in B$ by assumption. In this case we first use (5) with $w = a$, $y = b$, $z = a'$ to obtain

$$L_a R_b R_{a'} = L_a R_{a'b} - R_{a'b} L_a + R_b R_{a'} L_a + L_{(ab)a'} - L_{a(ba')} .$$

Thus if $S_{x_3} = R_{a'}$, then $(B^k)T = (B^k)L_a R_b R_{a'} S_{b'} \subseteqq (B^{k+1}) \sum S \cdots S$ utilizing our previous calculations for $L_a R_b S_{b'}$. Next we use (6) with $x = a$, $z = b$, $w = a'$ to obtain

$$L_a R_b L_{a'} = L_{a'a} R_b - L_b L_{a'a} + L_b L_a L_{a'} - L_{a'} R_b R_a + R_b L_{a'} R_a .$$

Then for $S_{x_3} = L_{a'}$ we have $(B^k)T = (B^k)L_a R_b L_{a'} S_{b'} \subseteqq (B^{k+1}) \sum S \cdots S$ using our preceding calculation for $L_a R_b R_{a'} S_{b'}$. Hence we now have shown that if $T = L_a R_b S_{x_3} S_{x_4}$, then $(B^k)T \subseteqq (B^{k+1}) \sum S \cdots S$ as required.

Thus far the argument applies for both generalized alternative and generalized $(-1, 1)$ rings. For generalized alternative rings the two remaining cases, $S_{x_1} S_{x_2} = R_a L_b$ and $S_{x_1} S_{x_2} = L_a L_b$, now follow by symmetry. Therefore at this point we can assume the ring A is generalized $(-1, 1)$. Let $S_{x_1} S_{x_2} = R_a L_b$. Then by (8) we have $(B^k)T = (B^k)T = (B^k)R_a L_b S_{x_3} S_{x_4} = (B^k)[R_a R_b - R_{ab} + L_{ab} - L_b L_a + L_b R_a] S_{x_3} S_{x_4}$. Since we have already established $(B^k) R_a R_b S_{x_3} S_{x_4} \subseteqq (B^{k+1}) S_{x_3} S_{x_4}$, this shows $(B^k) R_a L_b S_{x_3} S_{x_4} \subseteqq (B^{k+1}) \sum S \cdots S$. Finally, let $S_{x_1} S_{x_2} = L_a L_b$. Then again by (8) $(B^k)T = (B^k) L_a L_b S_{x_3} S_{x_4} = (B^k)[R_b R_a - R_{ba} + L_{ba} + L_a R_b - R_b L_a] S_{x_3} S_{x_4}$. Since we have also established $(B^k) L_a R_b S_{x_3} S_{x_4} \subseteqq (B^{k+1}) \sum S \cdots S$, this shows $(B^k) L_a L_b S_{x_3} S_{x_4} \subseteqq (B^{k+1}) \sum S \cdots S$, which completes the proof of the lemma.

LEMMA 3. *Let* $T = S_{x_1} \cdots S_{x_n} S_b$ *where* $b \in B$ *and* $n \geqq 1$. *Then* T *can be expressed as a sum of terms each of the form* $S_{b'} S_{y_1} \cdots S_{y_m}$ *or* $S_{y_1} S_{b'} S_{y_2} \cdots S_{y_m}$ *where* $b' \in B$.

Proof. The proof is by induction on n. For $n = 1$ the lemma is true immediately. Thus we assume $n \geqq 2$ and that the lemma holds for all values less than n. Then $T = S_{x_1} \cdots S_{x_{n-1}} S_{x_n} S_b$, and depending on the R's and L's in $S_{x_{n-1}} S_{x_n} S_b$ there are eight possible cases to consider. Using our induction assumption, first $R_{x_{n-1}} R_{x_n} R_b$ follows from (4) if we set $x = b$, $y = x_{n-1}$, $z = x_n$. Similarly $R_{x_{n-1}} R_{x_n} L_b$ follows from (5) taking $w = b$, $y = x_{n-1}$, $z = x_n$; and $L_{x_{n-1}} R_{x_{n-1}} R_b$ follows from (5) with $w = x_{n-1}$, $y = x_n$, $z = b$. Then letting $w = x_n$, $x = b$, $z = x_{n-1}$ in (6), $R_{x_{n-1}} L_{x_n} R_b$ reduces to $L_{x_n} R_{x_{n-1}} R_b$ which was just established.

So far our argument applies to both generalized alternative and generalized $(-1, 1)$ rings. For generalized alternative rings the four remaining cases, $R_{x_{n-1}} L_{x_n} L_b$, $L_{x_{n-1}} R_{x_n} L_b$, $L_{x_{n-1}} L_{x_n} R_b$, and $L_{x_{n-1}} L_{x_n} L_b$, now follow by symmetry. Therefore at this point we assume the ring A is generalized $(-1, 1)$. Then using (8) and the induction assumption we see that

$$R_{x_{n-1}}L_{x_n}L_b = R_{x_{n-1}}[R_bR_{x_n} - R_{bx_n} + L_{bx_n} + L_{x_n}R_b - R_bL_{x_n}]$$

reduces to $R_{x_{n-1}}L_{x_n}R_b$, and that

$$L_{x_{n-1}}R_{x_n}L_b = L_{x_{n-1}}[R_{x_n}R_b - R_{x_nb} + L_{x_nb} - L_bL_{x_n} + L_bR_{x_n}]$$

reduces to $L_{x_{n-1}}R_{x_n}R_b$, both of which have already been established. Similarly,

$$L_{x_{n-1}}L_{x_n}R_b = [R_{x_n}R_{x_{n-1}} - R_{x_nx_{n-1}} + L_{x_nx_{n-1}} + L_{x_{n-1}}R_{x_n} - R_{x_n}L_{x_{n-1}}]R_b$$

reduces to earlier cases, and then

$$L_{x_{n-1}}L_{x_n}L_b = L_{x_{n-1}}[R_bL_{x_n} - R_{bx_n} + L_{bx_n} + L_{x_n}R_b - R_bL_{x_n}]$$

reduces to $L_{x_{n-1}}L_{x_n}R_b$. This completes the proof of the lemma.

LEMMA 4. *Let $T = S_{x_1} \cdots S_{x_n}$ be such that k of the $x_i \in B$ where $1 \leqq k \leqq n$. Then T can be expressed as a sum of terms each of the form $S_{y_1} \cdots S_{y_m}S \cdots S$ where at least k of the $y_i \in B$, and for each $1 \leqq i < m$ either $y_i \in B$ or $y_{i+1} \in B$.*

Proof. The proof is by induction on k with the case $k = 1$ implied by Lemma 3. Thus assume $T = S \cdots S_{b_1} \cdots S_{b_{k-1}} \cdots S_{b_k} \cdots S$ where the $b_i \in B$, and that the lemma holds for values less than k. Then we apply this assumption to express $S \cdots S_{b_1} \cdots S_{b_{k-1}}$ as a sum of terms $S_{y_1} \cdots S_{y_m}S \cdots S$ each having the desired property for $k - 1$. This means T is now expressed as a sum of terms each of the form $S_{y_1} \cdots S_{y_m}S \cdots S_{b_k} \cdots S$. If $y_m \in B$, we apply Lemma 3 to $S \cdots S_{b_k}$. If $y_m \notin B$, then $y_{m-1} \in B$, and we apply Lemma 3 to $S_{y_m}S \cdots S_{b_k}$. In either case, T is then expressed as a sum of terms each having the desired property.

LEMMA 5. *Let $B^k = (0)$. If $T = S_{x_1} \cdots S_{x_n}$ where $4k + 1$ of the $x_i \in B$, then $T = 0$.*

Proof. For any $a \in A$, $(a)T \in (B)S_{x_2} \cdots S_{x_n}$ where at least $4k$ of the $x_i \in B$. By Lemma 4 $S_{x_2} \cdots S_{x_n}$ can be expressed as a sum of terms each of the form $S_{y_1} \cdots S_{y_m}S \cdots S$, where at least $4k$ of the $y_i \in B$ and for each $1 \leqq i < m$ either $y_i \in B$ or $y_{i+1} \in B$. Then by Lemma 2 $(a)T \in (B^2) \sum S_{z_1} \cdots S_{z_{j_t}}$, where in each term at least $4(k-1)$ of the $z_j \in B$. Again by Lemma 4 these $4(k-1)$ b's can be brought forward, if necessary, and again by Lemma 2 $(a)T \in (B^3) \sum S_{w_1} \cdots S_{w_{j_r}}$, where in each term at least $4(k-2)$ of the $w_j \in B$. Repeating this process k times, we arrive at $(a)T \in (B^k) \sum S \cdots S = (0)$. Thus $T = 0$, which completes the proof of the lemma.

At this point we adopt the following notation. If for operators T and T' we have $T - T' = \sum T_i$, where each operator T_i has a factor of the form S_{b_i} with $b_i \in B$, we shall write $T \equiv T'$.

LEMMA 6. *If* $T = S_{x_1} \cdots S_{x_j}$ *where* $m \geqq 1$ *of the* S*'s are* L*'s, then* $T \equiv \sum L_{y_1} \cdots L_{y_{m-1}} S \cdots S$.

Proof. For $j = 1$ or 2 the lemma is clearly true. Thus we assume $j \geqq 3$ and that the lemma holds for values less than j. Now for an operator of the form $LS \cdots S$ our induction assumption applies to the subword $S \cdots S$. Hence we can assume $T = RS \cdots S$ and consider the initial subword RSS of T. We shall show that in each of four possible cases, namely RRR, RRL, RLR, and RLL, we can make substitutions that reduce T to the form $T \equiv \sum LS \cdots S$. But then, as we have just indicated, the induction assumption can be applied to complete the proof.

First from (5) we see $RRL \equiv LRR$, and from (7) $RLL \equiv -LLR$. Next (6) implies

$$\text{(i)} \qquad RLR \equiv LRL + LRR - LLL .$$

Now if the ring A is generalized alternative, going to the opposite ring (i) gives

$$\text{(ii)} \qquad RRR \equiv RLR + RLL - LRL .$$

But RLR and RLL have already been reduced, so in this case the reduction of T to the form $T \equiv \sum LS \cdots S$ is complete. On the other hand, if A is generalized $(-1, 1)$, then using (8) we have

$$\text{(iii)} \qquad RRR \equiv [LL - LR + RL]R .$$

This likewise completes the reduction of T, and thereby the proof of the lemma.

LEMMA 7. *If the ring* A *is left nilpotent and* B *is nilpotent, then* A *itself is nilpotent.*

Proof. Suppose A is left nilpotent, so every product of say $m \geqq 1$ L's is zero, and consider an operator $T = S_{x_1} \cdots S_{x_{3(m+1)}}$. Thinking of T as a product of $m + 1$ blocks, each of length 3, it follows that T has an L in each block or else three consecutive R's in one or more blocks. In this latter case we can use (ii) or (iii) to substitute for each RRR, so that in either case $T \equiv \sum T_i$ where each T_i has $m+1$ blocks of length 3 and each block has at least one L. Then by Lemma 6 each $T_i \equiv \sum L_{y_1} \cdots L_{y_m} S \cdots S$. But each $L_{y_1} \cdots L_{y_m} = 0$, so $T \equiv 0$ if T is a product of $3(m+1)$ S's.

Now by assumption B is nilpotent, so suppose $B^k = (0)$. By the preceding argument it follows that every product of $3(m+1)(4k+1)$ S's is a sum of terms each containing $4k+1$ factors S_{x_i} with $x_i \in B$. Hence by Lemma 5 such a product is zero. This shows A^*, the associative ring generated by left and right multiplications of A, is nilpotent. Thus by Theorem 2.4 in [9] A itself is nilpotent.

THEOREM 1. *If A is a left or right nilpotent generalized alternative ring, then A is nilpotent.*

Proof. We assume first that A is a left nilpotent generalized alternative ring. Then A is solvable, and to prove A is nilpotent we induct on the index of solvability of A. To start, A is clearly nilpotent when $A^2 = A^{(1)} = 0$. Then by induction we can assume $B = A^2$ is nilpotent, since B is a left nilpotent generalized alternative ring with solvable index less than that of A. Hence by Lemma 7 A itself is nilpotent, which completes the induction.

On the other hand, if A is a right nilpotent generalized alternative ring, then the opposite ring of A is also generalized alternative but left nilpotent. Thus by the preceding argument the opposite ring of A must be nilpotent, which of course means A is nilpotent as well. This completes the proof of the theorem.

Now since Lemma 7 also applies to generalized $(-1, 1)$ rings, the above proof actually shows a left nilpotent generalized $(-1, 1)$ ring is likewise nilpotent. However, since the opposite ring of a generalized $(-1, 1)$ ring need not be generalized $(-1, 1)$, the above proof for the right nilpotent case does not apply to generalized $(-1, 1)$ rings. Consequently, we shall henceforth assume A is a generalized $(-1, 1)$ ring with characteristic $\neq 2$. For such an A we shall show we can replace left by right in Lemma 7. Then replacing left by right in the proof of Theorem 1, and again inducting on the index of solvability of A, it follows such a right nilpotent A is nilpotent. To make the indicated modification of Lemma 7, we first need to modify Lemma 6.

LEMMA 6′. *If $T = S_{x_1} \cdots S_{x_j}$ where $m \geqq 1$ of the S's are R's, then $T \equiv \sum R_{y_1} \cdots R_{y_{m-1}} S \cdots S$.*

Proof. The proof, which is by induction, is completely analogous to that of Lemma 6. However, this time our goal is to show that for $T = LS \cdots S$ we can substitute for the subword LSS to reduce

T to the form $T \equiv \sum RS \cdots S$, whence as in Lemma 6 the induction applied to the subwords $S \cdots S$ completes the proof.

We first use (5) and (7) to see $LRR \equiv RRL$ ann $LLR \equiv -RLL$. For operators T and T' we then introduce the notation $T \sim T'$ if $T \equiv T' + \sum RSS + \sum LRR + \sum LLR$. Using (8) we have $L_zL_yL_x \equiv [R_yR_z + L_zR_y - R_yL_z]L_x$, so that

$$\text{(iv)} \qquad L_zL_yL_x \sim L_zR_yL_x\ .$$

Using (8) again $L_xL_zL_y \equiv L_x[R_yR_z + L_zR_y - R_yL_z]$, so that also

$$\text{(v)} \qquad L_xL_zL_y \sim -L_xR_yL_z\ .$$

Next letting $w = y$ in (6) we obtain $L_zL_xL_y \equiv L_xR_zL_y + L_yR_zR_x - R_zL_yR_x$, which is

$$\text{(vi)} \qquad L_zL_xL_y \sim L_xR_zL_y\ .$$

Now applying (iv), (v), (vi), and (iv) in succession, we see that $L_zL_yL_x \sim L_zR_yL_x \sim -L_zL_xL_y \sim -L_xR_zL_y \sim -L_xL_zL_y$. Thus

$$\text{(vii)} \qquad L_zL_yL_x \sim -L_xL_zL_y\ .$$

Then applying (vii) repeatedly we obtain $L_zL_yL_x \sim -L_xL_zL_y \sim L_yL_xL_z \sim -L_zL_yL_x$, or $2L_zL_yL_x \sim 0$. Since characteristic $\neq 2$, this implies $LLL \equiv \sum RSS + \sum LRR + \sum LLR$; and so by (iv) also $LRL \equiv \sum RSS + \sum LRR + \sum LLR$. But the cases LRR and LLR have been established, and consequently this completes the proof of the lemma.

LEMMA 7′. *If the ring A is right nilpotent and B is nilpotent, then A itself is nilpotent.*

This is Lemma 7 with left replaced by right. Interchanging L's and R's, the proof of Lemma 7′ is the same as Lemma 7 with the following two adjustments. One uses (i) in order to substitute for each LLL, and Lemma 6′ is used in place of Lemma 6. As indicated after proving Theorem 1, we can now conclude

THEOREM 2. *Let A be a generalized $(-1, 1)$ ring. If A is either left nilpotent, or right nilpotent with characteristic $\neq 2$, then A is nilpotent.*

In [6] Pchelincev constructed the following example of a right nilpotent but not left nilpotent $(-1, 1)$ algebra with characteristic $= 2$. Let A be the vector space over Z_2 with countable basis $\{e_1, e_2, \cdots\}$. We define a multiplication on A by $e_1e_i = e_{i+1}$, $e_{2k}e_1 = e_{2k+1}$, and all other products of basis elements are zero. A straight forward verification shows that A is also a generalized $(-1, 1)$ algebra. Consequently, the restriction on characteristic in the right nilpotent case of Theorem 2 is necessary.

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