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RINGS ON CERTAIN MIXED ABELIAN GROUPS

DAVID R. JACKETT

This paper is concerned with the ring structures supported by certain mixed abelian groups. A class $\mathscr{M}$ of mixed abelian groups of torsion-free rank one is introduced, and properties of rings on groups in $\mathscr{M}$ are discussed. We provide complete descriptions of the absolute annihilator and the absolute radical of groups in $\mathscr{M}$. These absolute ideals are also investigated for cotorsion groups and reduced algebraically compact groups, thus providing a partial solution to Problem 94 of Fuchs (Infinite abelian groups, Vol. II). The results also allow us to answer a question raised by Rotman (J. Algebra, 9 (1968), 369–387) concerning completions of rings.

1. Preliminaries. All groups that we consider are additive abelian groups. A ring on a group A, denoted $(A, \cdot)$, is distributive, not necessarily associative, and may not have an identity.

A subgroup B of A is an *absolute ideal* of A if $(B, \cdot)$ is a (two sided) ideal of $(A, \cdot)$ for every ring $(A, \cdot)$ on A. The *absolute annihilator* of A, denoted $A(^*)$, is $\{a \in A \mid a \cdot A = 0 = A \cdot a$ for all rings $(A, \cdot)$ on $A\}$. If $(A, \cdot)$ is associative, its (Jacobson) radical is denoted $J(A, \cdot)$. The *absolute radical* of A is $J(A)$, the intersection of all $J(A, \cdot)$ over all associative rings $(A, \cdot)$ on A.

All other group and ring theoretical notation is standard and can be found in Fuchs [3] and Jacobson [6] respectively.

The structures of the absolute annihilator and the absolute radical of a torsion group are well known.

(1.1) (Fuchs [3] Vol. II, p. 289). *If A is a torsion group, then* $A(^*) = A^1 = \bigcap_n nA$, *and* $J(A) = \bigcap_p pA$, □

The following results, where A need not be torsion, are easily proved.

(1.2) *Suppose* $A = \bigoplus_{i \in I} A_i$. *Then* $A(^*) \subseteqq \bigoplus_{i \in I} A_i(^*)$, *and* $J(A) \subseteqq \bigoplus_{i \in I} J(A_i)$.

(1.3) *If B is an absolate ideal of A, then* $J(B) \subseteqq J(A)$. □

2. A class of mixed groups of torsion-free rank one. Let $\mathscr{M}$ denote the class of groups A such that A has torsion-free rank one and A can be embedded as a pure subgroup of $\prod_p A_p$, where

A_p is the p-primary component of $T(A)$, the torsion subgroup of A.

Suppose A is a mixed group. For $a \in \prod_p A_p$ let $\bar{a}$ denote the image of a under the natural map $\prod_p A_p \to \prod_p A_p/\bigoplus_p A_p = \prod_p A_p/T(A)$.

PROPOSITION 2.1.

(a) *If $A \in \mathscr{M}$, then $A/T(A) \cong Q$ and A_p is a direct summand of A for each prime p. Conversely, if A is a non-splitting mixed group for which $A/T(A) \cong Q$ and A_p is a direct summand of A for each prime p, then the reduced part of A is in $\mathscr{M}$.*

(b) *If $A \in \mathscr{M}$ and a is an element of infinite order in A, then A is the inverse image of $\langle \bar{a} \rangle_*$, the pure subgroup generated by $\bar{a}$, under the natural map $\prod_p A_p \to \prod_p A_p/\bigoplus_p A_p$. Conversely, for p-groups A_p and any element a in $\prod_p A_p$ of infinite order, the group A defined as the inverse image of $\langle \bar{a} \rangle_*$ under the natural map $\prod_p A_p \to \prod_p A_p/\bigoplus_p A_p$ is in $\mathscr{M}$.*

Proof. The only statement requiring more than elementary group theory is the second statement in (a), which can be proved using arguments found in Rajagopalan and Rotman [8]. □

A consequence of (a) is that if A is a reduced mixed group of torsion-free rank one, then various conditions on either the endormorphism ring of A, or the rings supported by A force A to be in $\mathscr{M}$. Examples abound in the literature, see for example Fuchs [2], Fuchs and Rangaswamy [4], Rangaswamy [9], Schultz [11], and Szele and Szendrei [13].

If $A \in \mathscr{M}$, then for each prime p there is a subgroup $A^{(p)}$ of A such that $A = A_p \oplus A^{(p)}$. Any ring $(A_p, \cdot)$ on A_p can be extended to a ring $(A, \cdot)$ on A by taking the ring direct sum of $(A_p, \cdot)$ with the trivial ring (all products are zero) on $A^{(p)}$. This method of extending a ring from a summand of a group to the group will be called *extending by zero* and will be used frequently throughout this paper. Clearly $(A, \cdot)^2 \subseteq T(A)$ in this case. Since there do not exist mixed nil groups, see Szele [12], it seems natural to ask which groups A in $\mathscr{M}$ have the property that all rings $(A, \cdot)$ on A satisfy $(A, \cdot)^2 \subseteq T(A)$. We can partially characterise such groups.

If $a = (a_2, a_3, \cdots, a_p, \cdots)$ in A has infinite order, define $\operatorname{supp}(a) = \{\text{primes } p \mid a_p \neq 0\}$.

LEMMA 2.2. *Let $A \in \mathscr{M}$ and $a = (a_2, a_3, \cdots, a_p, \cdots)$ be an element of infinite order in A. If for almost all $p \in \operatorname{supp}(a)$, $\langle a_p \rangle$ is a direct summand of A_p, then there is an associative ring $(A, \cdot)$ on A such that $(A, \cdot)^2 \nsubseteq T(A)$.*

Proof. If $\langle a_p \rangle$ is a summand of A_p define an associative ring $(\langle a_p \rangle, \cdot)$ on $\langle a_p \rangle$ by letting $a_p \cdot a_p = a_p$, and extend this by zero to obtain an associative ring $(A_p, \cdot)$ on A_p. If q is a prime for which $\langle a_q \rangle$ is not a summand of A_q, define $(A_q, \cdot)$ to be the trivial ring on A_q.

Now take the ring direct product of the rings $(A_p, \cdot)$ to obtain an associative ring $(\prod_p A_p, \cdot)$ on $\prod_p A_p$. For almost all $p \in \operatorname{supp}(a)$, $a_p \cdot a_p = a_p$, so $a \cdot a - a \in T(A)$. Since A has torsion-free rank one, (2.1)(b) shows $(A, \cdot)$ is a subring of $(\prod_p A_p, \cdot)$ with the desired property. $\square$

If $A \in \mathscr{M}$ and $a = (a_2, a_3, \cdots, a_p, \cdots)$ is an element of A, then for each prime p the p-indicator of a in A, $U_p(a) = (h_p(a), h_p(p^2 a), \cdots)$, is the indicator of a_p in A_p. Hence if $U_p(a)$ commences with an ordinal (and not ∞), then $U_p(a)$ contains at least one gap, namely the jump from ordinal to ∞.

Now let a have infinite order in A. For $p \in \operatorname{supp}(a)$, we say $U_p(a)$ is *reasonable* (*of type* I) if $U_p(a) = (\infty, \infty, \cdots)$, and $U_p(a)$ is *reasonable* (*of type* II) if $U_p(a)$ commences with 0 and contains only one gap. The first type can occur if $A = T(A) \oplus Q$ and $a \in Q$; the second type can occur if $\langle a_p \rangle$ is a summand of A. The height matrix $\mathscr{H}(A)$ is a *reasonable matrix* if, for almost all $p \in \operatorname{supp}(a)$, $U_p(a)$ is reasonable. $\mathscr{H}(A)$ is *very reasonable* if, for almost all $p \in \operatorname{supp}(a)$, $U_p(a)$ is reasonable of the same type. Since A has torsion-free rank one, if b is another element in A, $\mathscr{H}(c)$ is (very) reasonable if and only if $\mathscr{H}(b)$ is (very) reasonable.

PROPOSITION 2.3. *Suppose $A \in \mathscr{M}$ and a is an element of infinite order in A. If there is a ring $(A, \cdot)$ on A such that $(A, \cdot)^2 \nsubseteq T(A)$, then $\mathscr{H}(a)$ is reasonable. Conversely, if $\mathscr{H}(a)$ is very reasonable, then there is an associative ring $(A, \cdot)$ on A for which $(A, \cdot)^2 \nsubseteq T(A)$.*

Proof. Suppose $\mathscr{H}(a)$ is not reasonable and consider any ring $(A, \cdot)$ on A. For infinitely many $p \in \operatorname{supp}(a)$ there exist integers $k(p)$ and ordinals $\alpha_{k(p)}$ such that $h_p(p^{k(p)} a) = \alpha_{k(p)}$, where $k(p) < \alpha_{k(p)} < \infty$. In particular $p^{k(p)} a \in p^{k(p)+1} A$, so there is an $a' \in A$ for which $p^{k(p)}(a \cdot a) = p(a' \cdot p^{k(p)} a)$. Now $h_p(p^{k(p)}(a \cdot a)) \geqq k(p) + 1$, so $\mathscr{H}(a \cdot a)$ is not equivalent to $\mathscr{H}(a)$. Since any two elements of infinite order have equivalent height matrices, $(A, \cdot)^2 \nsubseteq T(A)$.

Next suppose $\mathscr{H}(a)$ is very reasonable, and consider the two cases.

(i) For almost all $p \in \operatorname{supp}(a)$, $U_p(a) = (\infty, \infty, \cdots)$. There is a positive integer n for which na belongs to the divisible part of A, so $A = T(A) \oplus A'$ for some subgroup A' of A, $A' \cong Q$. By defining the field on A' and extending by zero, we obtain the desired ring.

(ii) For almost all $p \in \operatorname{supp}(a)$, $U_p(a)$ commences with zero and contains only one gap. Writing $a = (a_2, a_3, \cdots, a_p, \cdots)$ it is clear that for almost all $p \in \operatorname{supp}(a)$, $U_p(a) = (0, 1, \cdots, n_p - 1, \infty, \infty, \cdots)$ where $n_p =$ order of $a_p \geqq 1$. $\langle a_p \rangle$ is now a summand of A_p, so simply apply Lemma 2.2. □

Complete descriptions of the absolute annihilators and the absolute radicals of groups in $\mathscr{M}$ can be given.

THEOREM 2.4. *Let $A \in \mathscr{M}$. If A is reduced $A(^*) = A^1$; otherwise $A(^*) = (T(A))^1$.*

Proof. Consider A reduced and let $a \in A$ have finite height. There is an integer i for which a gap occurs between $h_p(p^i a)$ and $h_p(p^{i+1}a)$, where $h_p(p^i a) = k_i$ is finite. There is now an $a' \in A$ such that $p^{i+1}a = pa'$ and $h_p(a') \geqq k_i + 1$, so $p^i a - a' \neq 0$ is an element of order p and height k_i. Writing $p^i a - a' = p^{k_i} a''$ where $a'' \in A$, $\langle a'' \rangle$ is a summand of A. Define $a'' \cdot a'' = a''$ and extend by zero to obtain a ring $(A, \cdot)$ on A. Now $(p^i a - a') \cdot a'' = p^i a \cdot a''$, since $h_p(a') \geqq k_i + 1$ and a'' has order p^{k_i+1}. But $(p^i a - a') \cdot a'' = (p^{k_i} a'') \cdot a'' \neq 0$, so $a \notin A(^*)$. Thus $A(^*) \subseteqq A^1$.

Next let $a \in A^1$, and suppose $\phi \in \operatorname{Hom}(A, E(A))$ defines the ring $(A, \cdot)$. Since $\phi(a)|_{T(A)} = 0$, $\phi(a)$ factors through $A/T(A)$, i.e., $\phi(A)$ is a composite $A \to A/T(A) \to A$. But $A/T(A)$ is divisible and A is reduced, so $\phi(a) = 0$. Thus $A(^*) = A^1$. (Notice that the latter argument actually shows that $A/T(A)$ divisible implies $A^1 \subseteqq A(^*)$ for every reduced group A (not necessarily in $\mathscr{M}$).)

Consider now A nonreduced. It suffices to prove $A(^*) \subseteqq (T(A))^1$. If A contains a divisible torsion subgroup D, write $A = D \oplus A'$ for some subgroup A' of A. Embed A' in its divisible hull $D' \oplus Q$, where D' is torsion, and consider the element a of infinite order in A. Let the nonzero components of a in A' and Q be a_1 and a_2 respectively. As in Szele [12] define an associative ring $(D \oplus Q, \cdot)$ on $D \oplus Q$ such that $a_2 \cdot a_2 \neq 0$ and $(D \oplus Q, \cdot)^2 \subseteqq D$. Extending this ring by zero we obtain an associative ring on $D \oplus D' \oplus Q$ which contains $(A, \cdot)$ as a subring. This ring also satisfies $a \cdot a_1 = a_2 \cdot a_2 \neq 0$, so $A(^*) \subseteqq (T(A))^1$.

If A does not contain a divisible torsion subgroup, then A splits, $A = T(A) \oplus A'$ for some subgroup A' of A, and $A' \cong Q$. Now (1.1) and (1.2) show $A(^*) \subseteqq (T(A))(^*) \oplus A'(^*) = (T(A))^1$. □

COROLLARY 2.5. *If $A \in \mathscr{M}$ is reduced and $A^1 \neq 0$, then there* *does not exist an identity in any ring on A.*

Proof. $A(*) \neq 0$ implies any ring on A cannot have an identity. □

THEOREM 2.6. *Suppose $A \in \mathscr{M}$, and $a \in A$ is an element of infinite order. Then $J(A) = \bigcap_p pA$ when $\mathscr{H}(a)$ is not a reasonable matrix and, for almost all primes p, $U_p(a)$ does not commence with zero. Otherwise $J(A) = \bigcap_p p(T(A))$.*

Proof. For the prime p write $A = A_p \oplus A^{(p)}$, where $A^{(p)}$ is some p-divisible subgroup of A. Then $J(A) \subseteqq J(A_p) \oplus J(A^{(p)}) \subseteqq pA$.

Suppose $\mathscr{H}(a)$ is not reasonable and for almost all p, $U_p(a)$ does not commence with zero, and consider an associative ring $(A, \cdot)$ on A. Clearly there is an integer n for which $na \in \bigcap_p pA$. Proposition 2.3 yields $(A, \cdot)^2 \subseteqq T(A)$, so for every $b \in A$, $na \cdot b \in \bigcap_p p(T(A))$. $T(A)$ is an absolute ideal of A, so (1.1) and (1.3) show $\bigcap_p p(T(A)) = J(T(A)) \subseteqq J(A, \cdot)$. Now $na \cdot b$ is a (right) quasi-regular element of $(A, \cdot)$. Since $J(A, \cdot)$ can be characterised as the set of all $a' \in A$ for which $a' \cdot b'$ is quasi-regular for all $b' \in B$ (see for example McCoy [7], p. 132), $na \in J(A, \cdot)$; that is $A/J(A, \cdot)$ is torsion. Thus $\bigcap_p p(A/J(A, \cdot)) = J(A/J(A, \cdot)) = 0$, so $\bigcap_p pA \subseteqq J(A, \cdot)$. Since the associative ring $(A, \cdot)$ was chosen arbitrarily, $\bigcap_p pA \subseteqq J(A)$.

The other case occurs when, for infinitely many primes p, $U_p(a)$ commences with zero, or for almost all primes p, $U_p(a) = (\infty, \infty, \cdots)$. In the former case $J(A) \subseteqq \bigcap_p pA$ shows $J(A)$ must be torsion, so $J(A) \subseteqq (\bigcap_p pA) \cap T(A) = \bigcap_p pT(A)$. But $J(T(A)) \subseteqq J(A)$, hence $J(A) = J(T(A))$. In the latter case A splits, $A = T(A) \oplus A'$ for some subgroup A' of A, $A' \cong Q$. (1.2) now yields $J(A) \subseteqq J(T(A)) \oplus J(A') = J(T(A))$, so again $J(A) = \bigcap_p p(T(A))$. □

3. Cotorsion groups, algebraically compact groups. A similarity exists between these groups and groups in $\mathscr{M}$; namely, if A is a reduced cotorsion group then A may be written uniquely in the form $A = \prod_p A_{(p)}$, where for each prime p, $A_{(p)}$ is a reduced cotorsion group which is a p-adic module. Such a group A is algebraically compact if and only if $A^1 = 0$, in which case each $A_{(p)}$ is a reduced algebraically compact group that is also complete in its p-adic topology. It should be noted that although these groups resemble groups in $\mathscr{M}$, they are seldom members of $\mathscr{M}$.

THEOREM 3.1. *If A is a cotorsion group, then $A(*) \subseteqq A^1$. If A is an adjusted cotorsion group, then $A(*) = A^1$.*

Proof. If we write $A = D \oplus R$ where D is divisible and R is reduced, (1.2) shows $A(*) \subseteqq D(*) \oplus R(*)$. Since $D(*) \subseteqq D = D^1$ we can assume A is reduced. If we now write $A = \prod_p A_{(p)}$ and apply the same argument, noting $\prod_{q \neq p} A_{(q)}$ is p-divisible, it is clear that we can further restrict our attention to reduced cotorsion groups A that are also p-adic modules, for some prime p.

Let $a \in A$ have finite p-height n. If B is a p-basic submodule of A then $A = B + p^{n+1} A$, so let $a = b + p^{n+1} a'$ where $b \in B$, $b \neq 0$ and $a' \in A$. Choose a cyclic submodule (and summand) B' of B for which b has a nonzero component b' in B'. Since B' is a pure submodule of A that is algebraically compact, B' is a summand of A.

B' is either a cyclic p-group or a copy of the p-adic integers. In either case it is possible to define a ring $(B', \cdot)$ on B' for which $b' \cdot b' \neq 0$. Extending this by zero to a ring $(A, \cdot)$ on A we see that $a \cdot b' = b' \cdot b' \neq 0$. Thus $A(*) \subseteqq A^1$.

If A is adjusted cotorsion then A is reduced and $A/T(A)$ is divisible. As in the proof of Theorem 2.4, $A^1 \subseteqq A(*)$. □

COROLLARY 3.2. *If A is reduced algebraically compact group, then $A(*) = 0$.* □

COROLLARY 3.3. *If a reduced cotorsion group A is the additive group of a ring with identity, then A is algebraically compact.*

Proof. The induced ring $(\hat{A}, \cdot)$ on $\hat{A} = \operatorname{Ext}(Q/Z, A)$ also has an identity, so $\hat{A}(*) = 0$. Thus $A^1 = 0$; that is, A is algebraically compact. □

THEOREM 3.4. *If A is a cotorsion group $J(A) \subseteqq \bigcap_p pA$.*

Proof. Again we restrict our attention to reduced cotorsion groups A that are also p-adic modules, for some prime p. We need to prove $J(A) \subseteqq pA$.

Suppose $a \notin pA$, and again let B be a p-basic submodule of A. Then $A = B + pA$, and we can select a cyclic submodule B' of B which is a direct summand of A for which the component of a in B' is not in pB'.

Since $J(Z_p^*) = pZ_p^*$ (Z_p^* being the ring of p-adic integers), and since B' is either finite cyclic or the p-adic integers, we can define an associative ring $(B', \cdot)$ on B' such that $J(B', \cdot) = pB'$. Extending this ring by zero to an associative ring $(A, \cdot)$ on A, it is clear that that $a \notin J(A, \cdot)$. □

COROLLARY 3.5. *If A is a reduced algebraically compact group*

$J(A) = \bigcap_p pA = \prod_p pA_{(p)}$.

Proof. Write $A = \prod_p A_{(p)}$ where each $A_{(p)}$ is a p-adic module complete in its p-adic topology. Since each $A_{(p)}$ is reduced, $\prod_{q \neq p} A_{(q)}$ is the maximal p-divisible subgroup of A. As such it is an absolute ideal of A, so any associative ring $(A, \cdot)$ decomposes as $(A, \cdot) = (A_{(p)}, \cdot) \oplus (\prod_{q \neq p} A_{(q)}, \cdot)$ where the direct sum is a ring direct sum. Clearly now $(A, \cdot)$ is the ring direct product of the associative rings $(A_{(p)}, \cdot)$. Thus it suffices to prove $pA \subseteqq J(A)$ when A is a p-adic module complete in its p-adic topology, for some prime p.

From Fuchs [3], Vol. I, p. 166 we know $A \cong \varprojlim_k A/p^kA$. If $(A, \cdot)$ is any associative ring on A, then A/p^kA inherits an associative ring structure we denote $(A/p^kA, \cdot)$, and $p(A/p^kA) \subseteqq J(A/p^kA, \cdot)$, for each positive integer k. With Z^+ denoting the set of positive integers it is readily checked that

$$A_1 = \{p(A/p^kA) \mid k \in Z^+\}$$

and

$$A_2 = \{J(A/p^kA, \cdot) \mid k \in Z^+\} ,$$

together with the maps of the inverse system $\{A/p^kA \mid k \in Z^+\}$ form two inverse systems for which there is a monomorphism $\phi\colon A_1 \to A_2$. Hence

$$\varprojlim_k p(A/p^kA) \subseteqq \varprojlim_k J(A/p^kA, \cdot) .$$

Theorem 1 of Ion [5] yields

$$\varprojlim_k J(A/p^kA, \cdot) = J(\varprojlim_k (A/p^kA, \cdot)) ,$$

and a trivial calculation proves

$$p(\varprojlim_k A/p^kA) \subseteqq \varprojlim_k p(A/p^kA) ,$$

so

$$pA = p(\varprojlim_k A/p^kA) \subseteqq J(\varprojlim_k (A/p^kA, \cdot)) = J(A, \cdot)$$

Since this is true for every associative ring $(A, \cdot)$ on A, $pA \subseteqq J(A)$. □

Corollary 3.5 allows us to answer in the negative the following question raised by Rotman [10]. If $(A, \cdot)$ is a semi-simple ring on a reduced group A, then is the induced ring $(\mathrm{Ext}(Q/Z, A), \cdot)$ on $\mathrm{Ext}(Q/Z, A)$ also semisimple?

PROPOSITION 3.6. *Suppose* $(A, \cdot)$ *is a semisimple ring on the reduced group* A. *If* A *is torsion-free, then* $J(\mathrm{Ext}(Q/Z, A), \cdot) \neq 0$. *However, if* A *is torsion or* A *is a mixed group such that* $A/T(A)$ *is divisible, then* $J(\mathrm{Ext}(Q/Z, A), \cdot) = 0$.

Proof. If A is torsion-free, $\mathrm{Ext}(Q/Z, A)$ is a reduced algebraically compact group, so we can write

$$\mathrm{Ext}(Q/Z, A) = \prod_p (\mathrm{Ext}(Q/Z, A))_{(p)}$$

where each $(\mathrm{Ext}(Q/Z, A))_{(p)}$ is a reduced algebraically compact group complete in its p-adic topology. Corollary 3.5 yields

$$J(\mathrm{Ext}(Q/Z, A)) = \prod_p p(\mathrm{Ext}(Q/Z, A))_{(p)} .$$

Since $p(\mathrm{Ext}(Q/Z, A))_{(p)} \neq 0$ for at least one prime p, $J(\mathrm{Ext}(Q/Z, A), \cdot) \neq 0$.

If A is a torsion group or A is a mixed group such that $A/T(A)$ is divisible, $\mathrm{Ext}(Q/Z, A)$ can be written uniquely $\mathrm{Ext}(Q/Z, A) = \prod_p \mathrm{Ext}(Z(p^\infty), A)$. For each prime p, $\mathrm{Ext}(Z(p^\infty), A) \cong \mathrm{Ext}(Z(p^\infty), T(A)) \cong \mathrm{Ext}(Z(p^\infty), A_p)$. From (1.1) and (1.3), $pA_p \subseteq J(A_p, \cdot) \subseteq J(A, \cdot) = 0$, so $\mathrm{Ext}(Z(p^\infty), A)$ is a subgroup of the p-component $(\mathrm{Ext}(Q/Z, A))_p$ of $\mathrm{Ext}(Q/Z, A)$. Since $\prod_{q \neq p} \mathrm{Ext}(Z(q^\infty), A)$ is p-divisible and $\mathrm{Ext}(Q/Z, A)$ is reduced, $\mathrm{Ext}(Z(p^\infty), A) = (\mathrm{Ext}(Q/Z, A))_p$. Thus for all p, $((\mathrm{Ext}(Q/Z, A))_p, \cdot) \cong (A_p, \cdot)$. Since A_p is reduced, $(\mathrm{Ext}(Q/Z, A), \cdot)$ is the ring direct product of the semisimple rings $((\mathrm{Ext}(Q/Z, A))_p, \cdot)$. Therefore $J(\mathrm{Ext}(Q/Z, A), \cdot) = 0$. □

Counter examples to Rotman's question now follow from the above, and Theorem 3.2 of Beaumont and Lawver [1]. Indeed, any ring $(Z, \cdot)$ on the integers Z is semisimple, so $J(\mathrm{Ext}(Q/Z, Z), \cdot) \neq 0$.

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UNIVERSITY OF TASMANIA
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