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OPERATOR-VALUED PICK'S CONDITIONS AND HOLOMORPHICITY

JACOB BURBEA

The classical Pick's conditions on disks or half-planes are extended in several directions. Specifically, these conditions are shown to be valid in any domain (or a complex manifold) in C^n, for operator-valued functions in the domain from a Hilbert space into another and for any holomorphic reproducing kernel in the domain. An interesting related result of Hindmarsh is also extended.

1. Introduction. The main purpose of this paper is to extend, in a variety of ways, a body of theorems, classically known as the Pick's conditions and holomorphicity, detailed below. We shall state these conditions in terms of the right half-plane $\mathscr{R}$ and note that, in view of their conformal invariance, they may be stated in terms of any simply-connected domain which is properly contained in the plane. Let $K_{\mathscr{R}}(z, \bar{\zeta}) = (z + \bar{\zeta})^{-1}$ be the Szegö reproducing kernel of $\mathscr{R}$ and let S be a complex-valued function on $\mathscr{R}$. Define

$$\mathscr{L}_S(z, \zeta) \equiv (z + \bar{\zeta})^{-1}[S(z) + \overline{S(\zeta)}];\ z, \zeta \in \mathscr{R} .$$

Clearly, $\mathscr{L}_S(z, z) \geqq 0$, $z \in \mathscr{R}$, if and only if $S(\mathscr{R}) \subset \hat{\mathscr{R}}$, $\hat{\mathscr{R}}$ being the closure of $\mathscr{R}$. Moreover,

THEOREM A. *If $S(\mathscr{R}) \subset \hat{\mathscr{R}}$ and S is holomorphic in $\mathscr{R}$, then $\mathscr{L}_S(\cdot, \cdot)$ is positive definite on $\mathscr{R} \times \mathscr{R}$.*

THEOREM B. *If $\mathscr{L}_S(\cdot, \cdot)$ is positive definite of order 2 on $\mathscr{R} \times \mathscr{R}$, then $S(\mathscr{R}) \subset \hat{\mathscr{R}}$ and S is continuous on $\mathscr{R}$.*

THEOREM C. *If $\mathscr{L}_S(\cdot, \cdot)$ is positive definite in $\mathscr{R} \times \mathscr{R}$, then S is holomorphic in $\mathscr{R}$ and $S(\mathscr{R}) \subset \hat{\mathscr{R}}$.*

THEOREM D. *If $\mathscr{L}_S(\cdot, \cdot)$ is positive definite of order 3 on $\mathscr{R} \times \mathscr{R}$, then S is holomorphic in $\mathscr{R}$ and $S(\mathscr{R}) \subset \hat{\mathscr{R}}$.*

Theorem A is known as Pick's theorem [9] (see also [7, p. 34] and [8]). Theorem B is rather trivial in this setting and may be also formulated in terms of the distance-decreasing property of S with respect to the Poincaré metric of $\mathscr{R}$. Theorem C [9] is known as the converse of Pick's theorem. Theorem D is, of course, stronger than Theorem C; this remarkable fact was first observed by Hindmarsh [8] (see also [7, pp. 36–38]).

Sometimes it is more convenient to deal with the disk version of these theorems. This may be expressed with the aid of the Szegö reproducing kernel $K_\Delta(z, \bar{\zeta}) = (1 - \bar{\zeta}z)^{-1}$ of the unit disk Δ. The disk version of $\mathscr{L}_S(z, \bar{\zeta})$ is then

$$\mathscr{K}_T(z, \zeta) \equiv (1 - \bar{\zeta}z)^{-1}[1 - \overline{T(\zeta)}T(z)];\ z, \zeta \in \Delta ,$$

where T is a complex-valued function on Δ (see [1, pp. 3–4]).

In this paper we shall extend the above theorems in the following directions: Instead of $\mathscr{R}$ or Δ we take any domain (or a complex manifold) D in C^n. Instead of $K_{\mathscr{R}}(z, \bar{\zeta})$ or $K_\Delta(z, \bar{\zeta})$ we take any positive-definite (reproducing) kernel $K(z, \bar{\zeta})$ which is holomorphic in $(z, \bar{\zeta})$ for $(z, \zeta) \in D \times D$. Finally, instead of S or T we take an accretive or contractive, respectfully, operator-valued function in D from a Hilbert space into another. The proofs we use seem to be even simpler than the classical ones. The contractive version of Theorem A was proved by us in [5, 6]. A more special case of this version, where $D = \Delta$, $K(z, \bar{\zeta}) = K_\Delta(z, \bar{\zeta})$ and thus $\mathscr{K}_T(z, \zeta) = [I - T(\zeta)^* T(z)]/(1 - \bar{\zeta}z)$, with $T(\cdot)$ being a contractive operator-valued holomorphic function in Δ from a Hilbert space U into a Hilbert space W, $T(\cdot)^*$ is its adjoint and I is the identity operator of U, was first proved by Rovnyak [10] (see also [13, p. 231]).

As expected the transition from a contractive version to an accretive one, and visa versa, is not particularly difficult for, we have the Cayley transforms at our disposal. Evidently, this also shows that we may adopt other versions as the dissipative version and so on. We shall not pursue these points here.

Section 2 is devoted to preliminaries and notation, which will be used in this paper. In § 3 we state the contractive version of Theorem A, proved in [5], and, establish its accretive version (Theorems 1 and 1′). We also prove the contractive and accretive versions of Theorem B (Theorems 2 and 2′). The generalizations of Theorem C are proved in § 4 (Theorems 3 and 3′). In § 5 we establish some auxilary facts on smooth kernels. This is done by, essentially, following the analysis of Hindemarsh [8], but the present set up is slightly more general. In § 6 we give the generalizations of Theorem D (Theorems 4, 4′, 5 and 5′).

2. Preliminaries and notation. Throughout this paper we shall adhere to the following notation: D is a domain (or a complex manifold) in C^n and $C^m(D)$, $0 \leqq m \leqq \infty$, is the class of continuously m-differentiable functions (or forms) in D. The class of holomorphic functions (or forms) in D is denoted by $H(D)$. We write $\bar{D} = \{\bar{z} \in C^n: z \in D\}$ and thus $H(\bar{D})$ is the family of anti-holomorphic functions (or forms) in D. By $H(D \times \bar{D})$ we mean the family of functions (or forms) $F(z, \bar{\zeta})$ so that $F(\cdot, \bar{\zeta}) \in H(D)$ and $F(z, \cdot) \in H(\bar{D})$ for any $z, \zeta \in D$.

The sets U and W stand for any two Hilbert spaces over C with inner products $(\ ,\ )_U$ and $(\ ,\ )_W$, respectively. The Banach space of bounded linear operators from U into W is denoted by $\mathscr{B}(U: W)$. By a contraction from U to W we mean a $T \in \mathscr{B}(U: W)$ with $\| Tu \|_W \leqq \| u \|_U$ for every $u \in U$. The family of all such contractions is denoted by $\mathscr{C}(U: W)$. Evidently, if $T \in \mathscr{C}(U: W)$ then its adjoint T^* is in $\mathscr{C}(W: U)$ and, in fact, $\| T \| = \| T^* \| \leqq 1$. We denote by $\mathscr{C}_1(U: U)$ the family of all $T \in \mathscr{C}(U: U)$ with T not having the eigenvalue 1. One shows that $T \in \mathscr{C}_1(U: U)$ if and only if $T^* \in \mathscr{C}_1(U: U)$. An operator $S \in \mathscr{B}(U: U)$ is said to be *accretive* if $\mathrm{Re}\,(Su, u)_U \geqq 0$ for every $u \in U$. The family of all accretive operators in $\mathscr{B}(U: U)$ is denoted by $\mathscr{A}(U: U)$. Clearly, $S \in \mathscr{A}(U: U)$ if and only if $S^* \in \mathscr{A}(U: U)$.

A function $A(z)$, $z \in D$, with values in the space $\mathscr{B}(U: W)$ will be called an *operator-valued function* in D, or in short $A(\cdot) \in \mathscr{B}(U: W)[D]$. In a similar fashion one introduces the classes $\mathscr{C}(U: W)[D]$, $\mathscr{C}_1(U: U)[D]$ and $\mathscr{A}(U: U)[D]$. The concepts of continuity, differentiability and holomorphicity extend to operator-valued functions. Thus, $\mathscr{B}(U: W)[C^m_w(D)]$, $\mathscr{B}(U: W)[C^m_s(D)]$ and $\mathscr{B}(U: W)[C^m_n(D)]$ denote the classes of weakly, strongly and normly, respectively, continuously m-differentiable operator-valued functions in $\mathscr{B}(U: W)[D]$. The corresponding classes where $\mathscr{B}(U: W)$ is replaced by $\mathscr{C}(U: W)$, $\mathscr{C}_1(U: U)$, and $\mathscr{A}(U: U)$ are defined in an analogous way. In the case of operator-valued holomorphic (or anti-holomorphic) functions the weakly, strongly and normly notions of holomorphicity coincide. Thus, $A(\cdot) \in \mathscr{B}(U: W)[H(D)]$, if for every $(u, w) \in U \times W$, $(A(\cdot)u, w)_W$ belongs to $H(D)$. When $U = W$, this definition of holomorphicity reduces in only requiring that $(A(\cdot)u, u)_U$ belongs to $H(D)$ for every $u \in U$. Evidently, $A(\cdot) \in \mathscr{B}(U: W)[H(D)]$ if and only if $A(\cdot)^* \in \mathscr{B}(W: U)[H(\bar{D})]$. The families $\mathscr{C}(U, W)[H(D)]$, $\mathscr{A}(U: U)[H(D)]$ and so on are defined in a similar way.

By an *operator-valued kernel*, or in short a *kernel*, $\mathscr{K} = \mathscr{K}(\cdot, \cdot)$ we mean any function $\mathscr{K}(\cdot, \cdot) \in \mathscr{B}(U: U)(D \times D)$. The kernel is said to be *hermitian* if $\mathscr{K}(z, \zeta)^* = \mathscr{K}(\zeta, z)$ for all $z, \zeta \in D$. The notions of continuity, differentiability and holomorphicity extend to operator valued kernels. For example, if $\mathscr{K}$ is hermitian and for each $\zeta \in D$, $\mathscr{K}(\cdot, \zeta) \in \mathscr{B}(U: U)[H(D)]$ then $\mathscr{K}(\cdot, \cdot) \in \mathscr{B}(U: U)[H(D \times \bar{D})]$. We may emphasize the last fact in writing $\mathscr{K}(z, \bar{\zeta})$ instead of $\mathscr{K}(z, \zeta)$, $z, \zeta \in D$. An hermitian $\mathscr{K} \in \mathscr{B}(U: U)(D \times D)$ is said to be *positive-definite of order N*, in short p.d. (N), if

$$\sum_{m,n} (\mathscr{K}(z_m, z_n)u_m, u_n)_U \geqq 0$$

for every finite system $\{z_m\}_{m=1}^M$ of points of D and every corresponding vectors $\{u_m\}_{m=1}^M$ of U, where $M = 1, \cdots, N$. The kernel is said to be *positive-definite* (p.d.), in short $\mathscr{K} \gg 0$, if it is p.d. (N) for all $N = 1, 2, \cdots$.

Let $K(z, \bar{\zeta})$ be a positive-definite (scalar) kernel so that $K(\cdot, \cdot) \in H(D \times \bar{D})$. As is well-known [2], the kernel $K(z, \bar{\zeta})$ determines a uniquely defined Hilbert space $\mathscr{H}(D)$ of elements in $H(D)$ with an inner product $(\ ,\ )$ and for which $k_\zeta(\cdot) \equiv K(\cdot, \bar{\zeta})$ is its reproducing kernel. Thus, for any $z, \zeta \in D$

$$f(\zeta) = (f, k_\zeta), \ f \in \mathscr{H}(D)$$

and

$$K(z, \bar{\zeta}) = k_\zeta(z) = (k_\zeta, k_z) = \overline{K(\zeta, \bar{z})}; \ K(z, \bar{z}) = (k_z, k_z) \geqq 0 .$$

The reproducing kernel $K(z, \bar{\zeta})$ is said to be of class $\mathscr{N}$, if $K(z, \bar{z}) > 0$ for every $z \in D$. Clearly, $K(z, \bar{\zeta})$ is of class $\mathscr{N}$ if and only if for each $z \in D$, there exists an $f \in \mathscr{H}(D)$ with $f(z) \neq 0$.

3. The Pick kernels. Let $K(z, \bar{\zeta})$ be the reproducing kernel of $\mathscr{H}(D)$, $T(\cdot) \in \mathscr{B}(U: W)[D]$ and $S(\cdot) \in \mathscr{B}(U: U)[D]$. For $z, \zeta \in D$, we define the operator-valued *Pick kernels*

$$\mathscr{K}_T(z, \zeta) = K(z, \bar{\zeta})[I - T(\zeta)^* T(z)] \tag{3.1}$$

and

$$\mathscr{L}_S(z, \zeta) = K(z, \bar{\zeta})[S(z) + S(\zeta)^*] , \tag{3.2}$$

where $I = I_U$ stands for the identity operator of U. These kernels belong to $\mathscr{B}(U: U)(D \times D)$ and they are hermitian.

In many instances the space $\mathscr{H}(D)$ may be realized as the space of *all* $f \in H(D)$ so that

$$\|f\|^2 = \int_{D_0} |f(z)|^2 d\mu(z) < \infty, \ \|f\| = \sqrt{(f, f)} .$$

Here μ is positive measure acting on D_0, where D_0 is either D or any part of the boundary ∂D which determines the holomorphic functions in D as, for example, the Šilov boundary of D. In the case that D_0 is not D, f in the last integral stands for the non-tangential boundary values of the holomorphic function $f(z)$, $z \in D$.

In this way we may regard $\mathscr{H}(D) \equiv H_2(D: \mu)$ as a closed subspace of $L_2(D_0: \mu)$ in a natural manner. The corresponding reproducing kernel $K(z, \bar{\zeta})$ of such a space $\mathscr{H}(D) = H_2(D: \mu)$ will be called a *μ-measure reproducing kernel.* The class of all such μ-measure reproducing kernels and their limits, via a cannonical exhaustion of the domain D, is denoted by $\mathscr{M}$. This class includes the familiar *weighted Bergman* and *Szegö kernels*, the *Rudin kernels* and the so called "*generalized Szegö kernels*" (see [3, 4, 11, 12] for details).

The following generalization of Pick's theorem is proved in [5] (see also [6, 10]):

THEOREM 1. *Let $K(z, \bar{\zeta})$ be a reproducing kernel of class $\mathscr{M}$ in the domain D and assume that $T(\cdot) \in \mathscr{C}(U: W)[H(D)]$. Then $\mathscr{K}_T(\cdot, \cdot) \in \mathscr{B}(U: U)[H(D \times \bar{D})]$ and $\mathscr{K}_T \gg 0$.*

A similar statement holds for the kernel $\mathscr{L}_S$ when $S(\cdot) \in \mathscr{A}(U: U)[H(D)]$. This will be done by relating the accretive and contractive operators via the *Cayley transforms.* More specifically, let $\Delta = \{z \in \boldsymbol{C}: |z| < 1\}$ be the unit disk and $\mathscr{R} = \{z \in \boldsymbol{C};\ \mathrm{Re} z > 0\}$ be the right half-plane. We write

$$g(z) = (1 + z)(1 - z)^{-1};\ h(z) = (z - 1)(z + 1)^{-1} \tag{3.3}$$

where, of course, g is a univalent holomorphic function of Δ onto $\mathscr{R}$ with h as its inverse. With these pair of functions one is able to establish the following relationship between the families $\mathscr{C}_1(U: U)$ and $\mathscr{A}(U: U)$ (see, for example, [13, p. 168]):

PROPOSITION 1. *The Cayley transform relations*

$$S = g(T) = (I + T)(I - T)^{-1};\ T = h(S) = (S - I)(S + I)^{-1}$$

establish a bijection between the operators T in $\mathscr{C}_1(U: U)$ and the operators S in $\mathscr{A}(U: U)$. Moreover, this bijection preserves the adjoint operation.

As a result of this we obtain:

COROLLARY 1. *The Cayley relations $g(\cdot) = g[T(\cdot)]$; $T(\cdot) = h[S(\cdot)]$, where $S(\cdot) \in \mathscr{A}(U: U)[D]$ and $T(\cdot) \in \mathscr{C}_1(U: U)[D]$, establish a bijection between the corresponding kernels $\mathscr{L}_S$ and $\mathscr{K}_T$ by the formulae:*

$$\mathscr{L}_S(z, \zeta) = 2[I - T(\zeta)^*]^{-1}\mathscr{K}_T(z, \zeta)[I - T(z)]^{-1}$$

and

$$\mathscr{K}_T(z, \zeta) = 2[S(\zeta)^* + I]^{-1}\mathscr{L}_S(z, \zeta)[S(z) + I]^{-1}$$

for any $z, \zeta \in D$.

This corollary, coupled with Theorem 1, leads to the accretive version of Pick's theorem:

THEOREM 1'. *Let* $K(z, \bar{\zeta})$ *be a reproducing kernel of class* $\mathscr{M}$ *in the domain* D *and assume that* $S(\cdot) \in \mathscr{A}(U: U)[H(D)]$. *Then* $\mathscr{L}_S(\cdot, \cdot) \in \mathscr{B}(U: U)[H(D \times \bar{D})]$ *and* $\mathscr{L}_S \gg 0$.

In order to deal with the converse of these two theorems, we let u be any unit vector of U ond consider the scalar kernels

$$k_T(z, \zeta: u) = (\mathscr{K}_T(z, \zeta)u, u)_U$$

and

$$\ell_S(z, \zeta: u) = (\mathscr{L}_S(z, \zeta)u, u)_U ,$$

where $K(z, \bar{\zeta})$, $z, \zeta \in D$, is any holomorphic reproducing kernel and $\mathscr{K}_T$, $\mathscr{L}_S$ are as in (3.1)–(3.2). Recall that $K(z, \bar{\zeta})$ is of class $\mathscr{N}$ if $K(z, \bar{z}) > 0$ for every $z \in D$. The following proposition is trivial:

PROPOSITION 2. *Let* $K(z, \bar{\zeta})$ *be of class* $\mathscr{N}$. *If for any unit vector* $u \in U$, $k_T(z, z: u) \geqq 0$ *for every* $z \in D$, *then* $T(\cdot) \in \mathscr{C}(U: W)[D]$. *Similarly, if for any unit vector* $u \in U$, $\ell_S(z, z: u) \geqq 0$ *for every* $z \in D$, *then* $S(\cdot) \in \mathscr{A}(U: U)[D]$.

We also have:

THEOREM 2. *Let* $K(z, \bar{\zeta})$ *be of class* $\mathscr{N}$ *such that for any unit vector* $u \in U$, $k_T(\cdot, \cdot: u)$ *is* p.d. (2) *on* $D \times D$. *Then*

$$\begin{aligned} &\|(T(z) - T(\zeta))u\|_W^2 - \|T(z)u\|_W^2 \|T(\zeta)u\|_W^2 + |(T(z)u, T(\zeta)u)_W|^2 \\ &\leqq \left\{1 - \frac{|K(z, \bar{\zeta})|^2}{K(z, \bar{z})K(\zeta, \bar{\zeta})}\right\} |1 - (T(z)u, T(\zeta)u)_W|^2 \end{aligned} \tag{3.4}$$

for any $z, \zeta \in D$ *and any unit vector* $u \in U$. *Moreover,* $T(\cdot) \in \mathscr{C}(U: W)[C_s^0(D)]$, i.e., $T(\cdot)$ *is a contraction from* U *to* W *which is also strongly continuous on* D.

Proof. Proposition 2 shows that $T(\cdot) \in \mathscr{C}(U: W)[D]$. Since $k_T(\cdot, \cdot: u)$ is p.d. (2) we have

$$|(\mathscr{K}_T(z, \zeta)u, u)_U|^2 \leqq (\mathscr{K}_T(z, z)u, u)_U(\mathscr{K}_T(\zeta, \zeta)u, u)_U$$

for $z, \zeta \in D$, and, a unit vector $u \in U$. Hence

$$\frac{|K(z,\bar\zeta)|^2}{K(z,\bar z)K(\zeta,\bar\zeta)}|1-(T(z)u,\,T(\zeta)u)_W|^2 \leqq [1-\|T(z)u\|_W^2]\cdot[1-\|T(\zeta)u\|_W^2]$$

and the inequality of the theorem follows. To prove the continuity assertion, we argue as follows: Let $\zeta\in U$ be fixed. We write $w(\zeta)=T(\zeta)u$ and $w(z)=T(z)u$, $z\in D$, and observe that $\|w(\zeta)\|_W$, $\|w(z)\|_W\leqq 1$. We have

$$\|w(z)-w(\zeta)\|_W^2-\|w(z)\|_W^2\|w(\zeta)\|_W^2+|(w(z),\,w(\zeta))_W|^2\leqq 4\alpha(z,\zeta)$$

where

$$\alpha(z,\zeta)=1-\frac{|K(z,\bar\zeta)|^2}{K(z,\bar z)K(\zeta,\bar\zeta)}$$

and thus $\lim_{z\to\zeta}\alpha(z,\zeta)=0$. We must show that $\lim_{z\to\zeta}\|w(z)-w(\zeta)\|_W=0$. We may assume that $w(\zeta)\neq 0$ for otherwise the result is immediate. In this case the left-hand side of (3.4) is precisely

$$\frac{1}{\|w(\zeta)\|_W^2}|(w(z)-w(\zeta),\,w(\zeta))_W|^2+\frac{1-\|w(\zeta)\|_W^2}{\|w(\zeta)\|_W^2}\{\|w(z)\|_W^2\|w(\zeta)\|_W^2 - |(w(z),\,w(\zeta))_W|^2\}\,.$$

Therefore, in view of (3.3) and the Cauchy-Schwarz inequality,

$$|(w(z)-w(\zeta),\,w(\zeta))_W|^2\leqq 4\alpha(z,\zeta) \tag{3.5}$$

and

$$[1-\|w(\zeta)\|_W^2]\{\|w(z)\|_W^2\|w(\zeta)\|_W^2-|(w(z),\,w(\zeta))_W|^2\}\leqq 4\alpha(z,\zeta)\,. \tag{3.6}$$

We distinguish two cases: (i) $\|w(\zeta)\|_W=1$ and (ii) $0<\|w(\zeta)\|_W<1$. In case (i), by (3.5), we have $\lim_{z\to\zeta}(w(z),\,w(\zeta))_W=1$. But,

$$|(w(z),\,w(\zeta))_W|^2\leqq\|w(z)\|_W^2\leqq 1$$

and, therefore, $\lim_{z\to\zeta}\|w(z)\|_W^2=1$. It follows that $\lim_{z\to\zeta}\|w(z)-w(\zeta)\|_W^2=0$. In the case of (ii), we have, by (3.5)–(3.6), that

$$\lim_{z\to\zeta}|(w(z)-w(\zeta),\,w(\zeta))_W|^2=0$$

and

$$\lim_{z\to\zeta}\{\|w(z)\|_W^2\|w(\zeta)\|_W^2-|(w(z),\,w(\zeta))_W|^2\}=0\,.$$

Since

$$\|w(z)-w(\zeta)\|_W^2=\frac{1}{\|w(\zeta)\|_W^2}\{|(w(z)-w(\zeta),\,w(\zeta))_W|^2 + [\|w(z)\|_W^2\|w(\zeta)\|_W^2-|(w(z),\,w(\zeta))_W|^2]\}\,,$$

we deduce that $\lim_{z\to\zeta}\|w(z)-w(\zeta)\|_W^2=0$. This concludes the proof.

The accretive version of this theorem is:

THEOREM 2′. *Let* $K(z,\bar\zeta)$ *be of class* $\mathscr{N}$ *such that for any unit vector* $u\in U$, $\mathscr{L}_S(\cdot,\cdot:u)$ *is* p.d. (2) *on* $D\times D$. *Then, for any* $z,\zeta\in D$ *and any unit vector* $u\in U$,

$$\frac{[s(z)+\overline{s(z)}]\cdot[s(\zeta)+\overline{s(\zeta)}]}{|s(z)+\overline{s(\zeta)}|^2}\geqq\frac{|K(z,\bar\zeta)|^2}{K(z,\bar z)K(\zeta,\bar\zeta)}\,;\quad s(z)\equiv(S(z)u,u)_U\,.$$

Moreover, $S(\cdot)\in\mathscr{A}(U\colon U)[C_w^0(D)]$, i.e., $S(\cdot)$ *is accretive in* U *and is weakly continuous on* D.

Proof. The distortion inequality is straightforward and Proposition 2 shows that $S(\cdot)$ is accretive. The weak continuity follows from the above inequality. In fact, since $s(z)\in\hat{\mathscr{R}}$ for every $z\in D$, writing $t(z)=h[s(z)]$, where h is given in (3.3), we obtain that $t(z)\in\hat\Delta$ and

$$\frac{[1-|t(z)|^2]\cdot[1-|t(\zeta)|^2]}{|1-\overline{t(\zeta)}t(z)|^2}\geqq\frac{|K(z,\bar\zeta)|^2}{K(z,\bar z)K(\zeta,\bar\zeta)}\,.$$

Therefore,

$$|t(z)-t(\zeta)|^2\leqq\left\{1-\frac{|K(z,\bar\zeta)|^2}{K(z,\bar z)K(\zeta,\zeta)}\right\}|1-\overline{t(\zeta)}t(z)|^2$$

and thus $\lim_{z\to\zeta}t(z)=t(\zeta)$. Consequently, $\lim_{z\to\zeta}(S(z)u,u)_U=(S(\zeta)u,u)_U$ and the proof is complete.

The following example (see also [7, p. 36]) shows that in Theorems 2 and 2′, one cannot expect that $T(\cdot)$ or $S(\cdot)$ to be holomorphic:

EXAMPLE. Let $D=\Delta$ be the unit disk, $U=W=\boldsymbol{C}$ and let $K(z,\bar\zeta)=(1-z\bar\zeta)^{-1}$ be the Szegö kernel of Δ. We choose $T(z)=|z|$ and observe that

$$\frac{|K(z,\bar\zeta)^2}{K(z,\bar z)K(\zeta,\bar\zeta)}=\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-z\bar\zeta|^2}\leqq\frac{(1-|z|^2)(1-|\zeta|^2)}{(1-|\zeta||z|)^2}$$
$$=\frac{[1-|T(z)|^2]\cdot[1-|T(\zeta)|^2]}{|1-\overline{T(\zeta)}T(z)|^2}\,.$$

This shows that $\mathscr{K}_T$ is p.d. (2) on $\Delta\times\Delta$ but $T(\cdot)$ is not holomorphic in Δ.

4. The converse of Pick's theorem. We now prove the following generalized converse of Pick's theorem. The present proof of this theorem (which may be regarded as a converse of Theorem 1) is even simpler than the classical proof for the less general case embodied in Theorem C of the introduction.

THEOREM 3. *Let $K(z, \bar{\zeta})$ be of class $\mathscr{N}$ such that for any unit vector $u \in U$, $k_T(\cdot, \cdot : u)$ is* p.d. *on $D \times D$. Then, for any $\zeta \in D$, $T(\zeta)^* T(\cdot) \in \mathscr{C}(U: U)[H(D)]$. In particular, if for some $\zeta_0 \in D$, $T(\zeta_0)^*$ is injective, then $T(\cdot) \in \mathscr{C}(U: W)[H(D)]$.*

Proof. Let $u \in U$ be a unit vector and consider the scalar kernel

$$r_T(z, \zeta : u) = (T(\zeta)^* T(z)u, u)_U : \quad a, \zeta \in D .$$

This kernel is clearly positive definite. It therefore follows that the kernel $K(z, \bar{\zeta}) r_T(z, \zeta : u)$ is positive definite on $D \times D$ as a product of two positive kernels (cf. [2, p. 36]) or [7, p. 93]). Now,

$$k_T(z, \zeta : u) \equiv K(z, \bar{\zeta}) - K(z, \bar{\zeta}) r_T(z, \zeta : u)$$

is by assumption positive definite on $D \times D$ and it is a difference of two positive definite (or reproducing) kernels. It follows, by a theorem of Aronszajn [2, p. 354], that the reproducing kernel space of $K(z, \bar{\zeta}) r_T(z, \zeta : u)$ is contained in that of $K(z, \bar{\zeta})$. But the reproducing kernel space of $K(z, \bar{\zeta})$ is the space $\mathscr{H}(D)$ which contains $H(D)$. In particular, for any fixed $\zeta \in D$, $K(\cdot, \bar{\zeta}) r_T(\cdot, \zeta : u) \in H(D)$. Consequently, $r_T(\cdot, \zeta : u) = (T(\zeta)^* T(\cdot)u, u)_U$ is meromorphic in D. However, by Theorem 2, $T(\cdot) \in \mathscr{C}(U: W)[C_s^0(D)]$. Therefore, $(T(\zeta)^* T(\cdot)u, u)_U$ is in fact holomorphic in D. Since $u \in U$ is an arbitrary unit vector we deduce that $T(\zeta) * T(\cdot) \in \mathscr{C}(U: U)[H(D)]$ for any $\zeta \in D$. Assume that for some $\zeta_0 \in D$, $T(\zeta_0)^*$ is injective. We have that $(T(\cdot)u, T(\zeta_0)u)_U$ is holomorphic in D for any $u \in U$. The injectivity of $T(\zeta_0)^*$ implies that the range of $T(\zeta_0)$ is dense in W. Consequently, $T(\cdot) \in \mathscr{C}(U: W)[H(D)]$ and the proof is complete.

REMARK. When the Hilbert space W is the scalar space $\boldsymbol{C}$, the condition of the theorem that $T(\zeta_0)^*$ is injective for some $\zeta_0 \in D$ means that $T(z)^*$ is not identically zero for $z \in D$. Here, for any $z \in D$, $|T(z)u| \leqq \|u\|_U$ for every $u \in U$ and $\|T(z)^*\|_U \leqq 1$.

The accretive version of this theorem is:

THEOREM 3'. *Let $K(z, \bar{\zeta})$ be of class $\mathscr{N}$ such that for any unit vector $u \in U$, $\ell_S(\cdot, \cdot : u)$ is* p.d. *on $D \times D$. Then $S(\cdot) \in \mathscr{A}(U: U)[H(D)]$.*

Proof. Let $u \in U$ be a fixed unit vecter and write

$$\ell_S(z, \zeta : u) = K(z, \bar{\zeta})[s(z) + \overline{s(\zeta)}]; \quad s(z) \equiv (S(z)u, u)_U$$

for $z, \zeta \in D$. As before, we let $t(z) = h[s(z)]$ with $t(z) \in \hat{\mathcal{I}}$ and $s(z) \in \hat{\mathscr{R}}$. Then

$$K(z, \bar{\zeta})[1 - \overline{t(\zeta)}t(z)] = \frac{1}{2}[1 - t(z)]\cdot[1 - \overline{t(\zeta)}]\ell_S(z, \zeta: u) .$$

It follows that $K(z, \bar{\zeta})[1 - \overline{t(\zeta)}t(z)]$ is p.d. on $D \times D$. As in Theorem 3, this implies that, for any fixed $\zeta \in D$, $\overline{t(\zeta)}t(\cdot) \in H(D)$. Let U_1 be the set of all $u \in U$ for which $(S(z)u, u)_U \equiv 1$ for all $z \in D$, and let $U_2 = U - U_1$. Evidently, $(S(\cdot)u, u)_U \in H(D)$ for all $u \in U_2$. Consequently, $S(\cdot) \in \mathscr{A}(U\colon U)[H(D)]$. This concludes the proof.

5. **Smooth kernels.** This section is devoted to some auxilary facts on smooth kernels which are of some interest in their own right and will be needed in this work. The present analysis is essentially similar to that of Hindmarsh [8] but it is slightly more general (see also [7, pp. 35–38]).

Let $K(x, \xi)$ be a complex-valued C^2-kernel defined for $x, \xi \in D$ where D is an open set in $\boldsymbol{R}^n$. For $u \in \boldsymbol{R}^n$, $u = (u^1, \cdots, u^n)$, we write

$$D_u = \sum_{k=1}^{n} u^k \partial_{x^k} , \quad x = (x^1, \cdots, x^n) \in \boldsymbol{R}^n .$$

For $v \in \boldsymbol{R}^n$ we write

$$D_v = \sum_{k=1}^{n} v^k \partial_{\xi^k} , \quad \xi = (\xi^1, \cdots, \xi^n) \in \boldsymbol{R}^n .$$

For a fixed point $(x, \xi) \in D \times D$, $u_1, \cdots, u_m \in \boldsymbol{R}^n$, $v_1, \cdots, v_m \in \boldsymbol{R}^n$ and for a small $\varepsilon > 0$ we form the $(m + 1) \times (m + 1)$ matrix $k = (k_{ij})$ defined by

$$k_{00} = K(x, \xi), \quad k_{ij} = K(x + \varepsilon u_i, \xi + \varepsilon v_j), \quad i, j = 1, \cdots, m .$$

We have

$$k_{ij} = k_{00} + \varepsilon(D_{u_i} + D_{v_j})K + \frac{\varepsilon^2}{2}(D_{u_i} + D_{v_j})^2 K + o(\varepsilon^2) ,$$

$$k_{0j} = k_{00} + \varepsilon D_{v_j}K + \frac{\varepsilon^2}{2}D_{v_j}D_{v_j}K + o(\varepsilon^2) ,$$

$$k_{i0} = k_{00} + \varepsilon D_{u_i}K + \frac{\varepsilon^2}{2}D_{u_i}D_{u_i}K + o(\varepsilon^2) ,$$

where $i, j \geqq 1$ and $K = K(x, \xi)$. We now form the matrix $\tilde{k}(\varepsilon) = (\tilde{k}_{ij})$ given by

$$\begin{aligned}
\tilde{k}_{00} &= k_{00} = K \\
\tilde{k}_{0j} &= \frac{1}{\varepsilon}(k_{0j} - k_{00}) = D_{v_j}K + o(1)\ , \\
\tilde{k}_{i0} &= \frac{1}{\varepsilon}(k_{i0} - k_{00}) = D_{u_i}K + o(1)\ , \\
\tilde{k}_{ij} &= \frac{1}{\varepsilon^2}(k_{ij} + k_{00} - k_{0j} - k_{i0}) = D_{u_i}D_{v_j}K + o(1)\ .
\end{aligned}$$

Let

$$E_m = \left[\begin{array}{c|c} 1 & -\varepsilon^{-1} \cdots -\varepsilon^{-1} \\ \hline 0 & \\ \vdots & \varepsilon^{-1}I_m \\ 0 & \end{array}\right]$$

be an $(m+1) \times (m+1)$ matrix where I_m is the identity matrix of order m. Then

$$\tilde{k}(\varepsilon) = E_m^* k E_m = \begin{bmatrix} K & D_{v_1}K & \cdots & D_{v_m}K \\ D_{u_1}K & D_{u_1}D_{v_1}K & \cdots & D_{u_1}D_{v_m}K \\ \vdots & & & \\ D_{u_m}K & D_{u_m}D_{v_1}K & \cdots & D_{u_m}D_{v_m}K \end{bmatrix} + o(1)\ , \tag{5.1}$$

In the case that the open set D is in $C^n \equiv R^{2n}$ and $K = K(z, \zeta)$ is a C^2-kernel in $D \times D$, we shall use the following notation: The points z and ζ will be written as $z = x + iy$ and $\zeta = \xi + i\eta$ with $x, y, \xi, \eta \in R^n$. We shall use vectors in R^{2n} of the form:

$$\begin{aligned}
u &= (u^1, \cdots, u^n;\ 0, \cdots, 0),\ v = (v^1, \cdots, v^n;\ 0, \cdots, 0)\ , \\
\mu &= (0, \cdots, 0;\ u^1, \cdots, u^n),\ \nu = (0, \cdots, 0;\ v^1, \cdots, v^n)\ .
\end{aligned}$$

We write

$$\begin{aligned}
D_u &= \sum_{k=1}^n u^k \partial_{x^k}, \quad D_v = \sum_{k=1}^n v^k \partial_{\xi^k}\ , \\
D_\mu &= \sum_{k=1}^n u^k \partial_{y^k}, \quad D_\nu = \sum_{k=1}^n v^k \partial_{\eta^k}\ .
\end{aligned}$$

Corresponding to (5.1) we now have the $(2m+1) \times (2m+1)$ matrix

$$
(5.2)\qquad \widetilde{k}(\varepsilon)=E_{2m}^{*}kE_{2m}=\begin{bmatrix} K & D_{v_1}K & \cdots & D_{v_m}K & D_{\nu_m}K \\ D_{u_1}K & D_{u_1}D_{v_1}K & \cdots & D_{u_1}D_{v_m}K & D_{u_1}D_{\nu_m}K \\ D_{\mu_1}K & D_{\mu_1}D_{v_1}K & \cdots & D_{\mu_1}D_{v_m}K & D_{\mu_1}D_{\nu_m}K \\ \vdots & & & & \\ D_{u_m}K & D_{u_m}D_{v_1}K & \cdots & D_{u_m}D_{v_m}K & D_{u_m}D_{\nu_m}K \\ D_{\mu_m}K & D_{\mu_m}D_{v_1}K & \cdots & D_{\mu_m}D_{v_m}K & D_{\mu_m}D_{v_m}K \end{bmatrix}+o(1)\ .
$$

We consider the $(2m+1)\times(2m+1)$ matrix

$$
B_{2m}=\left[\begin{array}{c|c} 1 & 0\cdots 0 \\ \hline 0 & \\ \vdots & C_{2m} \\ 0 & \end{array}\right]
$$

where

$$
C_{2m}=\frac{1}{2}\begin{bmatrix} J_2 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & J_2 \end{bmatrix}
$$

with

$$
J_2=\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}.
$$

Then

$$
B_{2m}\widetilde{k}(\varepsilon)B_{2m}^{*}=\begin{bmatrix} K & \bar{\partial}_{v_1}K & \cdots & \bar{\partial}_{v_m}K & \partial_{v_m}K \\ \partial_{u_1}K & \partial_{u_1}\bar{\partial}_{v_1}K & \cdots & \partial_{v_1}\bar{\partial}_{v_m}K & \partial_{u_1}\partial_{v_m}K \\ \bar{\partial}_{u_1}K & \bar{\partial}_{u_1}\bar{\partial}_{v_1}K & \cdots & \bar{\partial}_{u_1}\bar{\partial}_{v_m}K & \bar{\partial}_{u_1}\partial_{v_m}K \\ \vdots & & & & \\ \partial_{u_m}K & \partial_{u_m}\bar{\partial}_{v_1}K & \cdots & \partial_{u_m}\bar{\partial}_{v_m}K & \partial_{u_m}\partial_{v_m}K \\ \bar{\partial}_{u_m}K & \bar{\partial}_{u_m}\bar{\partial}_{v_1}K & \cdots & \bar{\partial}_{u_m}\bar{\partial}_{v_m}K & \bar{\partial}_{u_m}\partial_{v_m}K \end{bmatrix}+o(1)
$$

where

$$
\partial_u=2^{-1}(D_u-iD_\mu)=\sum_{k=1}^{n}u^k\partial_{z^k}\ ,
$$

$$
\bar{\partial}_u=2^{-1}(D_u+iD_\mu)=\sum_{k=1}^{n}u^k\bar{\partial}_{z^k}
$$

$$
\partial_v=2^{-1}(D_v-iD_\nu)=\sum_{k=1}^{n}v^k\partial_{\zeta^k}\ ,
$$

$$
\bar{\partial}_v=2^{-1}(D_v+iD_\nu)=\sum_{k=1}^{n}v^k\bar{\partial}_{\zeta^k}\ .
$$

Finally, we shall be needing the following result:

LEMMA 1. *Let D be an open set in C^n and let $K(z, \zeta)$ be a C^2-kernel which is* p.d. (3) *on $D \times D$. Then, for any $1 \leqq j \leqq n$, the matrix*

$$(5.4)\qquad \begin{bmatrix} K & \bar{\partial}_{\zeta j}K & \partial_{\zeta j}K \\ \partial_{zj}K & \partial_{zj}\bar{\partial}_{\zeta j}K & \partial_{zj}\partial_{\zeta j}K \\ \bar{\partial}_{zj}K & \bar{\partial}_{zj}\bar{\partial}_{\zeta j}K & \bar{\partial}_{zj}\bar{\partial}_{\zeta j}K \end{bmatrix}_{(z,\zeta)=(z,z)}$$

is p.d. (3) *for every $z \in D$. Here, $z = (z^1, \cdots, z^n)$, $\zeta = (\zeta^1, \cdots, \zeta^n) \in D$ and $K = K(z, \zeta)$.*

Proof. We use (5.1)–(5.3) by specialising $m = 1$ and $u_j = v_j = (0, \cdots, 1_j; 0, \cdots, 0)$. The result then follows in an obvious manner.

6. Positive-definitness of lower order. We now extend the result of Hindmarsh as described in Theorem D. Let $K(z, \bar{\zeta})$ be of class $\mathscr{N}$ and let $u \in U$ be an arbitary unit vector. From Theorem 2′ we know that if $\mathscr{L}_s(\cdot, \cdot : u)$ is p.d. (2) on $D \times D$, then $S(\cdot) \in \mathscr{A}(U: U)[C^0_w(D)]$. We also noted that, in general, the p.d. (2) property does not entail the holomorphicity of $S(\cdot)$. On the other hand, Theorem 3′ shows that if $\mathscr{L}_s(\cdot, \cdot : u)$ is p.d. of any order, then $S(\cdot) \in \mathscr{A}(U:U)[H(D)]$. It is, therefore, remarkable that under centain mild assumptions the replacement of p.d. (2) by p.d. (3) in Theorem 2′ entails the holomorphicity of $S(\cdot)$. For the classical case that D is the right half-plane $\mathscr{R}$, $K_{\mathscr{R}}(z, \bar{\zeta}) = (z + \bar{\zeta})^{-1}$ and $S(\cdot)$ maps $\mathscr{R}$ into itself, this fact was first observed by Hindmarsh [8] (see also [7, pp. 35–38]).

We begin with:

THEOREM 4. *Let $K(z, \bar{\zeta})$ be of class $\mathscr{N}$ such that for any unit vector $u \in U$, $\mathscr{L}_s(\cdot, \cdot : u)$ is* p.d. (3) *on $D \times D$. Assume further that $S(\cdot) \in \mathscr{B}(U: U)[C^2_w(D)]$. Then $S(\cdot) \in \mathscr{A}(U: U)[H(D)]$.*

Proof. Let $u \in U$ be a fixed unit vector. By assumption, the kernel

$$L(z, \zeta) \equiv \mathscr{L}_s(z, \zeta : u) \equiv K(z, \bar{\zeta})[s(z) + \overline{s(\zeta)}]; \quad s(z) = (S(z)u, u)_U ,$$

is a C^2-kernel on $D \times D$, and, it is p.d. (3) on $D \times D$. According to Lemma 1, for any $1 \leqq j \leqq n$, the matrix (5.4), with K replaced by $L = L(z, \zeta)$ is p.d. (3) for every $z \in D$, $D \subset C^n$. Now, in view of the Cauchy-Riemann equations

$$\bar{\partial}_{z^j} L(z, \zeta) = K(z, \bar{\zeta}) \bar{\partial}_{z^j} s(z)$$

and so

$$\bar{\partial}_{z^j} \partial_{\zeta^j} L(z, \zeta) = 0 .$$

Therefore, the element whose position is the entry (3, 3) in the matrix (5.4) is zero. This implies, since the matrix is positive definite, that, the elements with the positions (3, 3), (3, 2), (3, 1), (2, 3) and (1, 3) are all zero. In particular, for the element of the (3, 1)-position, we have

$$\bar{\partial}_{z^j} L(z, \zeta)|_{\zeta = z} = K(z, \bar{z}) \bar{\partial}_{z^j} s(z) = 0 .$$

Since $K(z, \bar{\zeta})$ is of class $\mathscr{N}$, $K(z, \bar{z}) > 0$ and, therefore, $\bar{\partial}_{z^j}(S(z)u, u)_U = 0$. This is true for any $z \in D$, any unit vector $u \in U$ and any $j = 1, \cdots, n$. Hence $S(\cdot)$ is holomorphic in D and, by Proposition 2, also $S(\cdot) \in \mathscr{A}(U: U)[H(D)]$. This concludes the proof.

The contractive version of this theorem is somewhat weaker:

THEOREM 4′. *Let $K(z, \bar{\zeta})$ be of class $\mathscr{N}$ and Let $T(\cdot) \in \mathscr{B}(U: \boldsymbol{C})[C_w^2(D)]$. Assume that for any unit vector $u \in U$, $k_T(\cdot, \cdot: u)$ is* p.d. (3) *on $D \times D$ and that $T(z)u \neq 1$ for every $z \in D$, then $T(\cdot) \in \mathscr{C}(U: \boldsymbol{C})[H(D)]$.*

Proof. Let $u \in U$ be a unit vector and $z, \zeta \in D$. In this case

$$k_T(z, \zeta: u) = K(z, \bar{\zeta})[1 - t(z) \cdot \overline{t(\zeta)}] ; \quad t(z) \equiv T(z)u$$

is a C^2-kernel on $D \times D$ and is p.d. (3) on $D \times D$. As in Corollary 1, we write $s(z) = g[t(z)]$. This gives

$$K(z, \bar{\zeta})[s(z) + \overline{s(\zeta)}] = 2[1 - \overline{t(\zeta)}]^{-1} k_T(z, \zeta: u)[1 - t(z)]^{-1}$$

and the proof proceeds as in that of Theorem 4.

In the case that $K(z, \bar{\zeta})$ is the reproducing kernel $K_{\mathscr{R}}(z, \bar{\zeta}) = (z + \bar{\zeta})^{-1}$ of the right half-plane $\mathscr{R}$ and $S(\cdot)$ maps $\mathscr{R}$ into itself, one is able, as is done in [8], to remove the assumption of $S(\cdot) \in C^2(\mathscr{R})$ in Theorem 4 by using a standard mollification argument. In the present more general case the removal of the assumption $S(\cdot) \in \mathscr{B}(U: U)[C_w^2(D)]$ requires some further mild assumptions on the kernel $K(z, \zeta)$, detailed below.

Before we proceed with the next theorem we briefly recall some standard facts on mollifiers in $\boldsymbol{C}^n$. We choose a C^∞-non-negative function ψ whose compact support B_ψ is inside the unit ball of $\boldsymbol{C}^n$ and such that

$$\int \psi(z)d\sigma(z) = 1$$

where $d\sigma(z)$ is the Lebesgue volume element in $\boldsymbol{C}^n$. For $\varepsilon > 0$ we define

$$\psi_\varepsilon(z) = \varepsilon^{-2n}\psi\left(\frac{z}{\varepsilon}\right).$$

Suppose that f is locally integrable in the domain D of $\boldsymbol{C}^n$. We may assume that $f = 0$ outside a compact set and thus $f \in L_1(\boldsymbol{C}^n)$. The mollification of f is

$$f_\varepsilon(\zeta) = (f*\psi_\varepsilon)(\zeta) = \int f(z)\psi_\varepsilon(\zeta - z)d\sigma(z) = \int \psi(z)f(\zeta - \varepsilon z)d\sigma(z).$$

As is well-known, $f_\varepsilon \in C^\infty(D)$. Moreover, if in addition f is continuous on D, then it is uniformly continuous on compacta of D, and, $\lim_{\varepsilon\to 0} f_\varepsilon = f$ uniformly on compacta of D.

For a fixed $t \in \boldsymbol{C}^n$ we define $D_t = \{z \in \boldsymbol{C}^n: z - t \in D\}$, and, for an operator-valued function $P(\cdot) \in \mathscr{B}(U: W)[D]$, we define $P_t(\cdot) \in \mathscr{B}(U: W)[D_t]$ by $P_t(z) \equiv P(z - t)$ for $z \in D_t$. Clearly, $D_0 = D$ and $P_0(\cdot) = P(\cdot)$. We now prove:

THEOREM 5. *Let $K(z, \bar{\zeta})$ be of class $\mathscr{N}$ such that for any unit vector $u \in U$, and for any fixed $t \in \boldsymbol{C}^n$ with $D \cap D_t \neq \phi$, $\ell_{S_t}(\cdot, \cdot: u)$ is* p.d. (3) *in $D \cap D_t$. Then $S(\cdot) \in \mathscr{A}(U: U)[H(D)]$.*

Proof. Since $S_0(\cdot) = S(\cdot)$, we deduce from Theorem 2′ that $S(\cdot) \in \mathscr{A}(U: U)[C_w^0(D)]$. For a fixed unit vector $u \in U$, we write

$$s(z) = (S(z)u, u)_U; \quad z \in D$$

and we consider the kernel

$$L_\varepsilon(z, \zeta) = K(z, \bar{\zeta})[s_\varepsilon(z) + \overline{s_\varepsilon(\zeta)}]; \quad z, \zeta \in D,$$

where $s_\varepsilon = s*\psi_\varepsilon$ is the mollification of s. This kernel is p.d. (3) on $D \times D$. Indeed, for any three points $z_1, z_2, z_3 \in D$ and corresponding scalars $\alpha_1, \alpha_2, \alpha_3 \in \boldsymbol{C}$, we have

$$\begin{aligned}\sum_{k,m=1}^{3} L_\varepsilon(z_k, z_m)\alpha_k\bar{\alpha}_m &= \sum_{k,m=1}^{3} \alpha_k\bar{\alpha}_m K(z_k, \bar{z}_m)[s_\varepsilon(z_k) + \overline{s_\varepsilon(z_m)}] \\ &= \sum_{k,m=1}^{3} \alpha_k\bar{\alpha}_m K(z_k, \bar{z}_m)\left[\int \psi(t)s(z_k - \varepsilon t)d\sigma(t) + \int \psi(t)\overline{s(z_m - \varepsilon t)}d\sigma(t)\right] \\ &= \int \psi(t)\left\{\sum_{k,m=1}^{3} \alpha_k\bar{\alpha}_m K(z_k, \bar{z}_m)[s(z_k - \varepsilon t) + \overline{s(z_m - \varepsilon t)}]\right\}d\sigma(t) \\ &= \int \psi(t)\left\{\sum_{k,m=1}^{3} \alpha_k\bar{\alpha}_m \ell_{S_{\varepsilon t}}(z_k, z_m: u)\right\}d\sigma(t)\end{aligned}$$

which is non-negative by assumption. Since $s_\varepsilon \in C^\infty(D)$, we deduce from Theorem 4 that $s_\varepsilon \in H(D)$. It follows, because of the continuity of s and the uniform convergence of s_ε to s, that $s \in H(D)$ and the proof is complete.

COROLLARY 2. *Let $K(z, \bar{\zeta})$ be of class $\mathscr{N}$ such that for any fixed $t \in \boldsymbol{C}^n$ with $D \cap D_t \neq \phi$, the scalar kernel $K(z, \bar{\zeta})/K(z - t, \overline{\zeta - t})$ is* p.d. (3) *in $D \cap D_t$. If for any unit vector $u \in U$, $\ell_S(\cdot, \cdot: u)$ is* p.d. (3) *on $D \times D$, then $S(\cdot) \in \mathscr{A}(U: U)[H(D)]$.*

Proof. Let $u \in U$ be any unit vector and $t \in \boldsymbol{C}^n$ with $D \cap D_t \neq \phi$. The kernel

$$\ell_{s_t}(z, \zeta: u) = K(z, \bar{\zeta})[s(z - t) + \overline{s(\zeta - t)}]; \quad s(z) = (S(z)u, u)_U ,$$

is p.d. (3) in $D \cap D_t$. Indeed, this kernel may be written as a product of two p.d. (3) kernels namely,

$$\frac{K(z, \bar{\zeta})}{K(z - t, \overline{\zeta - t})} K(z - t, \overline{\zeta - t})[s(z - t) + \overline{s(\zeta - t)}] ,$$

and therefore, in view of Schur's theorem [7, p. 9] it is p.d. (3) in $D \cap D_t$. The corollary now follows from Theorem 5.

The result of Hindmarsh, as stated in Theorem D of the introduction, is a special case of the following corollary:

COROLLARY 3. *Let $K(z, \bar{\zeta}) = [K_{\mathscr{R}}(z, \bar{\zeta})]^m$, where $K_{\mathscr{R}}(z, \bar{\zeta}) = (z + \bar{\zeta})^{-1}$, $z, \zeta \in \mathscr{R}$, is the Szegö kernel of the right-half plane $\mathscr{R}$ and $m \geqq 1$ is an integer. If for any unit vector $u \in U$, $\ell_s(\cdot, \cdot: u)$ is* p.d. (3) *on $\mathscr{R} \times \mathscr{R}$, then $S(\cdot) \in \mathscr{A}(U: U)[H(\mathscr{R})]$.*

Proof. In view of Corollary 2 it is sufficient to show that for any fixed $t \in \boldsymbol{C}$ with $\mathscr{R} \cap \mathscr{R}_t \neq \phi$, $[(z - t) + \overline{(\zeta - t)}]/(z + \bar{\zeta})$ is p.d. (3) in $\mathscr{R} \cap \mathscr{R}_t$. However, this is a trivial consequence of Pick's theorem as stated in Theorem A or Theorem 1′.

Finally, the contractive versions of Theorem 5 and its corollaries are proved in a similar way to that of Theorem 4′. We have:

THEOREM 5′. *Let $K(z, \zeta)$ be of class $\mathscr{N}$ and let $T(\cdot) \in \mathscr{B}(U: \boldsymbol{C})[D]$. Assume that for any unit vector $u \in U$, and, for any fixed $t \in \boldsymbol{C}^n$ with $D \cap D_t \neq \phi$, $k_{T_t}(\cdot, \cdot: u)$ is* p.d. (3) *in $D \cap D_t$ and that $T(z-t)u \neq 1$ for every $z \in D \cap D_t$. Then $T(\cdot) \in \mathscr{C}(U: \boldsymbol{C})[H(D)]$.*

COROLLARY 2′. *Let $K(z, \zeta)$ be of class $\mathscr{N}$ such that for any fixed $t \in \boldsymbol{C}^n$ with $D \cap D_t \neq \phi$, the scalar kernel $K(z, \zeta)/K(z - t, \overline{\zeta - t})$*

is p.d. (3) *in* $D \cap D_t$ *and assume that* $T(\cdot) \in \mathscr{B}(U: \boldsymbol{C})[D]$. *If for any unit vector* $u \in U$, $k_{T_t}(\cdot, \cdot: u)$ p.d. (3) *on* $D \times D$ *and* $T(z)u \neq 1$ *for every* $z \in D$, *then* $T(\cdot) \in \mathscr{C}(U: \boldsymbol{C})[H(D)]$.

COLOLLARY 3′. *Let* $K(z, \bar{\zeta}) = [K_\Delta(z, \bar{\zeta})]^m$ *where* $K_\Delta(z, \bar{\zeta}) = (1 - \zeta z)^{-1}$ *is the Szegö lernel of the unit disk* Δ *and* $m \geqq 1$ *is an integer. Assume also that* $T(\cdot) \in \mathscr{B}(U: \boldsymbol{C})[\Delta]$. *If for any unit vector* $u \in U$, $k_T(\cdot, \cdot: u)$ *is* p.d. (3) *on* $\Delta \times \Delta$ *and* $T(z)u \neq 1$ *for every* $z \in \Delta$, *then* $T(\cdot) \in \mathscr{C}(U: \boldsymbol{C})[H(\Delta)]$.

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UNIVERSITY OF PITTSBURGH
PITTSBURGH, PA 15260