PACIFIC JOURNAL OF MATHEMATICS
Vol. 98, No. 2, 1982

JOINS OF DOUBLE COSET SPACES

H. N. BHATTARAI AND J. W. FERNANDEZ

The double cosets of a group by a subgroup and the irreducible complex characters of a finite group have a structure which can be studied by means of hypergroups (alias convos, probability groups, and Pasch geometries). An external operation on these structures called the "join" is studied. Decomposition theorems are established in both the general hypergroup (Pasch geometry) case and the weighted hypergroup (probability group) case.

1. Geometries and Joins.

DEFINITION 1.1. A Pasch Geometry (see (3)) is a triple (A, Δ_A, e) where A is a set, $e \in A$ and $\Delta_A \subseteq A \times A \times A$ satisfying:

(I) $\forall a \in A, \exists$ unique $b \in A$ with $(a, b, e) \in \Delta_A$. Denote b by $a^\sharp$.

(II) $e^\sharp = e$ and $(a^\sharp)^\sharp = a \forall a \in A$.

(III) $(a, b, c) \in \Delta_A \Rightarrow (b, c, a) \in \Delta_A$.

(IV) (Pasch's axiom) $(a_1, a_2, a_3), (a_1, a_4, a_5) \in \Delta_A \Rightarrow \exists a_6 \in A$ with $(a_6, a_4^\sharp, a_2), (a_6, a_5, a_3^\sharp) \in \Delta_A$.

We may often write A for (A, Δ_A, e) and Δ for Δ_A if the context is clear. Also we use the word "geometry" for Pasch Geometry.

For $B \subseteq A$, we denote $B \backslash \{e\}$ by B^*. If $B \subseteq A$, B is called a *subgeometry* of A, and we write $B < A$, iff (i) $e \in B$ and (ii) $(b_1, b_2, a) \in \Delta_A$ with $b_1, b_2 \in B$ implies that $a \in B$. If A and C are geometries, a map $f: A \to C$ is called a *geometry morphism* iff $f(e) = e$ and $(x, y, z) \in \Delta_A$ implies $(f(x), f(y), f(z)) \in \Delta_C$.

1.2. Definition of $A//B$. If A is a geometry and $B<A$, it can be shown that the following relation $\sim$ is an equivalence relation on A:

$$x \sim y \text{ iff } \exists b_1, b_2 \in B, a \in A \quad \text{with} \quad (x, b_1, a^\sharp), (a, y^\sharp, b_2) \in \Delta .$$

If the equivalence class of $a \in A$ is denoted by $[a]$ and we let $A//B$ denote $\{[a]: a \in A\}$, then in fact $(A//B, \Delta, [e])$ is a geometry where $([x], [y], [z]) \in \Delta$ iff $\exists x' \in [x], y' \in [y], z' \in [z]$ with $(x', y', z') \in \Delta_A$.

If A is a group (with $\Delta = \{x, y, z): xyz = 1\}$) and B is a subgroup of A, then $A//B$ defines a geometry structure on the set of $B - B$ double cosets of A.

The following result is routine from the definitions.

PROPOSITION 1.3. *Suppose that A and B are geometries and $f: A \to B$ is a geometry morphism. Let $K_f = \{a \ni A: f(a) = e\}$ and*

$\mathrm{Im}_f = \bigcap_{f(A) \subseteqq C < B} C$.

(1) $K_f < A$, $\mathrm{Im}_f < B$.

(2) *The inclusion map* $i: K_f \to A$ *is a kernel of* f *(in the Geometry Category).*

(3) *The projection map* $p: B \to B//\mathrm{Im}_f$ *is a cokernel of* f.

1.4. Construction of $A \vee B$. Suppose A and B are geometries. We construct a geometry $A \vee B$ (see (4)) called the "join" of B by A as follows. Let $A \vee B = A \,\dot{\cup}\, B^*$. A function $p: A \vee B \to B$ is defined via

$$p(x) = \begin{cases} e & \text{if} \quad x \in A\ . \\ x & \text{if} \quad x \notin A\ . \end{cases}$$

Now let Δ be defined by requiring $(x, y, z) \in \Delta$ iff either $\{x, y, z\} \subseteqq A$ and $(x, y, z) \in \Delta_A$, or $\{x, y, z\} \not\subseteqq A$ and $(p(x), p(y), p(z)) \in \Delta_B$. It is a straightforward verification that $A \vee B$ is in fact a geometry.

The join of geometries has an interesting categorial property. In any category $\mathscr{C}$ with zero object, extensions are of interest where, if A and B are objects of $\mathscr{C}$, an extension of B by A is a triple (f, M, g) such that M is an object of $\mathscr{C}$, f and g are morphisms in $\mathscr{C}$ satisfying:

$$A \xrightarrow{f} M \xrightarrow{g} B$$

(1) f is a kernel of g and (2) g is a cokernel of f. For A and B fixed objects of $\mathscr{C}$, the class of exensions of B by A becomes a category $\mathrm{EXT}_{\mathscr{C}}(B, A)$ where a morphism from (f, M, g) to (f', N, g') is a $\mathscr{C}$-morphism θ which makes the following diagram commute:

$$\begin{tikzcd} & M \arrow[dr, "g"] \arrow[dd, "\theta"] & \\ A \arrow[ur, "f"] \arrow[dr, "f'"'] & & B \\ & N \arrow[ur, "g'"'] & \end{tikzcd}$$

A $\mathscr{C}$*-join of* B *by* A is defined as a terminal object in the category $\mathrm{EXT}_{\mathscr{C}}(B, A)$.

PROPOSITION 1.5. *Suppose* A *and* B *are geometries. With notation as above:*

(1) $(i, A \vee B, p)$ *is an extension of* B *by* A *(in the geometry category).*

(2) $(i, A \vee B, p)$ *is a* $\mathscr{G}$*-join of* B *by* A *(where* $\mathscr{G}$ *denotes the geometry category).*

Proof. Both assertions involve diagram chases using the definitions and are omitted.

The concept of 'join' can be extended. If Γ is a linearly ordered set and $\{A_\lambda: \lambda \in \Gamma\}$ is a Γ-indexed family of geometries, a Γ-join of the geometries $\{A_\lambda\}$, denoted by $\bigvee_{\lambda \in \Gamma} A_\lambda$, can be defined in the following way: Let $\bigvee_{\lambda \in \Gamma} A_\lambda = \dot{\bigcup} A_\lambda^* \dot{\cup} \{e\}$. For $\sigma \in \Gamma$ a map $f_\sigma: \dot{\bigcup}_{\tau \leqq \sigma} A_\tau^* \dot{\cup} \{e\} \to A_\sigma$ is defined via

$$f_\sigma(x) = \begin{cases} x & \text{if} \quad x \in A_\sigma^* \\ e & \text{otherwise} . \end{cases}$$

For $x, y, z \in \bigvee_{\lambda \in \Gamma} A_\lambda$, let $\Omega = \{\omega \in \Gamma: \{x, y, z\} \cap A_\omega^* \neq \phi\}$. Now we say $(x, y, z) \in \Delta$ iff either $\Omega = \phi$ or $\Omega \neq \phi$ and $(f_\lambda(x), f_\lambda(y), f_\lambda(z)) \in \Delta A_\lambda (\lambda = \sup \Omega)$. A straightforward check verifies that $\bigvee_{\lambda \in \Gamma} A_\lambda$ is a geometry, and moreover if $\Gamma = \{1, 2\}$, then $\bigvee_{\lambda \in \Gamma} A_\lambda = A_1 \vee A_2$.

The operation $\vee$ is easily shown to be associative in the obvious sense.

2. Decomposition of geometries.

Throughout this section, (A, Δ, e) is a geometry.

DEFINITION 2.1. Let $B \subseteqq A$. We call B a *weak subgeometry* of A if:

(1) $e \in B$.

(2) If $b \in B$, then $b^\# \in B$.

(3) If $b_1, b_2 \in B$, $b_1 \neq b_2^\#$, and $x \in A$ with $(b_1, b_2, x) \in \Delta$, then $x \in B$.

The following result is straightforward.

PROPOSITION 2.2. *If B is a weak subgeometry of A, then $(B, \Delta B, e)$ is a geometry, where $\Delta_B = (B \times B \times B) \cap \Delta_A$.*

For A a geometry, let $L_A = \{B: B$ is a subgeometry of A and $A \backslash B^*$ is a weak subgeometry of $A\}$.

If $a \in A$, set $A: a = \{x:$ whenever $(a, t, x) \in \Delta$, $t = a^\#\}$.

LEMMA 2.3. *With notation as above,*

(1) *L_A is linearly ordered by inclusion.*

(2) *For $B \in L_A$, $B \subseteqq A: a$ $\forall a \in A \backslash B$.*

Proof. (1) Let $B, C \in L_A$ with $B \nsubseteq C$. Then $\exists b \in B \backslash C$. Just suppose that $C \nsubseteq B$. Then $\exists c \in C \backslash B$. Let $t \in A$ with $(b, c, t) \in \Delta$ (such exists by an easy verification). It is easy to see that $(c, t, b) \in \Delta$. If $t \in C$, then $b \in C$ (since B is a subgeometry), a contradiction. Hence $t \notin C$. Similarly if $t \in B$, then $c \in B$ since $(t, b, c) \in \Delta$, which is again a contradiction. Thus $t \notin B \cup C$. But then $t \neq c^{\#}$ and hence $b \in A \backslash B^*$ (since $A \backslash B^*$ is a weak subgeometry of A), which is a contradiction.

(2) Let $a \in A \backslash B$, where $B \in L_A$. Let $b \in B$. Clearly $e \in A: a$, so we may assume that $b \neq e$. Suppose $t \in A$ with $(a, t, b) \in \Delta$. If $t \in B$, then $a \in B$ (since B is a subgeometry and $(t, b, a) \in \Delta$), which is a contradiction. Hence $t \in A \backslash B$. If $t \neq a^{\#}$, then $b \in A \backslash B$ (since $A \backslash B$ is a weak subgeometry of a), again a contradiction. Thus $t = a^{\#}$.

For $B \in L_A$, let $\bar{B} = \cup \{C: C \in L_A \text{ and } C \subsetneqq B\}$. Let $L_A^* = \{B \in L_A: \bar{B} \neq B\}$.

LEMMA 2.4. *$\forall x \in A^* \exists B \in L_A^*$ with $x \in B$ and if $C \in L_A$ such that $C \subsetneqq B$, then $x \notin C$.*

Proof. For $x \in A^*$, set $B = \bigcap_{D \in S(x)} D$, where $S(x) = \{D \in L_A: x \in D\}$. It is routine to show that $B \in L_A^*$ and has the desired property.

For $B \in L_A^*$, let $B^\wedge = B \backslash \bar{B}^*$.

THEOREM 2.5. *Let A be a geometry.*

(1) *$B^\wedge$ is a weak subgeometry of $A \forall B \in L_A^*$.*

(2) *$A = \bigcup_{B \in L^*A} B^\wedge$, and $B^\wedge \cap C^\wedge = \{e\}$ for $B, C \in L_A^*$ if $B \neq C$.*

(3) *If $B, C \in L_A^*$ and $B \neq C$, then $B^\wedge \subseteqq A: c \forall C \in C^*$* iff *$B \subseteqq C$.*

(4) *For the geometry $B^\wedge$, where $B \in L_A^*$, $L_{B^\wedge} = \{\{e\}, B\}$.*

Proof. (1) is routine.

(2) $A = \cup B$ follows from 2.4. If $B, C \in L_A^*$ with $B \neq C$, we may assume $B \subseteqq C$. Then $B \subseteqq \bar{C}$, and hence $(B \backslash B^*) \cap (C \backslash \bar{C}^*) = B^\wedge \cap C^\wedge = \{e\}$.

(3) This assertion is shown using 2.3(2).

(4) Suppose $B \in L_A^*$ and $C \in L_{B^\wedge}^*$ with $C \neq B$ and $C \neq \{e\}$. Let $B_1 = C \cup \bar{B}$. Then $\bar{B} \subsetneqq B_1 \subsetneqq B$. A contradiction to the definition of $\bar{B}$ is obtained after it is easily established that $B_1 \in L_A$.

A non-trivial geometry A is called *join-indecomposable* iff whenever $A \cong B \vee C$ for some geometries B and C, then either $B = \{e\}$ or $C = \{e\}$.

PROPOSITION 2.6. *A non-trivial geometry A is join-indecomposable* iff *$L_A = \{\{e\}, A\}$.*

Proof. ($\Leftarrow$). If $f: B \vee C \to A$ is a geometry isometry isomorphism, it is obvious from the definition of the join that $f(B) \in L_A$.

($\Rightarrow$) If $B \in L_A$ with $B \neq \{e\}$ and $B \neq A$, then it is routine to verify that $A = B \vee (A \backslash B^*)$.

If Γ is a partially ordered set with zero element $\lambda_0 \in \Gamma$, let Γ^*

denote $\Gamma\backslash\{\lambda_0\}$. The main decomposition theorem can now be easily obtained.

THEOREM 2.7. *Let A be a nontrivial geometry. There exists a linearly ordered set Γ and a Γ^*-family of weak subgeometries of A, $\{A_\lambda\}_{\lambda\in\Gamma^*}$ for which*

(1) $A_\lambda^* \neq \phi$ *for all* $\lambda \in \Gamma^*$.

(2) A_λ *is a join-indecomposable geometry* $\forall\lambda \in \Gamma^*$.

(3) $A = \bigvee_{\lambda\in\Gamma^*} A_\lambda$.

Moreover, if Ω is a linearly ordered set and $\{B_\omega\}$ is a Ω-family of nontrivial join-indecomposable geometries such that $A \cong \vee B_\omega$, then there is an isomorphism of partially ordered sets $f: \Gamma^* \to \Omega$ and a Γ^*-family of geometry isomorphisms $\{f_\lambda\}$ such that $f_\lambda: A_\lambda \to B_{f(\lambda)}$.

3. **Probability groups.** Many examples of geometries, such as the double cosets of finite groups, the set of irreducible complex characters of a finite group, and finite projective geometries, can be given additional structure, which is reflected in the following concept (see (3) and (6)):

DEFINITION 3.1. A discrete probability group is a pair (A, p^A) where A is a set and $p^A: A \times A \times A \to [0, 1]$ is a map which we will denote by $(a, b, c) \to p_c^A(a, b)$, and which satisfies the following axioms:

(0) For any $a, b \in A$, $p_c^A(a, b) = 0$ for all but finitely many $c \in A$ and $\sum_{c\in A} p_c^A(a, b) = 1$.

(1) (Associativity) For any $a, b, c, d \in A$,

$$\sum_{x\in A} p_x^A(a, b)p_d^A(x, c) = \sum_{y\in A} p_d^A(a, y)p_y^A(b, c) .$$

(2) (Identity) $\exists e \in A$ such that $p_c^A(a, e) = \delta_a(c)\forall a, c \in A$, where δ is the Kronecker delta.

(3) (Inverse) For each $a \in A$, there exists a unique $b \in A$ such that $p_e^A(b, a) \neq 0$. We denote b by $a^\#$.

(4) $(a^\#)^\# = a$ for all $a \in A$.

(5) $p_c^A(a, b) = p_{c^\#}^A(b^\#, a^\#)$ $\forall a, b, c \in A$.

Note that the $e \in A$ in (2) above is unique. The term "discrete probability group" will be abbreviated to simply "probability group" or even "prob. group". Also p will be used to denote p^A if the context is clear. For $B \subseteq A$, we use B^* to denote the set $B\backslash\{e\}$.

Useful properties of probability groups which follow easily from the axioms appear in the following statement.

PROPOSITION 3.2. *Let (A, p) be a* prob. *group. Then*

(1) $p_a(c^\#, a) = p_a(c, a)$ $\forall a, c \in A$.

(2) $p_a(a, a) \neq 1 \ \forall a \in A^*$.

(3) $P_{c^\#}(a, b)p_e(c^\#, c) = p_e(a, a^\#)p_{a^\#}(b, c)$. *In particular, if* $b = c^\#$, $p_b(a, b)p_e(b, b^\#) = p_e(a, a^\#)p_{a^\#}(b, b^\#)$, *which implies that whenever* $p_b(a, b) = 1$, $p_{a^\#}(b, b^\#) = p_e(b, b^\#)/p_e(a, a)^\#$.

(4) $p_c(a, b) \neq 0$ iff $p_{a^\#}(b, c^\#) \neq 0$.

If A is a finite prob. group, let $n_A = \sum_{x \in A} 1/p_e(x, x^\#)$. n_A is well defined since $p_e(x, x^\#) \neq 0$ and A is finite. Two probability groups (A, p^A) and (B, p^B) are called isomorphic iff there exists a bijective map $f: A \to B$ such that $p_z^A(x, y) = p^B_{f(z)}(f(x), f(y)) \ \forall x, y, z \in A$.

The notions of geometries and probability groups are related by the following result which can be obtained from the definitions (see (2)).

PROPOSITION 3.3. *If* (A, p) *is a* prob. *group, and* $\Delta = \{(a, b, c): p_{c^\#}(a, b) \neq 0\}$, *then* (A, Δ, e) *is a geometry, called the geometry induced by* (A, p).

If A is a prob. group, a subset B of A is called a sub-prob. *group* of A iff $e \in B$ and (B, p^B) is a prob. group, where p^B is the restriction of p to triples from B. It is easily seen that B is a sub-prob. group of A iff B is a subgeometry of the geometry A. In particular, B is a sub-prob. group of A iff $\forall b, c \in B$, whenever $p_a(b, c) \neq 0$, then $a \in B$ (assuming B non-empty).

4. **Joins of probability groups.** Suppose that (A, p^A) and (B, p^B) are probability groups, and also suppose that the set A is finite. A and B acquire geometry structures according to 3.3. Consider the following map (see (6)). $p: (A \mathbin{\dot\cup} B^*) \times (A \mathbin{\dot\cup} B^*) \times (A \mathbin{\dot\cup} B^*) \to [0, 1]$ is defined via

$$p_z(x, y) = \begin{cases} p_z^A(x, y) & \text{if } x, y, z \in A . \\ 0 & \text{if } x, y \in A \text{ and } z \in B^* . \\ \delta_x(z) & \text{if } x \in B^* \text{ and } y \in A . \\ \delta_y(z) & \text{if } x \in A \text{ and } y \in B^* . \\ p_z^B(x, y) & \text{if } x, y, z \in B^* . \\ 0 & \text{if } x, y \in B^*, z \in A \text{ and } y \neq x^\# . \\ \dfrac{p_e^B(x, x^\#)}{n_A p_e^A(z, z^\#)} & \text{if } x \in B^*, y = x^\#, z \in A . \end{cases}$$

A long but straightforward check shows the following:

PROPOSITION 4.1. *If A and B are probability groups with A* *finite, then* $(A \mathbin{\dot\cup} B^*, p)$ *is a* prob. *group whose induced geometry is* $(A \vee B, \Delta, e)$.

The probability group $(A \mathbin{\dot\cup} B^*, p)$ is called the join of (A, p^A) and (B, p^B), and is denoted by $(A \vee B, p)$ or simply by $A \vee B$. It will now be shown that there is uniqueness contained in the definition of the join.

THEOREM 4.2. *Suppose that (A, p^A) and (B, p^B) are probability groups with A finite. Suppose further that $(A \mathbin{\dot\cup} B^*, p')$ is a probability group such that*

(1) *The geometry induced from $(A \mathbin{\dot\cup} B^*, p')$ is $A \vee B$,*

(2) $p'_b(x, y) = p^B_b(x, y)$ $\forall b, x, y \in B^*$,

(3) $p'_a(x, y) = p^A_a(x, y)$ $\forall a, x, y \in A$.

Then $p' = p$, where p is the map defined above.

Proof. Clearly $p' = p$ except possibly in the last case of the definition of p. Hence we need only check that $p'_a(b, b^\#) = p_a(b, b^\#)$ for $a \in A, b \in B^*$. But we have

$$\sum_{a \in A} p'_a(b, b^\#) = 1 - \sum_{y \in B^*} p'_y(b, b^\#) = 1 - \sum_{y \in B^*} p^B_y(b, b^\#) = p^B_e(b, b^\#) .$$

Now by 3.2(3),

$$p'_a(b, b^\#) = \frac{p'_e(b, b^\#)}{p'_e(a, a^\#)} \cdot p'_{b^\#}(b^\#, a) = \frac{p'_e(b, b^\#)}{p^A_e(a, a^\#)} \; (1) .$$

Hence

$$\sum_{a \in A} p'_a(b, b^\#) = \sum_{a \in A} \frac{p'_e(b, b^\#)}{p'_e(a, a^\#)} = p'_e(b, b^\#) \cdot n_A .$$

Combining the two expressions for the same sum, we obtain that

$$p'_e(b, b^\#) = \frac{p^B_e(b, b^\#)}{n_A} .$$

Thus

$$p'_a(b, b^\#) = \frac{p'_e(b, b^\#)}{p'_e(a, a^\#)} = \frac{p^B_e(b, b^\#)}{n_A p^A_e(a, a^\#)} = p_a(b, b^\#) .$$

COROLLARY 4.3. *Suppose $(A_1, p^1), (A_2, p^2), \cdots, (A_n, p^n)$ are probability groups with $A_1, A_2, \cdots, A_{n-1}$ finite. For $i = 1, \cdots, n$ A_i has a geometry structure induced by the map p^i, and hence we may form the join of the geometries $\bigvee_{i=1}^n A_i$. There exists a unique probability* *map p on the set* $\dot{\bigcup}_{i=1}^{n} A_i^* \,\dot{\cup}\, \{e\}$ *which satisfies* (i) *p induces the join geometry* $\bigvee_{i=1}^{n} A_i$ *and* (ii) $p_x(y, z) = p_x^i(y, z)$ *whenever* $x, y, z \in A_i^*$.

Proof. A routine induction argument on n using the theorem establishes the corollary.

COROLLARY 4.4. *Suppose that* $\{(A, p^i)\}_{i=1}^{\infty}$ *is a family of finite probability groups. There exists a unique probability map p defined on* $\dot{\bigcup}_{i=1}^{\infty} A_i^* \,\dot{\cup}\, \{e\}$ *which satisfies* (1) *p induces the join geometry* $\bigvee_{i=1}^{\infty} A_i$, *and* (2) *for* $x, y, z \in A_i^*$, $p_x(y, z) = p_x^i(y, z)$.

Proof. This corollary follows immediately from 4.3, since $\bigvee_{i=1}^{\infty} A_i$ is a limit of $\bigvee_{i=1}^{n} A_i$.

Two notes on the above construction are important.

(1) The condition that all the probability groups be finite (except possibly the last one) is necessary because of the definition of a probability group, which stated that $p_c(a, b)$ is nonzero for only finitely many c. However, if $b = a^\# \in A_i^*$ and $c \in A_j$ where $j < i$, then the construction has $p_c(a, a^\#) \neq 0$. (2) The map p described in 4.3 and 4.4 (which is a generalization of the original construction) can be explicitly defined as follows, where n_k denotes $\sum_{x \in A_k} 1/p_e^k(x, x^\#)$:

$$(\dagger) \qquad p_c(a, b) = \begin{cases} p_c^i(a, b) \text{ if } a, b, c \in A_i \text{ and either } c \neq e \text{ or } b \neq a^\# \\ \left(\prod_{k<i} \dfrac{1}{n_k}\right) \cdot p_e^i(a, a^\#) \text{ if } a \in A_i^*, b = a^\#, c = e\,. \\ \delta_b(c) \text{ if } a \in A_i, b \in A_j^*, \text{ and } i < j\,. \\ \delta_a(c) \text{ if } a \in A_i^*, b \in A_j, \text{ and } j < i\,. \\ 0 \text{ if } a, b \in A_i^*, c \in A_j^*, b \neq a^\#, \text{ and } i \neq j\,. \\ \left(\prod_{j \leq k < i} \dfrac{1}{n_k}\right) \cdot \dfrac{p_e^i(a, a^\#)}{p_e^j(c, c^\#)} \text{ if } a^\# = b \in A_i^*, c \in A_j^* \text{ and} \\ \quad j < i\,. \\ \delta_c(e) \text{ if } a = b = e\,. \end{cases}$$

Combining the above results with the decomposition of geometries established in §2, a decomposition theorem is easily obtained.

THEOREM 4.5. *Let A be a probability group which is nontrivial. There exists a linearly ordered set Γ and a Γ-indexed family of nontrivial probability groups* $\{(A_i, p^i)\}$ *such that*

(1) *A_i is a join-indecomposable geometry* $\forall i \in \Gamma$.

(2) *Γ is either finite or order isomorphic to* Z^+.

(3) *The geometry A is the join of the geometries* $\{A_i\}_{i \in \Gamma}$.

Also, if Γ is finite then A_i is finite except possibly A_n, where Γ is identified with $\{1, 2, \cdots, n\}$, while if Γ is infinite, A_i is finite $\forall i \in \Gamma$. Finally, the original probability map p on A is given by the formula (†).

Proof. The main statements are merely a recapitulation of Theorem 2.7 and the calculations which appear above. The only statement which needs to be proved is that A_i has a probability structure which is "compatible" with the original map p. If we define the map p^i via:

$$p^i_z(x, y) = \begin{cases} p_z(x, y) & \text{if } \; x, y, z \in A^*_i \\ \delta_x(z) & \text{if } \; y = e \\ \delta_y(z) & \text{if } \; x = e \\ 0 & \text{if } \; z = e \text{ and } y \neq x^\# \\ 1 - \sum^*_{a \in A^*_i} p_a(x, y) & \text{if } \; z = e \text{ and } y = x^\# , \end{cases}$$

it is an easy exercise that (A_i, p^i) is the probability group that is needed.

Several examples of joins (of probability groups and geometries) can be given.

(1) The complex irreducible characters of a finite group G form a probability group if one defines the map p via the following procedure.

If $\hat{G} = \{\chi_1, \cdots, \chi_n\}$ is the set of complex irreducible characters, one has

$$\chi_i \cdot \chi_j = \sum_{k=1}^{n} n_k^{i,j} \chi_k \quad \text{for} \quad i, j = 1, \cdots, n .$$

Let $p_{\chi_k}(\chi_i, \chi_j) = n_k^{i,j} \chi_k(1)/\chi_i(1)\chi_j(1)$. Then $(\hat{G}, p)$ is a probability group (See (6) and (3)). If G is a nonabelian group of order p^3 for some prime p, it can be shown that G is the join of $Z_p \times Z_p$ and Z_p. Several other groups can be decomposed into joins, but it is not known to the authors which finite groups (non-simple, since a join does give a normal subgroup) decompose into joins.

(2) Projective geometries have a structure which may be studied as a hypergroup (see (5)) or as a Pasch geometry or probability group (3). A necessary and sufficient condition that a geometry decompose into a join of projective geometries is given in (1).

(3) Totally ordered groups (from which valuations are defined) can be given a geometry structure (although not a probability group structure except in the trivial case) which breaks up into the join of geometries consisting of two elements. If one considers more

general geometries in the same setting, a "generalized valuation" concept can be studied in the study of Rings and Fields (See (3)).

References

1. H. N. Bhattarai, *Orbit space geometries and probability groups*, Ph. D. Thesis, University of Oregon, 1977.

2. M. Dresher and O. Ore, *Theory of multigroups*, Amer. J. Math., **60** (1938), 705-733.

3. D. K. Harrison, *Double coset and orbit spaces*, Pacific J. Math., **80** (1979), 451-491.

4. R. I. Jewett, *Spaces with an abstract convolution of measures*, Advances in Math., **18** (1975), 1-101.

5. W. Prenowitz, *Projective geometries as multigroups*, Amer. J. Math., **65** (1943), 235-256.

6. R. L. Roth, *Character and conjugacy class hypergroups*, Annani di Mat. pura ed appl., **55** (1975), 295-311.

Received March 15, 1977 and in revised form February 27, 1981.

GENERAL MOTORS INSTITUTE
1700 WEST THIRD AVE.
FLINT, MI 48502