PACIFIC JOURNAL OF MATHEMATICS
Vol. 99, No. 1, 1982

CHARACTERIZATION AND ORDER PROPERTIES OF PSEUDO-INTEGRAL OPERATORS

A. R. SOUROUR

Let $(X, \mathscr{A}, m_1)$ and $(Y, \mathscr{B}, m_2)$ be separable σ-finite measure spaces. A linear transformation T from an order-ideal L of measurable functions on Y into the space of measurable functions on X is called a pseudo-integral operator if it is induced by a measure μ on $X \times Y$ via the equation

$$\int (Tf)(x)g(x)m_1(dx) = \iint f(y)g(x)\mu(dx, dy)$$

for sufficiently many functions g. Our main theorem states that T is a pseudo-integral operator if and only if $Tf_n \to 0$ a.e. whenever $0 \leqq f_n \leqq f \in L$ and $f_n \to 0$ a.e. We also study the order structure of the class of pseudo-integral operators showing that they form a band (order-closed ideal) in the space of order-bounded operators.

Introduction. Let $(X, \mathscr{A}, m_1)$ and $(Y, \mathscr{B}, m_2)$ be separable σ-finite measure spaces, and let $M(X)$ and $M(Y)$ be the linear spaces of equivalence classes of (real or complex) measurable functions on X and Y respectively. A linear operator T from a linear subspace L of $M(Y)$ into $M(X)$ is called an integral operator if there exists a measurable function k on $X \times Y$ (called the kernel of T) such that for every f in L, Tf is given by the equation $(Tf)(x) = \int f(y)k(x, y)m_2(dy)$ for m_1—almost every x. Arveson [2] introduced a more general class of operators, which he called pseudo-integral operators, associated with measures, rather than functions, on $X \times Y$. By a pseudo-integral operator we mean an operator given by the equation

$$\int (Tf)(x)g(x)m_1(dx) = \iint f(y)g(x)\mu(dx, dy)\,, \tag{0.1}$$

for sufficiently many functions g (this will be made precise later). If $(Y, \mathscr{B})$ is a standard Borel space, then T can be given explicitely by the equation

$$(Tf)(x) = \int f(y)\mu_x(dy) \tag{0.2}$$

where $\{\mu_x\}$ is a certain family of measures on Y, related to μ via the theorem on disintegration of measures.

The purpose of this paper is to give a necessary and sufficient condition for an operator between spaces of measurable functions to be a pseudo-integral operator. We also study the order structure of the class of pseudo-integral operators and show that they form a band in the vector lattice of order-bounded operators.

The "natural domain" of an integral or a pseudo-integral operator is an *order-ideal* of $M(Y)$, that is, a linear subspace E of $M(Y)$ such that $f \in E$ whenever $|f| \leqq g \in E$. In view of this, we will consider operators from an order-ideal L of $M(Y)$ into $M(X)$. Our main result (Theorem 5.2) states that T is a pseudo-integral operator if and only if it satisfies the condition: whenever $|f_n| \leqq f \in L$ and $f_n \to 0$ almost everywhere, we must have $Tf_n \to 0$ almost everywhere. A similar characterization of integral operators has been obtained by A. V. Bukhvalov [6].

THEOREM (*Bukhvalov*). *A linear operator T from L into $M(X)$ is an integral operator if and only if it satisfies the condition: $Tf_n \to 0$ almost everywhere whenever $|f_n| \leqq f \in L$ and $f_n \to 0$ in measure on every subset of Y of finite measure.*

Our approach is different from Bukhvalov's.

1. Preliminaries. For clarity of exposition we will deal first with standard measure spaces. The generalizations to general separable measure spaces will be indicated toward the end of the paper. Let $(X, \mathscr{A}, m_1)$ be an arbitrary measure space and $(Y, \mathscr{B}, m_2)$ a *standard Borel space*, i.e., $(Y, \mathscr{B})$ is Borel-isomorphic to a Borel subset of a complete separable metric space (see [13] or [3, Chapter 3]). The measures m_1 and m_2 are positive and finite. (Our results are valid for σ-finite measures and follow immediately from the finite case.) Let L be an order-ideal in $M(Y)$, and we will assume that $1 \in L$. This assumption is not essential (see Remark 2.7), but will simplify statements and proofs.

All linear spaces are over the real or the complex numbers. The proofs will be carried out only for the real case. As usual, equality between two members of $M(X)$ or $M(Y)$ will mean equivalence module sets of measure zero. The measure being m_1 or m_2. The characteristic function of a set E will be denoted by 1_E

DEFINITION 1.1. A *kernel* is a map $x \to \mu_x$ of X into the space of bounded Borel measures on Y satisfying the following two conditions.

(i) If $B \in \mathscr{B}$ and $m_2(B) = 0$, then $\mu_x(B) = 0$ for m_1-almost every x.

(ii) For every B in $\mathscr{B}$, the maps $x \to \mu_x(B)$ and $x \to |\mu_x|(B)$ are Borel function.

The *domain* of the kernel $x \to \mu_x$ is the ideal

$$L_\mu = \{f : f \in M(Y),\ f \in L^1(|\mu_x|) \text{ for } m_1\text{-almost every } x\} .$$

The kernel $x \to \mu_x$ will also be denoted by $\{\mu_x\}$.

DEFINITION 1.2. A linear operator T from an ideal $L \subset M(Y)$ into $M(X)$ is called a *pseudo-integral operator* if there is a kernel $\{\mu_x\}$ such that $L \subset L_\mu$ and

$$(Tf)(x) = \int f(y)\mu_x(dy) \quad \text{a.e.} \quad (m_1)$$

for every f in L.

For examples and counterexamples of pseudo-integral operators on $L^2(Y)$, see [17].

REMARKS 1.3. (i) The kernel is uniquely determined by the operator in the sense that if $\{\mu_x\}$ and $\{\nu_x\}$ are kernels of the same operator, the $\mu_x = \nu_x$ for m_1-almost every x. This follows from the fact that $\mathscr{B}$ has a countable generating family $\{B_n\}$, (see [13]), and the observation that $\mu(x, B_n) = (T1_{B_n})(x) = \nu(x, B_n)$ for m_1-almost every x.

(ii) The domain of $\{\mu_x\}$ is the same as the domain of $\{|\mu_x|\}$. Consequently, if T_μ is an integral operator from L into $M(X)$, then $\{|\mu_x|\}$ is the pseudo-kernel of an operator $T_{|\mu|}$ from L into $M(X)$.

(iii) Let $\{\mu_x\}$ be a family of measures on Y inducing an operator T on L by the equation $(Tf)(x) = \int f(y)\mu_x(dy)$. Then $\{\mu_x\}$ obviously satisfy condition 1.1(i). Furthermore, by modifying the measures $\{\mu_x\}$ for x in an m_1-null set, we obtain a kernel satisfying the measurability conditions 1.1(ii). This follows from our characterization theorem (Theorem 5.2), and we do not know of an independent proof. This problem is analogous to the question of measurability of the kernels of integral operators which was settled, in the affirmative, by Bukhvalov [6]. It is easy, however, to prove one half of 1.1(ii), namely the measurability of the maps $x \to \mu_x(B)$ after modifying μ_x for x in an m_1-null set if necessary. This follows from the equality $\mu_x(B) = (T1_B)(x)$ a.e., and the fact that $\mathscr{B}$ is countably generated.

2. Measure kernels. We have defined a kernel as a map $x \to \mu_x$ of X into the space of measures on Y. We wish to replace $\{\mu_x\}$ by one measure μ on the product space $(X \times Y, \mathscr{A} \otimes \mathscr{B})$. The measure μ is the product of m_1 and $\{\mu_x\}$. The only difficulty is the fact that $\int |\mu_x|(Y)m_1(dx)$ may be infinite. Therefore we must consider "measures" μ which are defined only on an ideal of sets $\mathscr{C}_0$ in $\mathscr{A} \otimes \mathscr{B}$. We will use the term *local measure* to refer to such an object, i.e., a countably additive complex-valued set function on an ideal of measurable sets. (This resembles a Radon measure except that we do not have any topology.) Equivalently, a local measure on a Borel space $(Z, \mathscr{C})$ is a product of a unimodular measurable function and a positive (extended real-valued) measure, that is $\mu(dz) = \phi(z)|\mu|(dz)$, where $|\phi(z)| = 1$. A local measure μ is called σ-finite if $|\mu|$ is σ-finite. For $f \in L^1(|\mu|)$, define $\int f d\mu$ by $\int f(z)\mu(dz) = \int f(z)\phi(z)|\mu|(dz)$.

Let $x \to \mu_x$ be a kernel, and let $X_n = \{x: n-1 \leqq |\mu_x|(Y) < n\}$. Thus $\{X_n\}$ are disjoint measurable sets and $X = \bigcup X_n$. The product μ of m_1 and $\{\mu_x\}$ can be defined on the Borel subsets of $X_n \times Y$ by $\mu(dx, dy) = \mu_x(dy)m_1(dx)$, that is, $\mu(E) = \int_X \int_Y 1_E(x, y)\mu_x(dy)m_1(dx)$, for every Borel set E in $X_n \times Y$. For the details of this construction, see [4, Theorem 2.6.2]. It is easy to see that μ extends to a local measure on $X \times Y$, and that $|\mu|(dx, dy) = |\mu_x|(dy)m_1(dx)$. It is also easy to see that $|\mu|$ vanishes on marginally null sets. (Recall that a measurable subsets of $X \times Y$ is called *marginally null* if it is a subset of a rectangle $A \times B$ with $m_1(A) = 0$ or $m_2(B) = 0$.)

The above construction is valid for any measure space. Under the assumption that Y is a standard Borel space, we can recover $\{\mu_x\}$ from μ as shown below.

DEFINITION 2.1. By a *measure kernel*, we mean a local measure μ on $X \times Y$ satisfying the following two properties:

(i) There are countably many disjoint measurable sets X_n such that $X = \bigcup X_n$, and $|\mu|(X_n \times Y) < \infty$ for every n.

(ii) $|\mu|$ vanishes on marginally null sets.

LEMMA 2.2. *Let μ be a measure kernel. Then there exists a map $x \to \mu_x$ of X into the set of all bounded Borel measures on Y such that*

(i) *For B in $\mathscr{B}$, the maps $x \to \mu_x(B)$ and $x \to |\mu_x|(B)$ are Borel functions, and are zero (m_1-a.e.) if $m_2(B) = 0$.*

(ii) $\mu(dx, dy) = \mu_x(dy)m_1(dx)$.

(iii) $|\mu|(dx, dy) = |\mu_x|(dy)m_1(dx)$.

Moreover, the measures μ_x are essentially unique, i.e., *if ν_x are measures satisfying* (i) *and* (ii), *then $\mu_x = \nu_x$ for m_1-almost every x.*

Proof. Let $|\mu|_1$ be the first marginal of $|\mu|$, that is, $|\mu|_1(A) =$

$|\mu|(A \times Y)$. Condition 2.1 (i) implies that $|\mu|_1$ is σ-finite, and condition 2.1 (ii) implies that it is absolutely continuous with respect to m_1. The theorem on disintegration of measures ([1, Theorem 2] or [5, pp. 59-63] together with [2, p. 461]) gives a map $x \to p_x$ from X into the space of probability measures on Y such that:

(a) For every B in $\mathscr{B}$, the map $x \to p_x(B)$ is a Borel function.

(b) $|\mu|(dx, dy) = p_x(dy)|\mu|_1(dx)$.

An examination of proofs in [1] and [5] shows that we need only require that Y is standard.

Let $\lambda_x(dy) = k(x)p_x(dy)$, where k is the Radon-Nikodym derivative of $|\mu|_1$ with respect to m_1. So $|\mu|(dx, dy) = \lambda_x(dy)_{m_1}(dx)$. Finally, let $\mu_x(dy) = \phi(x, y)\lambda_x(dy)$, where ϕ is the Radon-Nikodym derivative of μ with respect to $|\mu|$. Therefore $\mu(dx, dy) = \mu_x(dy)m_1(dx)$. It is also evident that $|\mu_x| = \lambda_x$.

To prove the uniqueness, let $\{\nu_x\}$ be another family of measures on Y such that maps $x \to \nu_x(B)$ are Borel functions, and $\mu(dx, dy) = \nu_x(dy)m_1(dx)$. Let $\{B_n\}$ be a countable generating family for $\mathscr{B}$. Then

$$\int \nu_x(B_n)1_A(x)m_1(dx) = \mu(A \times B_n) = \int \mu_x(B_n)1_A(x)m_1(dx)$$

whenever $A \in \mathscr{A}$, $|\mu|(A \times Y) < \infty$. Thus $\mu_x(B_n) = \nu_x(B_n)$ for almost every x and hence $\mu_x = \nu_x$ for almost every x. This ends the proof.

REMARK. It follows easily from conditions (ii) and (iii) of the Lemma that

$$\mu^+(dx, dy) = (\mu_x)^+(dy)m_1(dx), \qquad \text{and} \qquad \mu^-(dx, dy) = (\mu_x)^-(dy)m_2(dx) .$$

The term kernel will be used for both μ and $\{\mu_x\}$. When we wish to distinguish between the two, we will call μ the *measure kernel* and $\{\mu_x\}$ the *disintegrated kernel*.

Next we describe how the measure kernel μ directly induces an operator. For any f in $M(Y)$, let $\mathscr{A}_{f,\mu} = \{A: A \in \mathscr{A}, \text{ and } f(y)1_A(x) \in L^1(|\mu|)\}$ and $\mathscr{F}_{f,\mu} = \{g: g \in M(X), \text{ and } f(y)g(x) \in L^1(|\mu|)\}$. The *domain* L_μ of μ is defined to be the set of those functions f in $M(Y)$ for which $\mathscr{A}_{f,\mu}$ generates $\mathscr{A}$ as a σ-algebra (equivalently, the ideal of functions $\mathscr{F}_{f,\mu}$ has support X). It is obvious that L_μ is an order-ideal in $M(Y)$, and $1 \in L_\mu$.

LEMMA 2.3. *Let μ be a measure kernel. Then μ induces an operator T_μ from L_μ into $M(X)$ by the equation*

(i) $\int (T_\mu f)(x)1_A(x)m_1(dx) = \iint f(y)1_A(x)\mu(dx, dy)$ *for* $A \in \mathscr{A}_{f,\mu}$.

We also have

(ii) $\int (T_\mu f)(x)g(x)m_1(xd) = \iint f(y)g(x)\mu(dx, dy)$ *for* $g \in \mathscr{F}_{f,\mu}$.

Proof. Let $f \in L_\mu$, and let $X = \bigcup X_n$ where X_n are disjoint and $1_{X_n} \in \mathscr{A}_{f,\mu}$. Since it suffices to describe Tf on every X_n, we may assume that $1 \in \mathscr{A}_{f,\mu}$. For A in $\mathscr{A}$, let $\lambda_f(A) = \iint f(y)1_A(x)\mu(dx, dy)$. Thus λ_f is a countably additive bounded measure on X, absolutely continuous with respect to m_1. Let $T_\mu f$ be the Radon-Nikodym derivative of λ_f with respect to m_1. Then T_μ is the desired operator.

REMARK. The measure μ is uniquely determined by equation 2.3(i) since it determines μ uniquely on enough rectangles to generate $\mathscr{A} \otimes \mathscr{B}$.

Observe that in the preceding Lemma, the condition that Y is standard is not needed. In the case Y is standard, we will show that the class of operators induced by measure kernels agrees with the class of pseudo-integral operators induced by disintegrated kernels.

PROPOSITION 2.4. *Let μ be measure kernel and $\{\mu_x\}$ its disintegration. Then the domain L_μ of μ is the domain of $\{\mu_x\}$. Furthermore the operator T_μ induced by μ (Lemma 2.3) agrees with operator induced by $\{\mu_x\}$ (Definition 1.2).*

Proof. Apply the general Fubuini's theorem [4, Theorem 2.6.4].

COROLLARY 2.5. *The set of all pseudo-integral operators from a linear space.*

Note that there is some difficulty in proving this directly from Definition 1.2. The difficulty lies in proving the measurability condition 1.1(ii) for the variation of the sum of two disintegrated kernels.

PROPOSITION 2.6. *Let T be a pseudo-integral operator with measure kernel μ and let $\mu(dx, dy) = \mu_x(dy)m_1(dx)$. The following conditions are equivalent.*

(i) *T is an integral operator.*

(ii) *μ is absolutely continuous with respect to $m_1 \times m_2$.*

(iii) *μ_x is absolutely continuous with respect to m_2 for m_1-almost every x.*

Proof. The implication (i) $\Rightarrow$ (iii) follows from the uniqueness of the kernel. The implication (ii) $\Rightarrow$ (i) is obvious. To prove that (iii) $\Rightarrow$ (ii) assume that $\mu_x(dy) = k(x, y)m_2(dy)$. The function k is measurable in the second variable. (It may not be jointly measurable.) Let E be a Borel subset of $X \times Y$ with $(m_1 \times m_2)(E) = 0$, and let $E^x = \{y: (x, y) \in E\}$. Thus $m_2(E^x) = 0$ for m_1-almost every x, and hence $\mu_x(E^x) = 0$ for m_1-almost every x. Thus $\mu(E) = \int_Y \int_X \mu_x(E^x) m_1(dx) = 0$, that is, μ is absolutely continuous.

REMARK 2.7. The condition that $1 \in L$ is not essential. Let L be any order-ideal in $M(Y)$. By replacing Y by the support of L, we may assume that the support of L is all of Y (i.e., L is a foundation in the terminology of [18]). It is easy to see that there exists a disjoint countable family $\{Y_n\}$ of Borel subsets of Y such that $Y = \bigcup Y_n$ and $1_{Y_n} \in L$ for every n. In the definition of a disintegrated kernel, the measures μ_x must be replaced by local measures which are uniformly σ-infinite. Similarly, in the definition of a measure kernel μ, condition 2.1(i) must be weakened to the following form: There is a countable disjoint family $\{Y_n\}$ of measurable subsets of Y, and for every n, a countable disjoint family $\{X_{nj}: j = 1, 2, \cdots\}$ of measurable subsets of X such that $Y = \bigcup Y_n$, $X = \bigcup_j X_{nj}$ for every n, and $|\mu|(X_{nj} \times Y_n) < \infty$ for every n and j. With these modifications, it is easy to see that all the results in this paper are still valid without the assumption that $1 \in L$.

3. Positive operators.

In this section we prove the characterization theorem for *positive* operators, i.e., operators which map nonnegative functions into nonnegative functions.

As before we have finite measure spaces $(X, \mathscr{A}, m_1)$ and $(Y, \mathscr{B}, m_2)$ where Y is *standard*. We also have an order-ideal L of $M(Y)$ with $1 \in L$.

LEMMA 3.1. *A pseudo-integral operator is positive if and only if its measure kernel is positive.*

Proof. The "if" part is trivial. To prove the "only if" part, let T be a positive pseudo-integral operator with kernel μ. Let $A \in \mathscr{A}$, $B \in \mathscr{B}$ be such that $1_A T 1_B \in L^1(m_1)$, then $\mu(A \times B) = \int 1_A(x)(T1_B)(x) m_1(dx) \geqq 0$. There are enough such rectangles to generate $\mathscr{A} \otimes \mathscr{B}$, and so μ is a positive measure.

Note that μ is positive if and only if measures μ_x are positive for m_1-almost every x.

THEOREM 3.2. *Let T be a positive operator from L into $M(X)$. The following conditions are equivalent.*

(i) *T is a pseudo-integral operator.*

(ii) *T is order-continuous,* i.e., *if $0 \leqq f_n \leqq f \in L$, and $f_n \to 0$ almost everywhere (m_2), then $Tf_n \to 0$ almost everywhere (m_1).*

Proof. (i) $\Rightarrow$ (ii). Let T be a pseudo-integral operator with kernel $\{\mu_x\}$, and let $0 \leqq f_n \leqq f \in L$, $f_n \to 0$ almost everywhere. Let $X_j = \{x: (Tf)(x) \leqq j\}$. The dominated convergence theorem implies that $Tf_n \to 0$ a.e. on every X_j. This proves (ii).

(ii) $\Rightarrow$ (i). Every standard Borel space is Borel-isomorphic to a compact metric space [13], so we may assume that Y is a compact metric space and $\mathscr{B}$ is generated by the topology of Y. Let $C(Y)$ be the space of real-valued continuous functions on Y. Let $\{e_n: n = 0, 1, 2, \cdots\}$ be a countable linearly independent subset of positive continuous functions such that $e_0 = 1$ and the linear span $\mathscr{D}$ of $\{e_n\}$ is dense in $C(Y)$. Let $\mathscr{D}_r$ be the linear manifold, over the rationals, spanned by $\{e_n\}$. For every n, Let $\pi\{Te_n\}$ be a function in the equivalence class Te_n. Extend π by linearity to $T\mathscr{D}$. For every x, the map ϕ_x, defined by $\phi_x(f) = \pi(Tf)(x)$, is a linear map of $\mathscr{D}$ into the real numbers. There is an m_1-null set X_0 such that $\pi(Tf)(x) \geqq 0$ for every $x \in X \backslash X_0$ and every nonnegative f in $\mathscr{D}_r$.

We will show that ϕ_x is bounded on $\mathscr{D}$ for every x in $X \backslash X_0$. First consider f in $\mathscr{D}_r$ with $-1 \leqq f(y) \leqq 1$. By the positivity of T, we have $-h_0(x) \leqq \phi_x(f) \leqq h_0(x)$, where $h_0 = \pi(T1)$. Thus ϕ_x is bounded on $\mathscr{D}_r$ with norm $h_0(x)$. For every positive integer n, let $\mathscr{M}_n$ be the linear span of $\{e_1, e_2, \cdots, e_n\}$. The norm of the map $\phi_x | \mathscr{M}_n$ is determined on the dense set $\mathscr{D}_r \cap \mathscr{M}_n$, so the norm equals $h_0(x)$. Every f in $\mathscr{D}$ belongs to $\mathscr{M}_n$ for some n, so $|\phi_x(f)| \leqq h_0(x) \| f \|_\infty$ for every f in $\mathscr{D}$. Therefore ϕ_x extends to a bounded linear functional (still denoted by ϕ_x) on $C(Y)$ with norm $h_0(x)$.

For $x \in X \backslash X_0$, the map ϕ_x is positive since $\|\phi_x\| = \phi_x(1)$. By the Riesz Representation Theorem, there exists a positive Borel measure μ_x on Y such that $\pi(Tf)(x) = \int f(y)\mu_x(dy)$ for every f in $\mathscr{D}$. Finally, define μ_x for $x \in X_0$, by $\mu_x = 0$.

To show that $\{\mu_x\}$ is the required kernel, let $\mathscr{C}$ be the set of all f in L for which

$$(Tf)(x) = \int f(y)\mu_x(dy) \,, \quad \text{for } m_1\text{-almost every } x \,.$$

We must show that $\mathscr{C} = L$. First we show that $\mathscr{C}$ contains $C(Y)$. We have already established that $\mathscr{D} \subset \mathscr{C}$. Let $f \in C(Y)$ and let $f_n \in \mathscr{D}$ be such that $f_n \to f$ uniformly. Let $\alpha_n = \| f_n - f \|_\infty$, so $-\alpha_n \leqq f_n - f \leqq \alpha_n$ and hence $-\alpha_n T1 \leqq Tf_n - Tf \leqq \alpha_n T1$. So $Tf_n \to Tf$ almost everywhere. On the other hand, the dominated convergence theorem implies that $\int f_n(y)\mu_x(dy) \to \int f(y)\mu_x(dy)$. This shows that $f \in \mathscr{C}$.

The dominated convergence theorem and the order-continuity of T (condition (ii)) imply that $\mathscr{C}$ is a monotone class, i.e., if $f_n \in \mathscr{C}$,

$f_n \geqq 0$, $f_n \uparrow f$, and $f \in L$, then $f \in \mathscr{C}$. But every monotone class containing $C(Y)$ must also contain every characteristic function and hence must also contain L. Thus $\mathscr{C} = L$.

It remains only to show that μ_x can be chosen so that $\mu_x(B)$ is a Borel function of x for every $B \in \mathscr{B}$. To prove this, let $\{B_n\}$ be a countable generating family for $\mathscr{B}$. Since $\mu_x(B_n) = (T1_{B_n})(x)$ almost everywhere, there is an m_1-null set E such that the maps $x \to \mu_x(B_n)$ are Borel functions from $X \backslash E$ into the real numbers. Redefine μ_x for $x \in E$ to be 0, so the maps $x \to \mu_x(B_n)$ become Borel functions on X. The measurability of $x \to \mu_x(B)$ for any $B \in \mathscr{B}$ follows easily.

COROLLARY 3.3. *Let S and T be positive linear operators from L into $M(X)$ such that $S \leqq T$. If T is an integral (respectively, pseudo-integral) operator, then so is S.*

Proof. If T is a pseudo-integral operator, then T is order-continuous. The operator S must also be order-continuous and hence is a pseudo-integral operator.

If μ and ν are the measures inducing T and S respectively, then $\nu \leqq \mu$ by Lemma 3.1. If T is an integral operator, then μ, and hence also ν, is absolutely continuous with respect to $m_1 \times m_2$. Thus S is an integral operator.

COROLLARY 3.4. *Every positive operator from $L^p(Y)$ into $L^q(X)$, $1 \leqq p < \infty$, $1 \leqq q \leqq \infty$ is a pseudo-integral operator.*

Proof. Let T be a positive operator from $L^p(Y)$ into $L^q(X)$ and let $f_n \in L^p(Y)$, $f_n \downarrow 0$. By the monotone convergence theorem, $\| f_n \|_p \to 0$. Since T is automatically norm bounded [15, p. 84], we also have $\| Tf_n \|_q \to 0$. Since $\{Tf_n\}$ is a decreasing sequence, we must have $Tf_n \to 0$ almost everywhere. Thus T is order-continuous, and hence is a pseudo-integral operator.

The following result is stated in [10], but the proof given there seems to be incomplete.

COROLLARY 3.5. *Every bounded operator from $L^1(Y)$ into $L^1(X)$ is a pseudo-integral operator.*

Compare also [9, Corollary VI. 8.9] where X, rather than Y, is assumed to be a compact metric space.

Proof. For a bounded operator T from $L^1(Y)$ into $L^1(X)$, it is well-known that there are positive bounded operators T_1 and T_2 such that $T = T_1 - T_2$ (in the complex case, $T = T_1 - T_2 + iT_3 - iT_4$, where every T_j is a positive bounded operator), see [7]. The result follows from the preceding corollary.

4. Order properties. For terminology and notation concerning vector lattices and operators between them we refer to Schaefer [15]. When the vector lattices under consideration are lattices of equivalence classes of measurable functions, the notions of order convergence and order continuity can be reformulated in terms of the more familiar concepts of measure theory. In particular, in a vector lattice L of equivalence classes of measurable functions, a sequence of functions $\{f_n\}$ *order-converges* to 0 if and only if it converges to zero almost everywhere and dominatedly, i.e., $|f_n| \leqq f \in L$ and $f_n \to 0$ (a.e.). In the lattice $M(X)$, the condition of domination is automatically satisfied in the presence of almost everywhere convergence; see [15], p. 141, exercise 2(c).

Let $T: L \to L'$ be a linear operator between vector lattices L and L' of equivalence classes of measurable functions. Then T is *order-continuous* if and only if $0 \leqq f_n \leqq f \in L$, $f_n \to 0$ a.e. implies that $Tf_n \to 0$ a.e. and $|Tf_n| \leqq g \in L'$. Again if $L' = M(X)$, the condition that $|Tf_n| \leqq g$ is redundant. Order-continuous operators are called (0)-linear operators in [18], p. 214.

THEOREM 4.1. *Let T be a pseudo-integral operator from L into $M(X)$ with kernel μ. Then T is order-bounded (regular) and the operators T^+, T^-, and $|T|$ are pseudo-integral operators with kernels, μ^+, μ^-, and $|\mu|$ respectively.*

Proof. The measure $|\mu|$ induces a positive operator S from L into $M(X)$. Since $-|\mu| \leqq \mu \leqq |\mu|$, we must have $-S \leqq T \leqq S$. Therefore T is order-bounded and $|T| \leqq S$. By Corollary 3.3, $|T|$ is a pseudo-integral operator whose kernel ν satisfies $\nu \leqq |\mu|$. On the other hand, $0 \leqq 2T^- = |T| - T = T_{\nu-\mu}$, and so $\nu - \mu$ is a positive measure. Similarly $T_{\nu+\mu} = 2T^+$ and hence $\nu + \mu$ is a positive measure. Therefore $\nu \geqq |\mu|$. Thus we have $\nu = |\mu|$, and so $|T| = T_{|\mu|}$. It follows that $T^+ = (T + |T|)/2 = T_{\mu^+}$ and $T^- = T_{\mu^-}$.

COROLLARY 4.2 [12]. *Let T be an integral operator with kernel k. Then T^+, T^-, and $|T|$ are integral operators with kernels k^+, k^-, and $|k|$ respectively.*

Proof. Use Theorem 4.1 and Proposition 2.6.

COROLLARY 4.3. *Let T_μ and T_ν be pseudo-integral operators from L into $M(X)$. Then* $\sup(T_\mu, T_\nu) = T_{\mu\vee\nu}$ *and* $\inf(T_\mu, T_\nu) = T_{\mu\wedge\nu}$.

REMARK. $\mu \vee \nu$ and $\mu \wedge \nu$ are the usual supremum and infimum of μ and ν in the lattice of local measures, i.e., $\mu \vee \nu = \nu + (\mu - \nu)^+$, and $\mu \wedge \nu = \nu - (\mu - \nu)^-$.

Theorem 4.1 establishes that the set of pseudo-integral operators is an order-ideal in the vector lattice $\mathscr{L}_0(L, M(X))$ of all order-bounded (regular) operators from L into $M(X)$. We will show later (Corollary 5.3) that the pseudo-integral operators form a *band* (or a *component* in the terminology of [18]) in $\mathscr{L}_0(L, M(X))$. Recall that a band (a component) in a vector lattice $\mathscr{L}$ is by definition an order-ideal $\mathscr{I}$ with the property that whenever $x_\alpha \in \mathscr{I}$ and $x = \sup\{x_\alpha\}$ exists in $\mathscr{L}$, we must have $x \in \mathscr{I}$. (It would be more descriptive to call an ideal with this property an *order-closed ideal.*)

5. The characterization theorem.

LEMMA 5.1. *Let T be an order-continuous operator from L into $M(X)$. Then T is order-bounded, and the operators T^+, T^-, and $|T|$ are order-continuous.*

Proof. The proof is given in [18, p. 214 and p. 216] for operators between more general vector lattices. We give a sketch of the proof.

To prove that T is order-bounded it is enough to consider sequences [18, p. 154], i.e., if $|f_n| \leqq f \in L$, we must show that $\sup\{Tf_n\}$ exists. To prove this, it suffices to show that $\lambda_n Tf_n \to 0$ a.e. whenever $\{\lambda_n\}$ is a sequence of positive real numbers converging to 0. But this follows from the order-continuity of T.

To prove that T^+ is order-continuous, let $f_n \geqq 0$, $f_n \uparrow f \in L$. We must show that $T^+f = \sup\{T^+f_n\}$. If $0 \leqq g \leqq f$, then $(g \wedge f_n) \uparrow g$ and hence $Tg = \sup T(g \wedge f_n) = \sup T^+(g \wedge f_n) \leqq \sup T^+f_n$. So $T^+f = \sup\{Tg: 0 \leqq g \leqq f\} \leqq \sup T^+f_n$. The reverse inequality is trivial. This shows that T^+ is order-continuous and so T^- and $|T|$ must also be order-continuous.

THEOREM 5.2. *Let T be a linear operator from L into $M(X)$. The following conditions are equivalent.*

(i) *T is a pseudo-integral operator.*

(ii) *T is order-continuous,* i.e., *if $0 \leqq f_n \leqq f \in L$ and $f_n \to 0$* a.e., *then $Tf_n \to 0$* a.e.

Proof. If T is a pseudo-integral operator, then T^+ and T^- are pseudo-integral operators by Theorem 4.1. It follows from Theorem 3.2 that T^+ and T^- are order-continuous, and hence $T = T^+ - T^-$

is order-continuous.

Conversely, if T is order-continuous, apply Lemma 5.1 and Theorem 3.2 to conclude that T is the difference between two pseudo-integral operators, and so T itself must be a pseudo-integral operator.

COROLLARY 5.3. *The pseudo-integral operators form a band (an order-closed ideal) in the vector lattice of order-bounded operators from L into $M(X)$.*

Proof. It is known [18, p. 216], that the order-continuous operators between two vector lattices from a band in the lattice of regular operators.

We are now in a position to show that the measurability condition 1.1 (ii) is redundant as far as operators are concerned.

PROPOSITION 5.4. *Let $x \to \mu_x$ be a map of X into the space of bounded Borel measures on Y, and let L be an order-ideal of $M(Y)$ and T an operator from L into $M(X)$ such that*

(a) *every f in L belongs to $L^1(|\mu_x|)$ for almost every x,*

(b) $(Tf)(x) = \int f(y)\mu_x(dy)$ *for* $f \in L$.

Then there are measures ν_x such that

(c) $\nu_x = \mu_x$ *for almost every x,*

(d) *for every B in $\mathscr{B}$, the maps $x \to \nu(B)$ and $x \to |\nu_x|(B)$ are Borel functions.*

Proof. By the dominated convergence theorem, the operator T is order-continuous. Theorem 5.2 implies the existence of ν_x.

6. **More general measure spaces.** In this section we generalize our results to the case of a separable (not necessarily standard) measure space $(Y, \mathscr{B}, m_2)$. In this case, we will use the term "*pseudo-integral operator*" to mean an operator induced by a measure kernel μ as in Lemma 2.3. Since the theorem of disintegration of measures is not available in the present case, we may not be able to obtain an explicit representation of the operator as in (0.2). Operators given by (0.2) form a subclass of what we now call pseudo-integral operators.

Examination of the proofs of our previous results shows that they extend to the present general case once Theorem 3.2 has been so extended. In what follows $(Y, \mathscr{B}, m_2)$ is a separable finite measure space.

LEMMA 6.1. *Let T be a positive pseudo-integral operator from*

L into $M(X)$, then T is order-continuous.

Proof. Let $f_n \downarrow 0$. We must show that $Tf_n \downarrow 0$. Toward this end, let $g = \inf\{Tf_n\}$, and let $X_j = \{x: (Tf_1)(x) \leqq j\}$. If suffices to show that $g = 0$ a.e. on every X_j, and so we may assume that Tf_1 is a bounded function. Let μ be the kernel of T, and $A \in \mathscr{A}$, the monotone convergence theorem shows that

$$\begin{aligned}\int g(x)1_A(x)m_1(dx) &= \lim \int (Tf_n)(x)1_A(x)m_1(dx)\\ &= \lim \iint f_n(y)1_A(x)\mu(dx, dy) = 0 .\end{aligned}$$

Thus $g = 0$. This proves the Lemma.

For any separable measure space $(Y, \mathscr{B}, m_2)$, there is a compact metric space Y', a Borel measure m_2' on Y' and an isomorphism of the measure algebra (Y, m_2) onto the measure algebra of (Y', m_2') (see [11, p. 173] for the nonatomic case). This isomorphism induces a one-to-one positive linear map ψ of $M(Y)$ onto $M(Y')$, see [8, pp. 252–254] for details. It is easy to see that ψ preserves almost everywhere convergence, that is $f_n \to 0$ a.e. if and only if $\psi(f_n) \to 0$ a.e. This follows from the observation that ψ preserves the order structure and the fact that $f_n \to 0$ a.e. if and only if there is a decreasing sequence of positive functions $\{g_n\}$ such that $|f_n| \leqq g_n$ and $\inf\{g_n\} = 0$.

For an operator T from L into $M(X)$, let $\hat{T}$ be the operator from $\psi(L)$ into $M(X)$ defined by $\hat{T}(f) = T(\psi^{-1}(f))$. It is straightforward to see that $\hat{T}$ is positive if and only if T is positive and that $\hat{T}$ is order-continuous if and only if T is order-continuous.

THEOREM 6.2. *Let T be a positive operator from L into $M(X)$. The following conditions are equivalent.*

(i) *T is a pseudo-integral operator, in the sense that it has a measure kernel on $X \times Y$.*

(ii) *T is order-continuous.*

Proof. The implication (i)$\Rightarrow$(ii) has already been proved. Assume that T is order-continuous. Therefore $\hat{T}$ is order-continuous and so by Theorem 3.2 it is a pseudo-integral operator induced by a measure $\hat{\mu}$ on $X \times Y'$. Define μ on the measurable rectangles of $X \times Y$ by $\mu(A \times B) = \int 1_A(x)(T1_B)(x)m_1(dx)$. Thus $\mu(A \times B) = \hat{\mu}(A \times \psi(B))$. In order to show that μ extends to a countably additive measure on $\mathscr{A} \otimes \mathscr{B}$ it suffices to show that μ is countably additive on rectangles [14, p. 224]. But this is satisfied because $\hat{\mu}$ is countably additive.

In order to prove that T is a pseudo-integral operator induced by the measure μ, we must prove the following:

(a) For every f in L, the ideal of sets $\mathscr{A}_f = \{A: A \in \mathscr{A}, 1_A(x)f(y) \in L^1(\mu)\}$ generates $\mathscr{A}$ as a σ-algebra.

(b) $\int (Tf)(x)1_A(x)m_1(dx) = \iint f(y)g(x)\mu(dx, dy)$ for every $f \in L$ and $A \in \mathscr{A}_f$.

When f is a simple function, each of the conditions (a) and (b) is obviously satisfied in view of the definition of μ. Their validity for arbitrary f in L follows from the order-continuity of T and the monotone convergence theorem.

REFERENCES

1. M. B. Abrahamse and T. L. Kriete, *The spectral multiplicity of a multiplication operator*, Indiana Univ. Math. J., **22** (1973), 845-857.
2. W. Arveson, *Operator algebras and invariant subspace*, Ann. of Math., (2), **100** (1974), 433-532.
3. ———, *An Invitation to C*-Algebras*, Springer-Verlag, New York, 1976.
4. R. B. Ash, *Real Analysis and Probability*, Academic Press, New York, 1972.
5. N. Bourbaki, *Eléménts de Mathématique, Livre VI: Intégration. Chapitre 6: Intégration vectorielle*, Hermann, Paris, 1959.
6. A. V. Bukhvalov, *Integral representation of linear operators*, Zap. Naucn, Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **47** (1974), 5-14.
7. R. V. Chacon and U. Krengel, *Linear modulus of a linear operator*, Proc. Amer. Math. Soc., **15** (1964), 553-559.
8. J. L. Doob, *Stochastic Processes*, Wiley, New York, 1953.
9. N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1958.
10. H. Fakhoury, *Représentations d'opérateurs à valeurs dans* $L^1(X,\Sigma,\mu)$, Math. Ann., **240** (1979), 203-212.
11. P. R. Halmos, *Measure Theory*, Van Nostrand, New York, 1950.
12. W. A. J. Luxemburg and A. C. Zaanen, *The linear modulus of an order bounded linear transformation*, Indag. Math., **33** (1971), 422-447.
13. G. Mackey, *Borel structures in groups and their duals*, Trans. Amer. Math. Soc., **85** (1957), 265-311.
14. H. L. Royden, *Real Analysis*, Macmillan, New York, 1963.
15. H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer-Verlag, New York, 1974.
16. A. R. Schep, *Kernel operators*, Indag. Math., **41** (1979), 39-53.
17. A. R. Sourour, *Pseudo-integral operators*, Trans. Amer. Math. Soc., **253** (1979), 339-363.
18. B. Z. Vulikh, *Introduction to the Theory of Partially Ordered Spaces*, Wolters-Noordhoff, Groningen, 1967.

Received March 3, 1980 and in revised form July 30, 1980.

UNIVERSITY OF VICTORIA
VICTORIA, BRITISH COLUMBIA
CANADA V8W 2Y2