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ALTMAN'S CONTRACTORS AND FIXED POINTS OF MULTIVALUED MAPPINGS

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Let $P_i: D \subset X_1 \times \cdots \times X_n \to \mathrm{CL}(Y_i)$ be multivalued mappings where X_i, Y_i are Banach spaces and $\mathrm{CL}(Y_i)$ is the set of all nonempty closed subsets of Y_i, $i = 1, \cdots, n$. We prove a theorem ensuring that $\theta_i \in P_i(x_1, \cdots, x_n)$ for some $(x_1, \cdots, x_n) \in D$ and deduce the fixed point theorems for multivalued mappings proved earlier by Czerwik, Nadler and Reich as corollaries. Besides, generalizations for multivalued mappings of the existence theorems proved by Altman using his theory of contractors are also obtained.

1. **Introduction.** In [2] we showed how the fixed point theorems of Altman [1] and Matkowski ([5], [6]) can be unified in the set-up of Banach spaces. The present paper studies further the relationship between Altman's theory of contractors and Matkowski's fixed point theorem and offers an existence theorem for the multivalued operator equation $\theta \in Px$ on subsets of a Banach space. We deduce as a corollary a comprehensive fixed point theorem proved by Czerwik [4] for multivalued mappings. Czerwik's fixed point theorem generalized the earlier fixed point theorems for set valued transformations on metric spaces obtained by Nadler [7], Covitz and Nadler [3] and Reich [9]. Apart from Czerwik's theorem, our main result obtains as corollaries generalizations to multivalued mappings of Altman's existence theorems and Matkowski's fixed point theorem. Section 3 gives the main result of the paper, while § 2 provides preliminaries basic to § 3.

2. Let X be a Banach space. We employ the following notation of [7] and [8]:

$$(2.1) \qquad \mathrm{CL}(X) = \{C: C \text{ is a nonempty closed subset of } X\} .$$

$$(2.2) \qquad N(\varepsilon, C) = \{x \in X: \|x - c\| < \varepsilon \text{ for some } c \in C\} , \quad \varepsilon > 0 , \quad C \in \mathrm{CL}(X) .$$

$$(2.3) \qquad H(A, B) = \begin{cases} \inf \varepsilon > 0 , & A \subset N(\varepsilon, B) \text{ and } B \subset N(\varepsilon, A) , \\ & \text{if the infimum exists} , \\ \infty , & \text{otherwise} , \end{cases} \quad A, B \in \mathrm{CL}(X) .$$

The function H is called the generalized Hausdorff distance for

CL (X) induced by the norm of X.

$$D(x, A) = \inf \{\| x - a \| : a \in A\} . \tag{2.4}$$

The lemma given below is well-known and is used in § 3.

LEMMA 2.1. *Suppose $A, B \in$ CL (X) and $a \in A$. Then, for $q > 0$, there exists an element $b \in B$ such that*

$$d(a, b) \leqq H(A, B) + q .$$

A point $x \in X$ is said to be a fixed point for the multivalued mapping $f: X \to$ CL (X) if $x \in f(x)$.

We follow the notation of [6].

Let (a_{ik}) be an $n \times n$ nonnegative matrix. Define

$$a_{ik}^1 = \begin{cases} a_{ik} , & i \neq k , \\ 1 - a_{ik} , & i = k , \end{cases} \quad i, k = 1, \cdots, n . \tag{2.5}$$

$$a_{ik}^{l+1} = \begin{cases} a_{11}^l a_{i+1k+1}^l + a_{i+11}^l a_{1k+1}^l , & i \neq k \\ a_{11}^l a_{i+1k+1}^l - a_{i+11}^l a_{1k+1}^l , & i = k \end{cases} \tag{2.6}$$
$$l = 1, \cdots, n - 1 ; \quad i, k = 1, \cdots, n - l$$

Matkowski ([5], [6]) proved the following

LEMMA 2.2. *Let $a_{ik}^1 > 0$, $i, k = 1, \cdots, n$. The system of inequalities*

$$\sum_{k=1}^n a_{ik} r_k < r_i , \quad i = 1, \cdots, n , \tag{2.7}$$

has a solution $r_i > 0$, $i = 1, \cdots, n$ if and only if the following inequalities hold:

$$a_{ii}^l > 0 , \quad l = 1, \cdots, n ; \quad i = 1, \cdots, n + 1 - l . \tag{2.8}$$

Using this lemma he obtained the following fixed point theorem. (Actually Matkowski proved this theorem in the setting of complete metric spaces.)

THEOREM 2.1. *Let X_i be Banach spaces and $T_i: X_1 \times \cdots \times X_n \to X_i$, $i = 1, \cdots, n$ be mappings such that*

$$\| T_i(x_1, \cdots, x_n) - T_i(y_1, \cdots, y_n) \| \leqq \sum_{k=1}^n a_{ik} \| x_k - y_k \| , \tag{2.9}$$
$$i = 1, \cdots, n , \quad x_k, y_k \in X_k , \quad k = 1, \cdots, n ,$$

where $a_{ik} > 0$, $i, k = 1, \cdots, n$. If the numbers a_{ik}^l, $l = 1, \cdots, n$;

$i, k = 1, \cdots, n + 1 - l$ defined by (2.5) *and* (2.6) *fulfill* (2.8), *then the system of equations*

$$x_i = T_i(x_1, \cdots, x_n) , \quad i = 1, \cdots, n ,$$

has exacty one solution $x_i \in X_i$, $i = 1, \cdots, n$. Moreover,

$$x_i = \lim_{m\to\infty} x_i^m , \quad i = 1, \cdots, n ,$$

where $x_i^{m+1} = T_i(x_1^m, \cdots, x_n^m)$, $\mathring{x}_i \in X_i$, $i = 1, \cdots, n$ is arbitrarily chosen.

Considerations of some of the fundamental problems of numerical analysis and operator theory led Altman [1] to the concept of contractors.

DEFINITION 2.1 [p. 8, [1]]. Let $P: D(P) \subset X \to Y$, $D(P)$ being the domain of P, be a nonlinear operator from a Banach space X to a Banach space Y and $\Gamma(x): Y \to X$ be a bounded linear operator associated with $x \in X$. We say that P has a contractor $\Gamma(x)$ if there is a positive number $q < 1$ such that

$$(2.10) \qquad \| P(x + \Gamma(x)y) - Px - y \| \leqq q \| y \| ,$$

where $x \in D(P)$, $y \in Y$ are defined by the particular problem.

DEFINITION 2.2 [p. 10, [1]]. A contractor $\Gamma(x)$ is called regular if (2.10) is satisfied for all $y \in Y$ and $D(P) = \Gamma(x)(Y)$.

DEFINITION 2.3 [p. 6, [1]]. An operator $P: D(P) \subset X \to Y$ is closed on $D(P)$ if $x_n \in D(P)$, $x_n \to x$ and $Px_n \to y$ imply $x \in D(P)$ and $Px = y$.

Altman proved the following theorem:

THEOREM 2.2 [p. 13, Theorem 5.1, [1]]. *Suppose that the closed nonlinear operator $P: D(P) \subset X \to Y$ has a bounded contractor Γ such that*

$$(2.11)\ \text{(a)} \qquad x + \Gamma(x)y \in D(P) , \quad \textit{whenever} \quad x \in D(P) , \quad y \in Y ;$$

$$(2.11)\ \text{(b)} \qquad \| P(x + \Gamma(x)y) - Px - y \| \leqq q \| y \| , \quad y \in Y , \quad 0 < q < 1 ;$$

$$(2.12) \qquad \| \Gamma(x) \| \leqq B \quad \textit{for all} \quad x \in D(P) .$$

Then the equation $Px = y$ has a solution for $y \in Y$. When Γ is regular, the assumption (2.11)(a) *is readily satisfied and further, the solution is unique.*

3. We generalize the notion of a closed operator to multivalued mappings as follows:

DEFINITION 3.1. A nonlinear multivalued operator $P: D(P) \subset X \to \mathrm{CL}(Y)$, X, Y being Banach spaces, is closed on $D(P)$, if $x_n \to x$, $y_n \in Px_n$ and $y_n \to y$ imply that $x \in D(P)$ and $y \in Px$.

Let X_i, Y_i, $i = 1, \cdots, n$ be Banach spaces and $b_{ik}, c_{ik} \geqq 0$ $i, k = 1, \cdots, n$. Let $a_{ik} = b_{ik} + c_{ik}$, $i, k = 1, \cdots, n$, be positive and the numbers a^l_{ik} defined by (2.5) and (2.6) fulfill (2.8). Then, by Lemma 2.2, there exists a positive solution $r_1, \cdots, r_n$ of the system of inequalities (2.7). We define

$$q = \max_i \left(r_i^{-1} \sum_{k=1}^{n} a_{ik} r_k \right) . \tag{3.1}$$

Clearly, $0 < q < 1$, and

$$\sum_{k=1}^{n} a_{ik} r_k \leqq q r_i , \quad i = 1, \cdots, n . \tag{3.2}$$

THEOREM 3.1. *Suppose that the closed nonlinear transformations* $P_i: D \subset X_1 \times \cdots \times X_n \to \mathrm{CL}(Y_i)$, $i = 1, \cdots, n$ *fulfill the following*:

(3.3) *there exist bounded linear operators* $\Gamma_i(x_i): Y_i \to X_i$, $x_i \in X_i$, $i = 1, \cdots, n$ *such that*

$$\|\Gamma_i(x_i)\| \leqq B , \quad (x_1, \cdots, x_n) \in D , \quad i = 1, \cdots, n ;$$

(3.4) (a) $(x_1 + \Gamma_1(x_1)y_1, \cdots, x_n + \Gamma_n(x_n)y_n) \in D$ *whenever* $(x_1, \cdots, x_n) \in D$ *and* $y_i \in Y_i$, $i = 1, \cdots, n$;

(3.4) (b)
$$\begin{aligned} H_i[P_i(x_1 + \Gamma_1(x_1)y_1, \cdots, x_n + \Gamma_n(x_n)y_n), P_i(x_1, \cdots, x_n) + y_i] \\ \leqq \sum_{k=1}^{n} b_{ik}\|y_k\| + \sum_{k=1}^{n} c_{ik} D_k[y_k, y_k - P_k(x_1, \cdots, x_n)] \\ + cD_i[y_i, y_i \\ - \Gamma_i(x_i)P_i(x_1 + \Gamma_1(x_1)y_1, \cdots, x_n + \Gamma_n(x_n)y_n)] \\ \text{for} \quad (x_1, \cdots, x_n) \in D , \quad y_i \in Y_i , \quad i = 1, \cdots, n , \end{aligned}$$

where $a_{ik} = b_{ik} + c_{ik}$, $i, k = 1, \cdots, n$ *are positive and the numbers* a^l_{ik} *defined by* (2.5) *and* (2.6) *satisfy* (2.8);

(3.5) c *is a constant such that* $0 \leqq cB < 1 - q$.

Then there exists $(x_1, \cdots, x_n) \in D$ *such that* $\theta_i \in P_i(x_1, \cdots, x_n)$ $i = 1, \cdots, n$, *where* θ_i *is the zero element of the Banach space* Y_i, $i = 1, \cdots, n$.

Proof. Let $(\mathring{x}_1, \cdots, \mathring{x}_n) \in D$ be an arbitrary element. Choose

$\mathring{y}_i \in P_i(\mathring{x}_1, \cdots, \mathring{x}_n)$, $i = 1, \cdots, n$. We can assume without loss of generality that $\|\mathring{y}\| \leqq r_i$, $r_i \geqq 1$, $i = 1, \cdots, n$, since the set of solutions to the system (2.7) is closed with respect to multiplication by positive scalars. Define

$$x_i^1 = \mathring{x}_i - \Gamma_i(\mathring{x}_i)\mathring{y}_i \, , \quad i = 1, \cdots, n \, . \tag{3.6}$$

Replacing x_i by $\mathring{x}_i$ and y_i by $-\mathring{y}_i$ in (3.4)(b) and using (3.6) we get,

$$\begin{aligned} H_i[P_i(x_1^1, \cdots, x_n^1), P_i(\mathring{x}_1, \cdots, \mathring{x}_n) - \mathring{y}_i] &\leqq \sum_{k=1}^{n} b_{ik}\|\mathring{y}_k\| + \sum_{k=1}^{n} c_{ik}D_k[-\mathring{y}_k, -\mathring{y}_k - P_k(\mathring{x}_1, \cdots, \mathring{x}_n)] \\ &\quad + cD_i[-\mathring{y}_i, -\mathring{y}_i - \Gamma_i(\mathring{x}_i)P_i(x_1^1, \cdots, x_n^1)] \\ &\leqq \sum_{k=1}^{n} b_{ik}\|\mathring{y}_k\| + \sum_{k=1}^{n} c_{ik}\|\mathring{y}_k\| + cD_i[\theta_i, \Gamma_i(\mathring{x}_i)P_i(x_1^1, \cdots, x_n^1)] \, . \end{aligned} \tag{3.7}$$

As $\mathring{y}_i \in P_i(\mathring{x}_1, \cdots, \mathring{x}_n)$, $\theta_i \in P_i(\mathring{x}_1, \cdots, \mathring{x}_n) - \mathring{y}_i$, $i = 1, \cdots, n$. So, for $q > 0$, by Lemma 2.1, there exists an element $y_i^1 \in P_i(x_1^1, \cdots, x_n^1)$, $i = 1, \cdots, n$ such that

$$\|y_i^1 - \theta_i\| \leqq H_i[P_i(x_1^1, \cdots, x_n^1), P_i(\mathring{x}_1, \cdots, \mathring{x}_n) - \mathring{y}_i] + q \, . \tag{3.8}$$

From (3.7) and (3.8) we have

$$\begin{aligned} \|y_i^1\| &\leqq \sum_{k=1}^{n} (b_{ik} + c_{ik})\|\mathring{y}_k\| + c\|\Gamma_i(x_i)y_i^1\| + q \\ &\leqq \sum_{k=1}^{n} a_{ik}\|\mathring{y}_k\| + cB\|y_i^1\| + q \, , \quad \text{by (3.3)} \end{aligned}$$

$$\begin{aligned} (1 - cB)\|y_i^1\| &\leqq \sum_{k=1}^{n} a_{ik}r_k + q \, , \quad \text{by our assumption} \\ &\leqq qr_i + q \, , \quad \text{by (3.2)} \\ &\leqq 2qr_i \, , \quad \text{as} \quad r_i \geqq 1 \, , \quad i = 1, \cdots, n \quad \text{and} \quad 0 < q < 1 \, . \end{aligned}$$

Hence

$$\|y_i^1\| \leqq \frac{2q}{(1 - cB)} r_i \, , \quad i = 1, \cdots, n \, . \tag{3.9}$$

We shall now construct inductively sequences $\{x_i^m\}$ and $\{y_i^m\}$ $i = 1, \cdots, n$ such that

$$\text{(3.10) (a)} \qquad (x_1^m, \cdots, x_n^m) \in D \, , \quad y_i^m \in P_i(x_1^m, \cdots, x_n^m)$$

$$\text{(3.10) (b)} \qquad \|y_i^m\| \leqq (m + 1)\left(\frac{q}{1 - cB}\right)^m r_i \, , \quad i = 1, \cdots, n \, .$$

For $m = 1$, the above hypotheses are true, in view of (3.4)(a), (3.6) and (3.9). Assume the truth of (3.10)(a), (3.10)(b) for $m - 1 \in N$, i.e.,

$$(3.11)\quad (a)\qquad (x_1^{m-1}, \cdots, x_n^{m-1}) \in D\,,\quad y_i^{m-1} \in P_i(x_1^{m-1}, \cdots, x_n^{m-1})\,,$$

$$(3.11)\quad (b)\qquad \|y_i^{m-1}\| \leqq m\left(\frac{q}{1-cB}\right)^{m-1} r_i\,,\quad i = 1, \cdots, n\,.$$

Define

$$(3.12)\qquad x_i^m = x_i^{m-1} - \Gamma_i(x_i^m) y_i^{m-1}\,,\quad i = 1, \cdots, n\,.$$

In (3.4)(b), replacing x_i by x_i^{m-1}, y_i by $-y_i^{m-1}$ and using (3.12) we have

$$\begin{aligned}(3.13)\quad & H_i[P_i(x_1^m, \cdots, x_n^m), P_i(x_1^{m-1}, \cdots, x_n^{m-1}) - y_i^{m-1}] \\ & \leqq \sum_{k=1}^n b_{ik}\|y_k^{m-1}\| + \sum_{k=1}^n c_{ik} D_k[-y_k^{m-1}, -y_k^{m-1} - P_k(x_1^{m-1}, \cdots, x_n^{m-1})] \\ & \quad + cD_i[-y_i^{m-1}, -y_i^{m-1} - \Gamma_i(x_i^{m-1}) P_i(x_1^m, \cdots, x_n^m)] \\ & \leqq \sum_{k=1}^n b_{ik}\|y_k^{m-1}\| + \sum_{k=1}^n c_{ik}\|y_k^{m-1}\| \\ & \quad + cD_i[\theta_i, \Gamma_i(x_i^{m-1}) P_i(x_1^m, \cdots, x_n^m)]\,.\end{aligned}$$

Since $\theta_i \in P_i(x_1^{m-1}, \cdots, x_n^{m-1}) - y_i^{m-1}$, given $q/(1-cB)^{m-1} > 0$, there exists $y_i^m \in P_i(x_1^m, \cdots, x_n^m)$ such that

$$(3.14)\qquad \|y_i^m - \theta_i\| \leqq H_i[P_i(x_1^m, \cdots, x_n^m), P_i(x_1^{m-1}, \cdots, x_n^{m-1}) - y_i^{m-1}] + \frac{q^m}{(1-cB)^{m-1}}\,.$$

From (3.13) and (3.14) we get

$$\|y_i^m\| \leqq \sum_{k=1}^n (b_{ik} + c_{ik})\|y_k^{m-1}\| + cB\|y_i^m\| + \frac{q^m}{(1-cB)^{m-1}}$$

$$\begin{aligned}(1-cB)\|y_i^m\| & \leqq m\left(\frac{q}{1-cB}\right)^{m-1} \sum_{k=1}^n a_{ik} r_k + \frac{q^m}{(1-cB)^{m-1}} \\ & \leqq m a r_i \left(\frac{q}{1-cB}\right)^{m-1} + \frac{q^m}{(1-cB)^{m-1}}\,,\quad \text{by (3.2)}\end{aligned}$$

i.e.,

$$\begin{aligned}\|y_i^m\| & \leqq m\left(\frac{q}{1-cB}\right)^m r_i + \left(\frac{q}{1-cB}\right)^m r_i \\ & \leqq (m+1)\left(\frac{q}{1-cB}\right)^m r_i\,,\quad \text{as}\quad r_i \geqq 1\,,\quad i = 1, \cdots, n\,.\end{aligned}$$

Hence by induction (3.10)(b) holds for all $m = 0, 1, 2, \cdots$. From (3.12), and (3.4)(a), it follows that $(x_1^m, \cdots, x_n^m) \in D$. By construction, $y_i^m \in P_i(x_1^m, \cdots, x_n^m)$. Hence by induction (3.10)(a) holds. By (3.5), $0 \leqq q/(1-cB) < 1$ and it follows from (3.10)(b) that $y_i^m \to \theta_i$, as $m \to \infty$, $i = 1, \cdots, n$. From (3.12),

$$\begin{aligned}\|x_i^{m+1} - x_i^m\| &\leqq \|\Gamma_i(x_i^m)y_i^m\| \\ &\leqq B(m+1)\left(\frac{q}{1-cB}\right)^m r_i .\end{aligned}$$

Hence $\{x_i^m\}$ is a Cauchy sequence in X_i, $i = 1, \cdots, n$. Therefore, $x_i^m \to x_i$, $i = 1, \cdots, n$. As the operator P_i is closed, $y_i^m \in P_i(x_1^m, \cdots, x_n^m)$, $y_i^m \to \theta_i$, $x_i^m \to x_i$, $i = 1, \cdots, n$, imply that $(\theta_1, \cdots, \theta_n) \in D$ and $\theta_i \in P(x_1, \cdots, x_n)$, $i = 1, \cdots, n$. □

We now deduce Czerwik's Theorem [4] as a corollary in the set-up of Banach spaces. Czerwik proved his theorem for multivalued mappings on complete metric spaces.

THEOREM 3.2 [Theorem, [4]]. *Let X_i, $i = 1, \cdots, n$ be Banach spaces and b_{ik}, $c_{ik} \geqq 0$ for $i, k = 1, \cdots, n$. Let $a_{ik} = b_{ik} + c_{ik}$, $i, k = 1, \cdots, n$ be positive and let the numbers a_{ik}^l defined by* (2.5) *and* (2.6) *fulfill* (2.8). *Suppose that the transformations $F_i: X_1 \times \cdots \times X_n \to \mathrm{CL}(X_i)$, $i = 1, \cdots, n$ fulfill*

$$\begin{aligned}(3.15)\qquad H_i[F_i(x_1, \cdots, x_n), F_i(z_1, \cdots, z_n)] &\leqq \sum_{k=1}^n b_{ik}\|x_k - z_k\| + \sum_{k=1}^n c_{ik} D_k[x_k, F_k(x_1, \cdots, x_n)] \\ &\quad + cD_i[z_i, F_i(z_1, \cdots, z_n)], \quad i = 1, \cdots, n ;\end{aligned}$$

for $x_j, z_j \in X_j$, $j = 1, \cdots, n$, where c fulfills the condition $0 \leqq c < 1 - q$, q being defined by

$$q = \max_i \left(r_i^{-1} \sum_{k=1}^n a_{ik} r_k\right) .$$

Then the system $(F_1, \cdots, F_n)$ has a fixed point, i.e., *there exist points $x_i \in X_i$, $i = 1, \cdots, n$ such that $x_i \in F_i(x_1, \cdots, x_n)$, $i = 1, \cdots, n$.*

Proof. That the above theorem follows as a corollary for the Theorem 3.1 can be seen if we put $\Gamma_i(x_i) = I$, $(x_i \in X_i)$, the identity operator on X_i and $P_i(x_1, \cdots, x_n) = x_i - F_i(x_1, \cdots, x_n)$ $i = 1, \cdots, n$ in (3.4)(b) and observe that it reduces to

$$\begin{aligned}H_i[x_i + y_i - F_i(x_1 + y_1, \cdots, x_n + y_n), x_i + y_i - F_i(x_1, \cdots, x_n)] \\ \leqq \sum_{k=1}^n b_{ik}\|y_k\| + \sum_{k=1}^n c_{ik} D_k[y_k, y_k - x_k + F_k(x_1, \cdots, x_n)] \\ + cD_i[x_i + y_i, x_i + y_i - (x_i + y_i) + F_i(x_1 + y_1, \cdots, x_n + y_n)]\end{aligned}$$

i.e.,

$$\begin{aligned}H_i[F_i(x_1 + y_1, \cdots, x_n + y_n), F_i(x_1, \cdots, x_n)] \\ \leqq \sum_{k=1}^n b_{ik}\|y_k\| + \sum_{k=1}^n c_{ik}[y_k, y_k - x_k + F_k(x_1, \cdots, x_n)] \\ + cD_i[x_i + y_i, F_i(x_1 + y_1, \cdots, x_n + y_n)] .\end{aligned}$$

Taking $x_i + y_i = z_i$, $i = 1, \cdots, n$, we have

$$\begin{aligned} H_i[F_i(x_1, \cdots, x_n), F_i(z_1, \cdots, z_n)] &\leqq \sum_{k=1}^{n} b_{ik}\|x_k - z_k\| + \sum_{k=1}^{n} c_{ik}D_k[x_k, F_k(x_1, \cdots, x_n)] \\ &\quad + cD_i[z_i, F_i(z_1, \cdots, z_n)] , \end{aligned}$$

which is nothing but condition (3.15). It can be similarly shown that (3.15) implies (3.4)(b) in this case. To prove that the operator F_i, $i = 1, \cdots, n$ is closed in the sense of Definition 3.1, observe that we have shown in the proof of Theorem 3.1 that $x_i^m \to x_i$ and $y_i^m \in P_i(x_1^m, \cdots, x_n^m)$, i.e., $y_i^m \in x_i^m - F_i(x_1^m, \cdots, x_n^m)$ $i = 1, \cdots, n$, and $y_i^m \to 0$ as $m \to \infty$, i.e., $x_i^m - y_i^m \to x_i$, $i = 1, \cdots, n$. It remains to show that $x_i \in F_i(x_1, \cdots, x_n)$, $i = 1, \cdots, n$.

$$\begin{aligned} &D_i[x_i, F_i(x_1, \cdots, x_n)] \\ &\leqq \|x_i - x_i^m\| + D_i[x_i^m, F_i(x_1, \cdots, x_n)] \\ &\leqq \|x_i - x_i^m\| + H_i[F_i(x_1^{m-1}, \cdots, x_n^{m-1}), F_i(x_1, \cdots, x_n)] \\ &\leqq \|x_i - x_i^m\| + \sum_{k=1}^{n} b_{ik}\|x_k^{m-1} - x_k\| \\ &\quad + \sum_{k=1}^{n} c_{ik}D_k[x_k^{m-1}, F_k(x_1^{m-1}, \cdots, x_n^{m-1})] + cD_i[x_i, F_i(x_1, \cdots, x_n)] \\ &\leqq \|x_i - x_i^m\| + \sum_{k=1}^{n} b_{ik}\|x_k^{m-1} - x_k\| \\ &\quad + \sum_{k=1}^{n} c_{ik}\|x_k^{m-1} - x_k^m\| + cD_i[x_i, F_i(x_1, \cdots, x_n)] . \end{aligned}$$

As $0 < c < 1$ and

$$\begin{aligned} &D_i[x_i, F_i(x_1, \cdots, x_n)] \\ &\leqq \frac{1}{1-c}\left[\|x_i - x_i^m\| + \sum_{k=1}^{n} b_{ik}\|x_k^{m-1} - x_k\| + \sum_{k=1}^{n} c_{ik}\|x_k^{m-1} - x_k^m\|\right] \end{aligned}$$

it follows that $D_i[x_i, F_i(x_1, \cdots, x_n)] = 0$. Since $F_i(x_1, \cdots, x_n)$ is a closed set, $x_i \in F_i(x_1, \cdots, x_n)$. □

Theorem 3.3 below is a generalization of Matkowski's Theorem 2.1 to multivalued mappings.

THEOREM 3.3. *Let X_i, $i = 1, \cdots, n$ be Banach spaces and a_{ik} $i, k = 1, \cdots, n$ be positive and a_{ik}^l be defined by* (2.5) *and* (2.6) *and fulfill* (2.8). *Suppose that the transformations $F_i: X_1 \times \cdots \times X_n \to$ CL (X_i), $i = 1, \cdots, n$ satisfy*

$$H_i[F_i(x_1, \cdots, x_n), F_i(z_1, \cdots, z_n)] \leqq \sum_{k=1}^{n} a_{ik}\|x_k - z_k\|$$

for all $x_j, z_j \in X_j$, $j = 1, \cdots, n$. *Then the operator* $F = (F_1, \cdots, F_n)$ *has a fixed point,* i.e., *there exist points* $x_i \in X_i$, *such that* $x_i \in F_i(x_1, \cdots, x_n)$ *for all* $i = 1, \cdots, n$.

Proof. In Theorem 3.2, let $c_{ik} = 0$, $b_{ik} = a_{ik}$, $i, k = 1, \cdots, n$ and $c = 0$. □

The following is the multivalued version of Altman's Theorem 2.2.

THEOREM 3.4. *Suppose a nonlinear closed operator* $P: D(P) \subset X \to \mathrm{CL}(Y)$ *has a bounded contractor* Γ *satisfying*

$$(3.16)\quad (a)\quad x + \Gamma(x)y \in D(P),\ \textit{whenever}\ x \in D(P),\ y \in Y,\ D(P)\ \textit{being the domain of}\ P;$$

$$(3.16)\quad (b)\quad H[P(x + \Gamma(x)y), Px + y] \leqq q\|y\|,\quad x \in D(P),\quad y \in Y,\quad 0 < q < 1;$$

$$(3.17)\qquad \|\Gamma(x)\| \leqq B,\quad x \in D(P).$$

Then there exists $x \in D(P)$ *such that* $\theta \in Px$, *where* θ *is the zero element of* Y.

Proof. For $n = 1$, Theorem 3.1 reduces to the above theorem for the choice $c_{ik} = 0$, $b_{ik} = a_{ik}$, $i, k = 1, \cdots, n$ i.e., $b_{11} = a_{11} = q < 1$, and $c = 0$. □

Besides, Theorem 3.1 yields as corollaries several fixed point theorems for single-valued mappings including the following theorem proved elsewhere (Theorem 2.1, [2]).

THEOREM 3.5. *Let* X_i, Y_i, $i = 1, \cdots, n$ *be Banach spaces and* $T_i: D \subset X_1 \times \cdots \times X_n \to Y_i$, $i = 1, \cdots, n$ *be closed non-linear operators. Suppose that there exist bounded linear operators* $\Gamma_i(x_i): Y_i \to X_i$, $i = 1, \cdots, n$ *such that*

$$(3.18)\quad (a)\quad (x_1 + \Gamma_1(x_1)y_1, \cdots, x_n + \Gamma_n(x_n)y_n) \in D \quad \textit{whenever}\ (x_1, \cdots, x_n) \in D,\quad y_i \in Y_i,\quad i = 1, \cdots, n;$$

$$(3.18)\quad (b)\quad \|T_i(x_1 + \Gamma_1(x_1)y_1, \cdots, x_n + \Gamma_n(x_n)y_n) - T_i(x_1, \cdots, x_n) - y_i\| \leqq \sum_{k=1}^{n} a_{ik}\|y_k\|,$$

where the nonnegative numbers a_{ik}, $i, k = 1, \cdots, n$ *are defined by* (2.5) *and* (2.6) *and fulfill* (2.8);

(3.19) $$\|\Gamma_i(x_i)\| \leqq B\,, \quad i = 1, \cdots, n\,, \quad (x_i, \cdots, x_n) \in D\,.$$

Then the system of operator equations

(3.20) $$T_i(x_1, \cdots, x_n) = y_i\,, \quad i = 1, \cdots, n$$

has a solution in D for arbitrary $y_i \in Y_i$, $i = 1, \cdots, n$.

Proof. In Theorem 3.1, let $c_{ik} = 0$, $b_{ik} = a_{ik}$, $i, k = 1, \cdots, n$ and $c = 0$. Define $P_i(x_1, \cdots, x_n) = \{T_i(x_1, \cdots, x_n)\}$ for $(x_1, \cdots, x_n) \in D$, $i = 1, \cdots, n$. Clearly the assumptions of Theorem 3.1 are satisfied and hence the system of equations (3.20) has a solution in D. □

The above Theorem proved in [2] unified, in the setting of Banach spaces, Altman's extension of the contraction principle and Matkowski's fixed point theorem.

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