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REALIZING AUTOMORPHISMS OF QUOTIENTS OF PRODUCT σ-FIELDS

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Let $(X_\alpha)_{\alpha\in I}$ be a family of Polish spaces, $X = \prod_{\alpha\in I} X_\alpha$, and $\mathfrak{B}$ the product of the Borel fields of the spaces X_α. For $K\subset I$ let $X_K = \prod_{\alpha\in K} X_\alpha$ and let $\pi_K: X \to X_K$ be the canonical projection. Moreover, let $\mathfrak{n}$ be a σ-ideal in $\mathfrak{B}$ satisfying the following Fubini type condition:
$N\in\mathfrak{n}$ if and only if $\pi_J^{-1}(\{z\in X_J \mid \pi_{I\setminus J}^{-1}(\{y\in X_{I\setminus J} \mid (z, y)\in N\})\notin\mathfrak{n}\})\in\mathfrak{n}$ for every nonempty $J\subset I$. Then, given an automorphism Φ from $\mathfrak{B}/\mathfrak{n}$ onto itself, there exists a bijection $f: X\to X$ such that f and f^{-1} are measurable and

$$[f^{-1}(B)] = \Phi([B]) , \qquad [f(B)] = \Phi^{-1}([B])$$

for all $B\in\mathfrak{B}$.

1. **Introduction.** Let $(X_\alpha)_{\alpha\in I}$ be an arbitrary family of Polish spaces and, for every $\alpha\in I$, μ_α a Borel measure on X_α. Let $X = \prod_{\alpha\in I} X_\alpha$ be equipped with the Baire σ-field $\mathfrak{B}(X)$ which is equal to the product of the Borel fields of the spaces X_α. Moreover, let μ be the product measure on $\mathfrak{B}(X)$ and $\mathfrak{n}$ the σ-ideal of μ-nullsets. D. Maharam [5] showed that every automorphism of $\mathfrak{B}(X)/\mathfrak{n}$ onto itself is induced by an invertible $\mathfrak{B}(X)$-measurable point mapping of X. In [6] D. Maharam proved the same result in the case that $\mathfrak{n}$ is the σ-ideal of first category sets in $\mathfrak{B}(X)$. It is the purpose of this note to give a common generalization of these two results: We shall show that for σ-ideals $\mathfrak{n}$ in $\mathfrak{B}(X)$ which satisfy a certain Fubini type condition the conclusions of Maharam's theorems still hold.

Choksi [1], [2] generalized Maharam's first result to arbitrary Baire measures on $X = \prod X_\alpha$. Our methods of proof consist in a slight modification of those used by Choksi [2] (cf. also Choksi [3]). We shall formulate our lemmas in such a way that we can also reprove Choksi's theorem.

Our basic tool in the proofs of the results stated above consists in the following generalization of a theorem due to Sikorski (cf. [8], p. 139, 32.5): Each σ-homomorphism from $\mathfrak{B}(\prod X_\alpha)$ to an arbitrary quotient of a σ-field on any set Y (w.r.t. a σ-ideal) is induced by a measurable map from Y to $X = \prod X_\alpha$.

This last result is also used to deduce a characterization of injective measurable spaces first given by Falkner [4] (cf. §3).

2. **Notation.** In what follows $(X_\alpha)_{\alpha\in I}$ is always a family of Polish spaces. For a subset J of I let X_J stand for $\prod_{\alpha\in J} X_\alpha$ and X

for X_I. For $K \subset J \subset I$ let π_{JK} denote the canonical projection from X_J onto X_K. If $J = I$ we write π_K instead of π_{JK}. For an arbitrary completely regular Hausdorff space Y let $\mathfrak{B}(Y)$ denote the σ-field of Baire sets in Y. We will write $\mathfrak{B}$ for $\mathfrak{B}(X)$. $\mathfrak{B}$ is equal to the product σ-field of the Borel fields $\mathfrak{B}(X_\alpha)$. A map $f\colon X \to X$ is called measurable if it is $\mathfrak{B}$-$\mathfrak{B}$-measurable.

3. Realizing σ-homomorphisms. The following theorem is a generalization of a result due to Sikorski (cf. [8], p. 139, 32.5) and provides the basic tool for deriving the results in the later sections.

THEOREM 3.1. *Let $X = \prod X_\alpha$, $\mathfrak{B} = \mathfrak{B}(X)$. Moreover let $(Y, \mathfrak{A})$ be a measurable space, $\mathfrak{n}$ a σ-ideal in $\mathfrak{A}$, and $\Phi\colon \mathfrak{B} \to \mathfrak{A}/\mathfrak{n}$ a σ-homomorphism. Then there exists an $\mathfrak{A}$-$\mathfrak{B}$-measurable map $f\colon Y \to X$ with $f^{-1}(B) \in \Phi(B)$ for all $B \in \mathfrak{B}$,* i.e. *Φ is induced by f.*

Proof. For every $\alpha \in I$ define $\Phi_\alpha\colon \mathfrak{B}(X_\alpha) \to \mathfrak{A}/\mathfrak{n}$ by $\Phi_\alpha(B) = \Phi(\pi_\alpha^{-1}(B))$. Then Φ_α is obviously a σ-homomorphism. It follows from Sikorski [8], p. 139, 32.5 that there exists an $\mathfrak{A}$-$\mathfrak{B}(X_\alpha)$-measurable map $f_\alpha\colon Y \to X_\alpha$ with $f_\alpha^{-1}(B) \in \Phi_\alpha(B)$ for all $B \in \mathfrak{B}(X_\alpha)$. Define $f\colon Y \to X$ by $f(y) = (f_\alpha(y))_{\alpha \in I}$. Then f is $\mathfrak{A}$-$\mathfrak{B}$-measurable and for every $B \in \mathfrak{B}$ with $B = \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(B_{\alpha_i})$, $B_{\alpha_i} \in \mathfrak{B}(X_{\alpha_i})$ one has $f^{-1}(B) = \bigcap_{i=1}^n f_{\alpha_i}^{-1}(B_{\alpha_i})$. Since $f_{\alpha_i}^{-1}(B_{\alpha_i}) \in \Phi_{\alpha_i}(B_{\alpha_i}) = \Phi(\pi_{\alpha_i}^{-1}(B_{\alpha_i}))$ we deduce

$$\bigcap_{i=1}^n f_{\alpha_i}^{-1}(B_{\alpha_i}) \in \Phi\Big(\bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(B_{\alpha_i})\Big) = \Phi(B) ,$$

hence

$$f^{-1}(B) \in \Phi(B) .$$

Since the sets of the above form generate $\mathfrak{B}$ as a σ-field and since Φ is a σ-homomorphism it follows that $f^{-1}(B) \in \Phi(B)$ for all $B \in \mathfrak{B}$.

Before we shall go on with our main subject let us use the above theorem to derive a characterization of injective measurable spaces. Essentially the same characterization has been given first by Falkner [4]. It is also possible to deduce Theorem 3.1 from Falkner's results.

DEFINITION 3.2.

(a) A measurable space $(Z, \mathfrak{C})$ is called *separated* iff for all $z, z' \in Z$ with $z \neq z'$ there exists a set $C \in \mathfrak{C}$ with $z \in C$ and $z' \notin C$.

(b) Two measurable spaces $(Y, \mathfrak{A})$ and $(Z, \mathfrak{C})$ are called *point-isomorphic* iff there exists a bijection g from Y onto Z such that g and g^{-1} are measurable. g is called a *point-isomorphism.*

(c) A measurable space $(Y, \mathfrak{A})$ is called a *retract* of a measurable space $(Z, \mathfrak{C})$ iff there exists a subset Z_0 of Z and an $\mathfrak{A}$-$\mathfrak{C} \cap Z_0$-measurable map $h: Z \rightarrow Z_0$ with $h_{|Z_0} = id_{Z_0}$ such that $(Y, \mathfrak{A})$ is point-isomorphic to $(Z_0, \mathfrak{C} \cap Z_0)$, where $\mathfrak{C} \cap Z_0 = \{C \cap Z_0 | C \in \mathfrak{C}\}$.

(d) A measurable space $(Z, \mathfrak{C})$ is called *injective* iff for every measurable space $(Y, \mathfrak{A})$, for every subset $Y_0 \subset Y$, for every $\mathfrak{A} \cap Y_0 -$ $\mathfrak{C}$-measurable map $f: Y_0 \rightarrow Z$ there exists an $\mathfrak{A}$-$\mathfrak{C}$-measurable map $\tilde{f}: Y \rightarrow Z$ with $\tilde{f}_{|Y_0} = f$.

LEMMA 3.3. *Let $(Z, \mathfrak{C})$ be a separated measurable space and let $\mathfrak{E}$ be a subset of $\mathfrak{C}$ generating $\mathfrak{C}$ as a σ-field. Then there exists a set $B \subset [0, 1]^{\mathfrak{E}}$ such that $(Z, \mathfrak{C})$ is point-isomorphic to $(B, \mathfrak{B}([0, 1]^{\mathfrak{E}}) \cap B)$.*

Proof. Define $g: Z \rightarrow [0, 1]^{\mathfrak{E}}$ by $g(z) = (1_E(z))_{E \in \mathfrak{E}}$. Then g is $\mathfrak{C} - \mathfrak{B}([0, 1]^{\mathfrak{E}})$-measurable and one-to-one. Let $B = g(Z)$. For $E_0 \in \mathfrak{E}$ we have $g(E_0) = \{(s_E)_{E \in \mathfrak{E}} \in g(Z) | s_{E_0} = 1\}$, hence $g(E_0) \in \mathfrak{B}([0, 1]^{\mathfrak{E}}) \cap B$, which proves g to be a point-isomorphism of $(Z, \mathfrak{C})$ and $(B, \mathfrak{B}([0, 1]^{\mathfrak{E}}) \cap B)$.

REMARK 3.4. Let I be an index set and $\emptyset \neq B \in \mathfrak{B}([0, 1]^I)$. Then $(B, \mathfrak{B}([0, 1]^I) \cap B)$ is a retract of $([0, 1]^I, \mathfrak{B}([0, 1]^I))$.

Proof. Let $x_0 \in B$ be given. Define $h: [0, 1]^I \rightarrow B$ by

$$h(x) = \begin{cases} x, & x \in B \\ x_0, & x \notin B . \end{cases}$$

Then h is measurable and $h_{|B} = id_B$.

It remains an open question whether every retract of $([0, 1]^I, \mathfrak{B}([0, 1]^I))$ is point-isomorphic to a Baire subset of some generalized cube $[0, 1]^K$. (For $K = I$ this is not true in general.)

COROLLARY 3.5. (cf. *Falkner* [4], *Theorem* 3.2.) *For a separated measurable space $(Z, \mathfrak{C})$ the following conditions are equivalent:*

(i) *$(Z, \mathfrak{C})$ is injective.*

(ii) *There exists an index set I such that $(Z, \mathfrak{C})$ is a retract of* $([0, 1]^I, \mathfrak{B}([0, 1]^I))$.

(iii) *For every measurable space $(Y, \mathfrak{A})$ and every σ-ideal $\mathfrak{n}$ of $\mathfrak{A}$ each σ-homomorphism $\Phi: \mathfrak{C} \rightarrow \mathfrak{A}/\mathfrak{n}$ is induced by an $\mathfrak{A}$-$\mathfrak{C}$-measurable map $f: Y \rightarrow Z$.*

If $(Z, \mathfrak{C})$ is countably generated, in addition, then the conditions (i) *to* (iii) *are also equivalent to*

(iv) *$(Z, \mathfrak{C})$ is point-isomorphic to $(B, \mathfrak{B}([0, 1]^N) \cap B)$ for some $B \in \mathfrak{B}([0, 1]^N)$.*

Proof. (i) $\Rightarrow$ (ii): According to Lemma 3.3 we may assume $Z \subset [0, 1]^I$ and $\mathfrak{C} = \mathfrak{B}([0, 1]^I) \cap Z$ for some I. Let $f = id_Z$. Since $(Z, \mathfrak{C})$ is injective there exists a $\mathfrak{B}([0, 1]^I) - \mathfrak{C}$-measurable map $\tilde{f}\colon [0, 1]^I \to Z$ with $\tilde{f}_{|Z} = id_Z$. Hence $(Z, \mathscr{C})$ satisfies condition (ii).

(ii) $\Rightarrow$ (iii): Without loss of generality we may assume that $Z \subset [0, 1]^I$, $\mathfrak{C} = \mathfrak{B}([0, 1]^I) \cap Z$, and that there is a $\mathfrak{B}([0, 1]^I) - \mathfrak{C}$-measurable map $h\colon [0, 1]^I \to Z$ with $h_{|Z} = id_Z$. Let $(Y, \mathfrak{A})$ be any measurable space, $\mathfrak{n} \subset \mathfrak{A}$ a σ-ideal, $\Phi\colon \mathfrak{C} \to \mathfrak{A}/\mathfrak{n}$ a σ-homomorphism. Define $\Phi_0\colon \mathfrak{B}([0, 1]^I) \to \mathfrak{A}/\mathfrak{n}$ by $\Phi_0(B \cap Z)$. Then Φ_0 is a σ-homomorphism and according to Theorem 3.1 there exists an $\mathfrak{A} - \mathfrak{B}([0, 1]^I)$-measurable map $f_0\colon Y \to [0, 1]^I$ which induces Φ. Let $f = h \circ f_0$. Then f is $\mathfrak{A} - \mathfrak{C}$-measurable and obviously induces Φ.

(iii) $\Rightarrow$ (i): Let $(Y, \mathfrak{A})$ be any measurable space, $Y_0 \subset Y$, and $f\colon Y_0 \to Z$ an $\mathfrak{A} \cap Y_0 - \mathfrak{C}$-measurable map. Let $\mathfrak{n} = \{A \in \mathfrak{A}\colon A \cap Y_0 = \emptyset\}$. Then $\mathfrak{n}$ is a σ-ideal in $\mathfrak{A}$. Define $\Phi(C)$ to be the residual class in $\mathfrak{A}/\mathfrak{n}$ of any $A \in \mathfrak{A}$ with $A \cap Y_0 = f^{-1}(C)$. Then $\Phi\colon \mathfrak{C} \to \mathfrak{A}/\mathfrak{n}$ is a σ-homomorphism. According to (iii) there exists an $\mathfrak{A} - \mathfrak{C}$-measurable map $\tilde{f}\colon Y \to Z$ which induces Φ. From the definition of Φ it follows immediately that $\tilde{f}_{|Y_0} = f$.

Now let $(Z, \mathfrak{C})$ be countably generated.

(ii) $\Rightarrow$ (iv): Without loss of generality we may assume that $Z \subset [0, 1]^N$, $\mathfrak{C} = \mathfrak{B}([0, 1]^N) \cap Z$, and that there is a $\mathfrak{B}([0, 1]^N) - \mathfrak{C}$-measurable map $h\colon [0, 1]^N \to Z$ with $h_{|Z} = id_Z$ (cf. Lemma 3.3 and the proof of (i) $\Rightarrow$ (ii)). $\mathfrak{B}([0, 1]^N)$ has a countable subset $\mathfrak{E}$ such that for all $x, x' \in [0, 1]^N$ there exists an $E \in \mathfrak{E}$ with $x \in E$ and $x' \notin E$. For $x \in [0, 1]^N \backslash Z$ there, therefore, exists an $E \in \mathfrak{E}$ with $x \in E$ and $h(x) \notin E$. Since $h_{|Z} = id_Z$ we deduce $x \in E \backslash h^{-1}(E) \subset [0, 1]^N \backslash Z$, hence $[0, 1]^N \backslash Z = \bigcup_{E \in \mathfrak{E}} E \backslash h^{-1}(E)$ belongs to $\mathfrak{B}([0, 1]^N)$.

(iv) $\Rightarrow$ (ii) follows immediately from Remark 3.4.

4. Realizing automorphisms. In this section $\mathfrak{n}$ is always a σ-ideal in $\mathfrak{B}(X)$, $X = \prod X_\alpha$. For $B \in \mathfrak{B}(X)$ the symbol $[B]$ stands for the residual class of B in $\mathfrak{B}(X)/\mathfrak{n}$. We say that a subset B of X depends only on $J \subset I$ if $B = \pi_J^{-1}(\pi_J(B))$. It is a well-known fact that every $B \in \mathfrak{B}(X)$ depends only on a countable subset of I.

DEFINITION 4.1.

(a) $\mathfrak{n}$ is said to satisfy condition (F) iff a set $N \in \mathfrak{B}(X)$ belongs to $\mathfrak{n}$ if and only if for every nonempty $J \subset I$

$$\pi_J^{-1}(\{z \in X_J \,|\, \pi_{I\backslash J}^{-1}(\{y \in X_{I\backslash J} \,|\, (z, y) \in N\}) \notin \mathfrak{n}\}) \in \mathfrak{n} .$$

(b) $\mathfrak{n}$ is said to satisfy condition (D) iff for all countable nonempty $J_1, J_2 \subset I$ with $J_1 \cap J_2 = \emptyset$ there exists an $N \in \mathfrak{n}$ such that N depends only on $J_1 \cup J_2$ and, for all $z \in X_{J_1}$, $\pi_{J_1 \cup J_2, J_1}^{-1}(z) \cap \pi_{J_1 \cup J_2}(N)$ is uncountable and of second category in $\pi^{-1}_{J_1 \cup J_2, J_1}(z)$.

REMARK 4.2.

(1) For every $\alpha \in I$ let μ_α be a finite measure on $\mathfrak{B}(X_\alpha)$. Let μ be the product measure on $\mathfrak{B}(X)$ obtained from the μ_α's and let $\mathfrak{n}$ be the σ-ideal of μ-nullsets. Then it follows from Fubini's theorem that $\mathfrak{n}$ satisfies condition (F).

(2) Let $\mathfrak{n}$ be the σ-ideal of all sets of first category in $\mathfrak{B}(X)$. Then $\mathfrak{n}$ satisfies condition (F). This is a consequence of Theorem 1 in [6].

(3) If there exists a σ-ideal $\mathfrak{n}$ in $\mathfrak{B}$ satisfying condition (D) then each of the X_α's has to be uncountable.

(4) Let μ be a σ-finite measure on $\mathfrak{B}(X)$ and $\mathfrak{n}$ the σ-ideal of μ-nullsets. If each X_α is uncountable then $\mathfrak{n}$ satisfies condition (D). This follows from Lemma B (and proof) in [2].

Let us now state our main theorem.

THEOREM 4.3. *Let* $\mathfrak{n}$ *be a* σ*-ideal in* $\mathfrak{B}(X) = \mathfrak{B}(\prod X_\alpha)$ *satisfying conditon* (F) or (D). *Let* Φ *be an automorphism of* $\mathfrak{B}(X)/\mathfrak{n}$ *onto itself. Then there exists a bijection* $f\colon X \to X$ *such that* f *and* f^{-1} *are measurable and* $[f^{-1}(B)] = \Phi([B])$, $[f(B)] = \Phi^{-1}([B])$ *for all* $B \in \mathfrak{B}(X)$.

The ingredients of the proof will be provided by a series of lemmas. Let us first make the following definition:

Given a measurable map $g\colon X \to X$ a subset J of I is called g-invariant iff, for all $x, y \in X$, the identity $\pi_J(x) = \pi_J(y)$ implies $\pi_J(g(x)) = \pi_J(g(y))$.

LEMMA 4.4. *Let* $g, h\colon X \to X$ *be measurable mappings. Then, for every countable* $J_0 \subset I$, *there exists a countable set* $J \subset I$ *which contains* J_0 *and is* h- *and* g-*invariant.*

Proof. Let $\mathscr{B}_0$ be a countable base for the topology of X_{J_0}. For $B \in \mathscr{B}_0$ let $J(B)$ be the smallest subset J of I such that $\pi^{-1}_{J_0}(B)$, $g^{-1}(\pi^{-1}_{J_0}(B))$, and $h^{-1}(\pi^{-1}_{J_0}(B))$ depend only on J. Then $J(B)$ is countable. Define $J_1 = \cup\{J(B) \mid B \in \mathscr{B}_0\}$ and let $\mathscr{B}_1$ be a countable base for the topology of X_{J_1}. Then one constructs J_2 from $\mathscr{B}_1$ as J_1 has been constructed from $\mathscr{B}_0$. Continuing this process we get an increasing sequence (J_n) of subsets of I and, for each $n \in N$, a countable base $\mathscr{B}_n$ for the topology of X_{J_n}. Let $J = \bigcup_{n \in N} J_n$. Then J is at most countable and $J_0 \subset J$. We shall show that J is g- and h-invariant. To this end let $x, y \in X$ be such that $\pi_J(x) = \pi_J(y)$. Assume $\pi_J g(x) \neq \pi_J g(y)$. Then there is a $k \in N$ with $\pi_{J_k} g(x) \neq \pi_{J_k} g(y)$. Hence there exists a $\mathscr{B}_k$ with $\pi_{J_k} g(x) \in B$ and $\pi_{J_k} g(y) \notin B$ which implies $x \in g^{-1} \pi_{J_k}^{-1}(B)$ and $y \notin g^{-1} \pi_{J_k}^{-1}(B)$. Since, by definition, $g^{-1}(\pi_{J_k}^{-1}(B))$ depends only on J_{k+1} there is a $j \in J_{k+1}$ with $\pi_j(x) \neq \pi_j(y)$. But this is a contradiction since $j \in J_{k+1} \subset J$. Thus we deduce $\pi_J g(x) = \pi_J g(y)$. In the same way one shows $\pi_J h(x) = \pi_J h(y)$.

LEMMA 4.5. *Let $\mathfrak{n}$ be a σ-ideal in $\mathfrak{B}$ satisfying condition* (F). *Let $q: X \to X$ be a measurable map with $q^{-1}(N) \in \mathfrak{n}$ for all $N \in \mathfrak{n}$. Moreover, let J be a q-invariant subset of I. Define $q_J: X \to X$ by $q_J(x) = (\pi_J q(x), \pi_{I \setminus J}(x))$. Then q_J is measurable with $q_J^{-1}(N) \in \mathfrak{n}$ for all $N \in \mathfrak{n}$.*

Proof. From the definition it follows immediately that q_J is measurable. Now, let $N \in \mathfrak{n}$ be given. Since $\mathfrak{n}$ satisfies condition (F) we have

$$P: = \pi_J^{-1}(\{z \in X_J \mid \pi_{I \setminus J}^{-1}(\{y \in X_{I \setminus J} \mid (z, y) \in N\}) \notin \mathfrak{n}\}) \in \mathfrak{n} \ .$$

We will show

$$R: = \pi_J^{-1}(\{z' \in X_J \mid \pi_{I \setminus J}^{-1}(\{y' \in X_{I \setminus J} \mid (z', y') \in q_J^{-1}(N)\}) \notin \mathfrak{n}\}) \in \mathfrak{n} \ .$$

To this end let $x \in R$ be given. Then we have

$$S_x: = \pi_{I \setminus J}^{-1}(\{y' \in X_{I \setminus J} \mid (\pi_J(x), y') \in q_J^{-1}(N)\}) \notin \mathfrak{n} \ .$$

Since

$$\begin{aligned} S_x &= \pi_{I \setminus J}^{-1}(\{y' \in X_{I \setminus J} \mid q_J((\pi_J(x), y')) \in N\}) \\ &= \pi_{I \setminus J}^{-1}(\{y' \in X_{I \setminus J} \mid (\pi_J q(x), y') \in N\}) \end{aligned}$$

this implies $q(x) \in P$; hence $R \subset q^{-1}(P)$. Because of $P \in \mathfrak{n}$ and, therefore, $q^{-1}(P) \in \mathfrak{n}$, this implies $R \in \mathfrak{n}$, which, according to condition (F), leads to $q_J^{-1}(N) \in \mathfrak{n}$.

LEMMA 4.6. (cf. *Choksi* [3], p. 115.) *Let Y and Z be uncountable Polish spaces, $q: Y \to Y$ a bijection such that q and q^{-1} are $\mathfrak{B}(Y)-\mathfrak{B}(Y)$-measurable, and $B \in \mathfrak{B}(Y \times Z)$ such that for each $y \in Y$ the set $B_y = \{z \in Z \mid (y, z) \in B\}$ is uncountable and of second category in Z. Then there exists a bijection $r: B \to B$ such that r and r^{-1} are $\mathfrak{B}(Y \times Z) \cap B - \mathfrak{B}(Y \times Z) \cap B$-measurable and such that, for each $y \in Y$, $r(y, \cdot)$ maps $\{y\} \times B_y$ onto $\{q(y)\} \times B_{q(y)}$.*

Proof. According to Mauldin [7], Theorem 2.7 there exists a set $E \in \mathfrak{B}(Z)$ and a point-isomorphism g from $(Y \times E, \mathfrak{B}(Y \times Z) \cap Y \times E)$ onto $(B, \mathfrak{B}(Y \times Z) \cap B)$ such that, for each $y \in Y$, $g(y, \cdot)$ maps E onto $\{y\} \times B_y$. Define $r: B \to B$ by $r(y, z) = g(q(y'), z')$, where

$(y', z') = g^{-1}(y, z)$. Then r is a bijection and r as well as r^{-1} are $\mathfrak{B}(Y \times Z) \cap B - \mathfrak{B}(Y \times Z) \cap B$-measurable. For each $y \in Y$, $g^{-1}(y, \cdot)$ is a map from B_y onto $\{y\} \times E$, and $(y, z) \mapsto (q(y), z)$ defines a map from $\{y\} \times E$ onto $\{q(y)\} \times E$. Since g maps $\{q(y)\} \times E$ onto $\{q(y)\} \times B_{q(y)}$ we, therefore, deduce that $r(y, \cdot)$ is a map from B_y onto $\{q(y)\} \times B_{q(y)}$.

LEMMA 4.7. *Let $\mathfrak{n}$ be a σ-ideal in $\mathfrak{B}$ satisfying condition* (F) or (D). *Let $g, h: X \to X$ be measurable maps such that $g^{-1}(N) \in \mathfrak{n}$ and $h^{-1}(N) \in \mathfrak{n}$ for all $N \in \mathfrak{n}$ and such that $h^{-1}g^{-1}(B) \triangle B \in \mathfrak{n}$ as well as $g^{-1}h^{-1}(B) \triangle B \in \mathfrak{n}$ for all $B \in \mathfrak{B}$. Let $J \subset I$ be h- and g-invariant with $\pi_J \circ h \circ g = \pi_J = \pi_J \circ g \circ h$. Moreover, let $\alpha_0 \in I$ be given. Then there exist measurable maps $\tilde{g}, \tilde{h}: X \to X$ and a subset $K \subset I$ with the following properties*:

(i) $J \cup \{\alpha_0\} \subset K$

(ii) *K is $\tilde{g}$- and $\tilde{h}$-invariant.*

(iii) $\pi_K \circ \tilde{g} \circ \tilde{h} = \pi_K = \pi_K \circ \tilde{h} \circ \tilde{g}$

(iv) $\pi_J \circ \tilde{g} = \pi_J \circ g$ *and* $\pi_J \circ \tilde{h} = \pi_J \circ h$

(v) $\tilde{g}^{-1}(B) \triangle g^{-1}(B) \in \mathfrak{n}$ *and* $\tilde{h}^{-1}(B) \triangle h^{-1}(B) \in \mathfrak{n}$ *for all* $B \in \mathfrak{B}$.

Proof. According to Lemma 4.4 there exists a countable g- and h-invariant subset J_0 of I with $\alpha_0 \in J_0$. Define $K = J \cup J_0$. Then K is obviously g- and h-invariant. Define

$$N = \{x \in X \mid \pi_K \circ g \circ h(x) \neq \pi_K(x) \text{ or } \pi_K \circ h \circ g(x) \neq \pi_K(x)\} .$$

We will show $N \in \mathfrak{n}$.
Since $\pi_J \circ g \circ h = \pi_J = \pi_J \circ h \circ g$ and since K is g- and h-invariant the set N depends only on J_0. Let $\mathscr{B}$ be a countable base for the topology of X_{J_0}. Then we have

$$\begin{aligned} N &= \{x \in X \mid \exists B \in \mathscr{B}: \pi_{J_0} \circ g \circ h(x) \in B \text{ and } \pi_{J_0}(x) \notin B\} \\ &\quad \cup \{x \in X \mid \exists B \in \mathscr{B}: \pi_{J_0} \circ h \circ g(x) \in B \text{ and } \pi_{J_0}(x) \notin B\} \\ &= \bigcup_{B, B' \in \mathscr{B}} ((h^{-1}g^{-1}\pi_{J_0}^{-1}(B) \backslash \pi_{J_0}^{-1}(B)) \cup (g^{-1}h^{-1}\pi_{J_0}^{-1}(B') \backslash \pi_{J_0}^{-1}(B')) . \end{aligned}$$

Since, according to our assumptions, $h^{-1}g^{-1}\pi_{J_0}^{-1}(B) \backslash \pi_{J_0}^{-1}(B) \in \mathfrak{n}$ and $g^{-1}h^{-1}\pi_{J_0}^{-1}(B) \backslash \pi_{J_0}^{-1}(B) \in \mathfrak{n}$ we deduce $N \in \mathfrak{n}$.

Case 1. Let $\mathfrak{n}$ satisfy condition (F).
Let h_J and g_J be defined in the same way as q_J has been defined in Lemma 4.5. Define

$$N_0 = \bigcup_{m \in \mathbf{N}} \bigcup \{h_J^{-\nu_m} h^{-\lambda_m} g_J^{-\rho_m} g^{-\kappa_m} \cdots h_J^{-\nu_1} h^{-\lambda_1} g_J^{-\rho_1} g^{-\kappa_1}(N) \mid \\ \nu_1, \cdots, \nu_m, \lambda_1, \cdots, \lambda_m, \rho_1, \cdots, \rho_m, \kappa_1, \cdots, \kappa_m \in \mathbf{N}\} .$$

From Lemma 4.5 we deduce $N_0 \in \mathfrak{n}$, and it follows that $h_J^{-1}(N_0) \subset N_0$, $h^{-1}(N_0) \subset N_0$, $g_J^{-1}(N_0) \subset N_0$, and $g^{-1}(N_0) \subset N_0$.
Define $\tilde{h}: X \to X$ by

$$\tilde{h}(x) = \begin{cases} h(x), & x \notin N_0 \\ h_J(x), & x \in N_0 \end{cases}$$

and $\tilde{g}: X \to X$ by

$$\tilde{g}(x) = \begin{cases} g(x), & x \notin N_0 \\ g_J(x), & x \in N_0 . \end{cases}$$

Then $\tilde{g}$ and $\tilde{h}$ are obviously measurable.

(1) We will show that K is $\tilde{g}$- and $\tilde{h}$-invariant.
To this end let $x, y \in X$ be such that $\pi_K(x) = \pi_K(y)$. If $x \in N_0$ then there exist $\nu_1, \cdots, \nu_m, \lambda_1, \cdots, \lambda_m, \rho_1, \cdots, \rho_m, \kappa_1, \cdots, \kappa_m \in N \cup \{0\}$ with

$$g^{\kappa_1} \circ g^{\rho_1} \circ h^{\lambda_1} \circ h_J^{\nu_1} \circ \cdots \circ g^{\kappa_m} \circ g_J^{\rho_m} \circ h^{\lambda_m} \circ h_J^{\nu_m}(x) \in N .$$

Since K is g- and h-invariant it is also g_J- and h_J-invariant. This fact implies

$$\begin{aligned} &\pi_K \circ g^{\kappa_1} \circ g_J^{\rho_1} \circ h^{\lambda_1} \circ h_J^{\nu_1} \circ \cdots \circ g^{\kappa_m} \circ g_J^{\rho_m} \circ h^{\lambda_m} \circ h_J^{\nu_m}(x) \\ &\quad = \pi_K \circ g^{\kappa_1} \circ g_J^{\rho_1} \circ h^{\lambda_1} \circ h_J^{\nu_1} \circ \cdots \circ g^{\kappa_m} \circ g_J^{\rho_m} \circ h^{\lambda_m} \circ h_J^{\nu_m}(y) . \end{aligned}$$

Since N depends only on K this implies

$$g^{\kappa_1} \circ g_J^{\rho_1} \circ h^{\lambda_1} \circ h_J^{\nu_1} \circ \cdots \circ g^{\kappa_m} \circ g_J^{\rho_m} \circ h^{\lambda_m} \circ h_J^{\nu_m}(y) \in N ;$$

hence $y \in N_0$.
Since K is g_J-invariant we deduce

$$\pi_K(\tilde{g}(x)) = \pi_K(g_J(x)) = \pi_K(g_J(y)) = \pi_K(\tilde{g}(y)) .$$

If $x \notin N_0$ it follows by the same arguments that $y \notin N_0$. Hence, the g-invariance of K implies

$$\pi_K(\tilde{g}(x)) = \pi_K(g(x)) = \pi_K(g(y)) = \pi_K(\tilde{g}(y)) .$$

In the same way one can show that K is $\tilde{h}$-invariant.

(2) Next we will show that $\pi_K \circ \tilde{g} \circ \tilde{h} = \pi_K = \pi_K \circ \tilde{h} \circ \tilde{g}$.
If $x \in N_0$ then we have $\tilde{h}(x) = h_J(x)$. Since

$$g_J \circ h_J(x) = (\pi_J \circ g \circ h_J(x), \pi_{I \setminus J} \circ h_J(x)) = (\pi_J \circ g \circ h(x), \pi_{I \setminus J}(x)) = x$$

we get $h_J(x) \in g_J^{-1}(N_0) \subset N_0$; hence $\tilde{g} \circ \tilde{h}(x) = g_J \circ h_J(x) = x$; in particular $\pi_K \circ \tilde{g} \circ \tilde{h}(x) = \pi_K(x)$.

If $x \notin N_0$ then we have $\tilde{h}(x) = h(x)$. From $h^{-1}(N_0) \subset N_0$ it follows that $h(x) \notin N_0$; hence $\tilde{g} \circ \tilde{h}(x) = g \circ h(x)$. Since $N \subset N_0$ we get $x \notin N$ and, therefore, $\pi_K \circ g \circ h(x) = \pi_K(x)$; hence $\pi_K \circ \tilde{g} \circ \tilde{h}(x) = \pi_K(x)$.

In the same way one can show that $\pi_K \circ \tilde{h} \circ \tilde{g} = \pi_K$.

(3) From the definition of $\tilde{g}$ and $\tilde{h}$ it follows immediately that $\pi_J \circ \tilde{g} = \pi_J \circ g$ and $\pi_J \circ \tilde{h} = \pi_J \circ h$.

(4) Let $B \in \mathfrak{B}$ be given. Then we have $\tilde{g}^{-1}(B) \triangle g^{-1}(B) \subset N_0$; hence $\tilde{g}^{-1}(B) \triangle g^{-1}(B) \in \mathfrak{n}$.

In the same way one can deduce that $\tilde{h}^{-1}(B) \triangle h^{-1}(B) \in \mathfrak{n}$.

Case 2. Let $\mathfrak{n}$ satisfy condition (D).

If $J \cap J_0 \neq \emptyset$ then, according to condition (D), there exists a set $N' \in \mathfrak{n}$ such that N' depends only on J_0 and such that $\pi^{-1}_{J_0, J_0 \cap J}(u) \cap \pi_{J_0}(N')$ is uncountable and of second category in $\pi^{-1}_{J_0, J_0 \cap J}(u)$ for all $u \in X_{J_0 \cap J}$.
If $J \cap J_0 = \emptyset$ define $N' = \emptyset$.

We will show that $J_0 \cap J$ is g- and h-invariant. Let $x, y \in X$ be such that $\pi_{J_0 \cap J}(x) = \pi_{J_0 \cap J}(y)$. Then, due to the g-invariance of J_0 and J, we have

$$\pi_{J_0} \circ g(x) = \pi_{J_0} \circ g((\pi_{J_0}(x), \pi_{I \setminus J_0}(y)))$$

and

$$\begin{aligned} \pi_J \circ g((\pi_{J_0}(x), \pi_{I \setminus J_0}(y))) &= \pi_J \circ g((\pi_{J_0 \cap J}(x), \pi_{J_0 \setminus J}(x), \pi_{I \setminus J_0}(y))) \\ &= \pi_J \circ g((\pi_{J_0 \cap J}(y), \pi_{J_0 \setminus J}(x), \pi_{I \setminus J_0}(y))) \\ &= \pi_J \circ g((\pi_J(y), \pi_{J_0 \setminus J}(x), \pi_{I \setminus (J_0 \cup J)}(y))) \\ &= \pi_J \circ g(y) ; \end{aligned}$$

hence $\pi_{J \cap J_0} \circ g(x) = \pi_{J \cap J_0} \circ g(y)$.

In the same way one can show that $J \cap J_0$ is h-invariant.

Define $g_0 \colon X_{J \cap J_0} \to X_{J \cap J_0}$ by $g_0(u) = \pi_{J \cap J_0} g(u, w)$, where $w \in X_{I \setminus (J \cap J_0)}$ is arbitrary. Since $J \cap J_0$ is g-invariant g_0 is a well-defined map. From $\pi_J \circ g \circ h = \pi_J = \pi_J \circ h \circ g$ it follows that g_0 is a bijection. It is also easy to check that g_0 and g_0^{-1} are $\mathfrak{B}(X_{J \cap J_0}) - \mathfrak{B}(X_{J \cap J_0})$-measurable.

Define

$$N_0 = \bigcup_{m \in N} \bigcup \{g^{-\nu_m} h^{-\lambda_m} \cdots g^{-\nu_1} h^{-\lambda_1}(N \cup N') \mid \nu_1, \cdots, \nu_m, \lambda_1, \cdots, \lambda_m \in N \cup \{0\}\} .$$

From our assumptions concerning g and h we deduce $N_0 \in \mathfrak{n}$, $N \cup N' \subset N_0$, $g^{-1}(N_0) \subset N_0$, and $h^{-1}(N_0) \subset N_0$. Since N and N' depend only on J_0 and since J_0 is g- and h-invariant the set N_0 also depends only on J_0. If $J_0 \cap J = \emptyset$ define $\tilde{g} \colon X \to X$ by

$$\tilde{g}(x) = \begin{cases} g(x), & x \notin N_0 \\ x, & x \in N_0 \end{cases}$$

and $\tilde{h} \colon X \to X$ by

$$\tilde{h}(x) = \begin{cases} h(x), & x \notin N_0 \\ x, & x \in N_0 . \end{cases}$$

Then $\tilde{g}$ and $\tilde{h}$ obviously satisfy conditions (i) to (v) in Lemma 4.7. If $J_0 \cap J \neq \emptyset$ then according to our assumptions (cf. Remark 4.2.3) $X_{J_0 \cap J}$ and $X_{J_0 \setminus J}$ are uncountable Polish spaces. In this case we have $\pi_{J_0}(N_0) \in \mathfrak{B}(X_{J_0})$ and, for each $u \in X_{J_0 \cap J}$, the set $\pi^{-1}_{J_0, J_0 \cap J}(u) \cap \pi_{J_0}(N_0)$ is uncountable and of the second category in $\pi^{-1}_{J_0, J_0 \cap J}(u)$. According to Lemma 4.6 there exists a bijection $r: \pi_{J_0}(N_0) \to \pi_{J_0}(N_0)$ such that r and r^{-1} are measurable and such that, for each $w \in X_{J_0}$, we have

$$\pi_{J_0, J_0 \cap J} \circ r(w) = g_0 \circ \pi_{J_0, J_0 \cap J}(w) .$$

Since $\pi_J \circ h \circ g = \pi_J = \pi_J \circ g \circ h$ this implies

$$\pi_{J_0, J_0 \cap J} r^{-1}(w) = h_0 \circ \pi_{J_0, J_0 \cap J}(w) ,$$

where h_0 is defined in an analogous way as g_0.
Define $\tilde{g}: X \to X$ by

$$\tilde{g}(x) = \begin{cases} g(x), & x \notin N_0 \\ (\pi_{I \setminus J_0} \circ g(x),\ r \circ \pi_{J_0}(x)), & x \in N \end{cases}$$

and $\tilde{h}: X \to X$ by

$$\tilde{h}(x) = \begin{cases} h(x), & x \notin N_0 \\ (\pi_{I \setminus J_0} \circ h(x),\ r^{-1} \circ \pi_{J_0}(x)), & x \in N_0 . \end{cases}$$

Then $\tilde{g}$ and $\tilde{h}$ are measurable.

(1) We will show that K is $\tilde{g}$- and $\tilde{h}$-invariant.

Let $x, y \in X$ be such that $\pi_K(x) = \pi_K(y)$. Since N_0 depends only on $J_0 \subset K$ either x and y are both in N_0 or x and y are both in $X \setminus N_0$. In the first case we have $\pi_K \circ \tilde{g}(x) = \pi_K(\pi_{I \setminus J_0} \circ g(x), r \circ \pi_{J_0}(x))$ and, due to the g-invariance of K combined with $\pi_{J_0}(x) = \pi_{J_0}(y)$,

$$\pi_K(\pi_{I \setminus J_0} \circ g(x), r \circ \pi_{J_0}(x)) = \pi_K(\pi_{I \setminus J_0} \circ g(y), r \circ \pi_{J_0}(y)) = \pi_K \tilde{g}(y) .$$

In the second case the g-invariance of K implies

$$\pi_K \circ \tilde{g}(x) = \pi_K \circ g(x) = \pi_K \circ g(y) = \pi_K \circ \tilde{g}(y) .$$

In the same way one can show that K is $\tilde{h}$-invariant.

(2) We will show that $\pi_K \circ \tilde{g} \circ \tilde{h} = \pi_K = \pi_K \circ \tilde{h} \circ \tilde{g}$.

Since N_0 depends only on J_0 we have $\tilde{h}(N_0) \subset N_0$ and $\tilde{g}(N_0) \subset N_0$. Because $g^{-1}(N_0) \subset N_0$ and $h^{-1}(N_0) \subset N_0$ we also have $g(X \setminus N_0) \subset X \setminus N_0$ and $h(X \setminus N_0) \subset X \setminus N_0$.

We, therefore, deduce that, for each $x \in N_0$,

$$\begin{aligned} \pi_K \circ \tilde{h} \circ \tilde{g}(x) &= \pi_K \circ \tilde{h}(\pi_{I \setminus J_0} g(x), r \circ \pi_{J_0}(x)) \\ &= \pi_K(\pi_{I \setminus J_0} \circ h(\pi_{I \setminus J_0} \circ g(x), r \circ \pi_{J_0}(x)), r^{-1} \circ r \circ \pi_{J_0}(x)) . \end{aligned}$$

Since $\pi_{J_0, J_0 \cap J} \circ r \circ \pi_{J_0}(x) = g_0 \circ \pi_{J_0 \cap J}(x) = \pi_{J_0 \cap J} \circ g(x)$ and since J is h-invariant we have

$$\pi_J \circ h(\pi_{I\setminus J_0} \circ g(x),\ r \circ \pi_{J_0}(x)) = \pi_J \circ h \circ g(x) .$$

Because of $\pi_J \circ h \circ g = \pi_J$ and $K\setminus J_0 \subset J$ this implies

$$\pi_K \circ \tilde{h} \circ \tilde{g}(x) = \pi_K(\pi_{I\setminus J_0} \circ h \circ g(x),\ \pi_{J_0}(x)) = (\pi_{K\setminus J_0} \circ h \circ g(x),\ \pi_{J_0}(x)) = \pi_K(x) .$$

For $x \notin N_0$ it follows from $N \subset N_0$ that

$$\pi_K \circ \tilde{h} \circ \tilde{g}(x) = \pi_K \circ h \circ g(x) = \pi_K(x) .$$

In the same way one can show that $\pi_K \circ \tilde{g} \circ \tilde{h} = \pi_K$.

(3) We will show that $\pi_J \circ \tilde{g} = \pi_J \circ g$ and $\pi_J \circ \tilde{h} = \pi_J \circ h$.

For $x \in X\setminus N_0$ these identities obviously hold.

For $x \in N_0$ we deduce

$$\begin{aligned}\pi_J \circ \tilde{g}(x) &= \pi_J(\pi_{I\setminus J_0} \circ g(x),\ r \circ \pi_{J_0}(x)) \\ &= (\pi_{J\setminus J_0} \circ g(x),\ \pi_{J_0, J_0 \cap J} \circ r \circ \pi_{J_0}(x)) \\ &= (\pi_{J\setminus J_0} \circ g(x),\ g_0 \circ \pi_{J_0 \cap J}(x)) \\ &= \pi_J \circ g(x) .\end{aligned}$$

In the same way one can show that $\pi_J \circ \tilde{h} = \pi_J \circ h$.

(4) Property (v) in Lemma 4.7 follows from the fact that $\tilde{g}$ and g as well as $\tilde{h}$ and h differ only in a subset of $N_0 \in \mathfrak{n}$.

Proof of Theorem 4.3. Let $\mathfrak{S}$ be the collection of the triples (J, g, h), where $g, h: X \to X$ are measurable such that $[g^{-1}(B)] = \Phi([B])$ and $[h^{-1}(B)] = \Phi^{-1}([B])$ for all $B \in \mathfrak{B}$, and J is a g- and h-invariant subset of I with $\pi_J \circ h \circ g = \pi_J = \pi_J \circ g \circ h$.

We define the following preorder on $\mathfrak{S}$:

$(J_1, g_1, h_1) \leqq (J_2, g_2, h_2)$ iff $J_1 \subset J_2$, $\pi_{J_1} \circ g_2 = \pi_{J_1} \circ g_1$, and $\pi_{J_1} \circ h_2 = \pi_{J_1} \circ h_1$.

According to Theorem 3.1 there are measurable maps g_0 and h_0 from X into itself such that g_0 induces Φ and h_0 induces Φ^{-1}. Thus $(\varnothing, g_0, h_0)$ belongs to $\mathfrak{S}$ and $\mathfrak{S}$ is not empty.

We claim that the preorder $\leqq$ is inductive. To show this let $(J_\lambda, g_\lambda, h_\lambda)_{\lambda \in \Lambda}$ be a (nonempty) chain in $\mathfrak{S}$ and let $\lambda_0 \in \Lambda$ be fixed. Define $J = \bigcup_{\lambda \in \Lambda} J_\lambda$ and $g: X \to X$ by

$$\pi_\alpha(g(x)) = \begin{cases} \pi_\alpha(g_\lambda(x)), & \alpha \in J_\lambda \\ \pi_\alpha(g_{\lambda_0}(x)), & \alpha \notin J . \end{cases}$$

Let h be defined in an analogous way.

Then g and h are obviously measurable.

Next we will show that g induces Φ. To prove this it is enough to prove $[g^{-1}(\pi_{\alpha_0}^{-1}(B))] = \Phi([\pi_{\alpha_0}^{-1}(B)])$ for all $\alpha_0 \in I$ and all $B \in \mathfrak{B}(X_{\alpha_0})$. For $\alpha_0 \in J$ and $B \in \mathfrak{B}(X_{\alpha_0})$ there exists a $\lambda \in \Lambda$ with $\alpha_0 \in J_\lambda$; hence

$$g^{-1}(\pi_{\alpha_0}^{-1}(B)) = \{x \in X \mid \pi_{\alpha_0} \circ g(x) \in B\}$$

$$= \{x \in X \mid \pi_{\alpha_0} \circ g_\lambda(x) \in B\}$$
$$= g_\lambda^{-1}(\pi_{\alpha_0}^{-1}(B)) .$$

Since $(J_\lambda, g_\lambda, h_\lambda) \in \mathfrak{S}$ this implies $[g^{-1}(\pi_{\alpha_0}^{-1}(B))] = \Phi([\pi_{\alpha_0}^{-1}(B)])$. For $\alpha_0 \in I \backslash J$ one has to replace λ by λ_0 in the above argument. In the same way one can see that h induces Φ^{-1}.

By standard arguments it can be shown that J is g- and h-invariant and that

$$\pi_J \circ g \circ h = \pi_J = \pi_J \circ h \circ g .$$

Thus (J, g, h) is an upper bound of $(J_\lambda, g_\lambda, h_\lambda)_{\lambda \in \Lambda}$ in $\mathfrak{S}$.

By Zorn's lemma there exists a maximal element (J', g', h') in $\mathfrak{S}$. Using Lemma 4.7 we conclude $J' = I$. Since g' induces Φ and h' induces Φ^{-1} the equality $g' \circ h' = h' \circ g' = id_X$ yields that $f := g'$ is a bijection with the desired properties.

References

1. J. Choksi, *Automorphisms of Baire measures on generalized cubes,* Z. Wahrscheinlichkeitstheorie verw. Geb., **22** (1972), 195-204.

2. ————, *Automorphisms of Baire measures on generalized Cubes II,* Z. Wahrscheinlichkeitstheorie verw. Geb., **23** (1972), 97-102.

3. ————, *Measurable transformations on compact groups,* Trans. Amer. Math. Soc. **184** (1973), 101-124.

4. N. Falkner, *Generalizations of standard and analytic measurable spaces,* preprint 1978.

5. D. Maharam, *Automorphisms of products of measure spaces,* Proc. Amer. Math. Soc., **9** (1958), 702-707.

6. ————, *Realizing automorphisms of category algebras of product spaces,* General Topology and Appl., **10** (1979), 161-174.

7. R. D. Mauldin, *Borel Parameterizations,* Trans. Amer. Math. Soc., **250** (1979), 223-234.

8. Sikorski, *Boolean Algebras,* 3rd ed., Springer, Bern-Heidelberg-New York 1969.

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