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ON THE REGULARITY UP TO THE BOUNDARY FOR SECOND ORDER NONLINEAR ELLIPTIC SYSTEMS

M. GIAQUINTA, J. NEČAS, O. JOHN, AND J. STARÁ

It is proved the regularity up to the boundary of the uniformly Lipschitz-continuous weak solutions of a boundary value problem for the elliptic system

(1.1) $\quad -D_i a_i^r(x, u, Du) + \bar{a}^r(x, u, Du) = f^r \; ; \qquad r = 1, \cdots, m$

from the Liouville properties of the system.

In (1.1) $u = \{u^r\}_{r=1,\cdots,m}$ is a vector function and $Du = \{D_i u^r\}_{\substack{i=1,\cdots,n \\ r=1,\cdots,m}}$ is its gradient. We write $D_i u^r = \partial u^r / \partial x_i$ and the summation convention is used throughout the paper.). We follow the ideas of our previous work (see [1-4]) where interior regularity was shown to be equivalent (in some sense) to the Liouville property (L) (see Definition 2.2). In the present paper, regularity up to the boundary is shown to be, essentially, equivalent to the previous (L) together with a certain "boundary" Liouville property (L^+) (see Definition 2.3).

2. **Notation and assumptions.** Let $\boldsymbol{R}^n$ be an n-dimensional Euclidean space; for $x = (x_1, \cdots, x_{n-1}, x_n) = (x', x_n) \in \boldsymbol{R}^n$ let $|x| = \max\{|x_i|; i = 1, \cdots, n\}$; further let $\boldsymbol{R}^n_+ = \{x \in \boldsymbol{R}^n; x_n > 0\}$; $\Omega = \{x \in \boldsymbol{R}^n_+; |x| < 1\}$; $\Gamma = \{x \in \boldsymbol{R}^n; |x'| < 1; x_n = 0\}$; $B(x_0, R) = \{x \in \Omega; |x - x_0| < R\}$; $\Gamma(x_0, R) = \overline{B(x_0, R)} \cap \Gamma$.

Let us denote

$$a(x, u, Du) = \{a_i^r(x, u, Du)\}_{\substack{i=1,\cdots,n \\ r=1,\cdots,m}}$$

$$\bar{a}(x, u, Du) = \{\bar{a}^r(x, u, Du)\}_{r=1,\cdots,m}$$

$$f(x) = \{f^r(x)\}_{r=1,\cdots,m} ,$$

where a, $\bar{a}$ are once continuously differentiable functions on $\bar{\Omega} \times \boldsymbol{R}^m \times \boldsymbol{R}^{nm}$, and $f \in [W^{1,p/2}(\Omega)]^m$ for some p, $p > n$.

REMARK. In what follows we omit the notation of the Cartesian product. So we write $f \in W^{1,p}(\Omega)$ instead of $f \in [W^{1,p}(\Omega)]^{mn}$ etc.

In this notation the system (1.1) can be rewritten as

$$-\operatorname{div}(a(x, u, Du)) + \bar{a}(x, u, Du) = f(x) \tag{2.1}$$

on Ω. We suppose that the strong ellipticity condition holds:

$$\frac{\partial a_i^r}{\partial \eta_j^s}(x, \xi, \eta)\zeta_i^r \zeta_j^s > 0 \tag{2.2}$$

for every $\zeta \neq 0$ and each $(x, \xi, \eta) \in \bar{\Omega} \times \boldsymbol{R}^m \times \boldsymbol{R}^{nm}$.

To describe the boundary conditions we introduce two disjoint sets M, N of positive integers such that $M \cup N = \{1, \cdots, m\}$ (both the cases $M = \varnothing$ and $N = \varnothing$ being admissible). Let $\{b_{rs}\}_{r \in M, s \in N}$ be the set of real constants. The stable boundary operators B_r $(r \in M)$ are given by the formulas

$$B_r u = u^r - \sum_{s \in N} b_{rs} u_s \, .$$

Put

$$\boldsymbol{C} = \{c_{rs}\}_{r,s=1,\cdots,m} \, , \quad \text{where} \quad c_{rs} = \begin{cases} -b_{rs} \, , & r \in M, \ s \in N \, , \\ \delta_{rs} \, , & r, s \in M \, , \\ 0 \, , & r \in N \, , \end{cases}$$

$$\boldsymbol{C}^* = \{c^*_{rs}\}_{r,s=1,\cdots,m} \, , \quad \text{where} \quad c^*_{rs} = \begin{cases} b_{sr} \, , & r \in N, \ s \in M \, . \\ -\delta_{rs} \, , & r, s \in N \, , \\ 0 \, , & r \in M \, , \end{cases} \tag{2.3}$$

$$\mathscr{F}(x, u, Du) = \{a^r_n(x, u, Du) + h^r(x, u) - g^r(x) - f^r_n(x)\}_{r=1,\cdots,m} \, ,$$

where h and g are given functions; $h \in C^1(\bar{\Gamma} \times \boldsymbol{R}^m)$, $g \in W^{1,\infty}(\Gamma)$.

Let, finally, $u_0 = \{u^r_0\}_{r=1,\cdots,m}$ be a given function from $W^{2,p}(\Omega)$. We consider the following boundary value problem for the system (2.1) (in its classical formulation):

$$\begin{aligned} &\boldsymbol{C}(u - u_0) = 0 \quad \text{on} \quad \Gamma \, , \\ &\boldsymbol{C}^* \mathscr{F}(x, u, Du) = 0 \quad \text{on} \quad \Gamma \, , \\ &u - u_0 = 0 \quad \text{on} \quad \partial\Omega \backslash \Gamma \, . \end{aligned} \tag{2.4}$$

Denote the scalar product in $\boldsymbol{R}^n$ as well as in $\boldsymbol{R}^{nm}$ by (,) and put

$$V = \{v \in W^{1,2}(\Omega) \, ; \quad \boldsymbol{C}v = 0 \quad \text{on} \quad \Gamma \, ; \quad v = 0 \quad \text{on} \quad \partial\Omega \backslash \Gamma\} \, . \tag{2.5}$$

A function $u \in W^{1,2}(\Omega)$ is said to be a weak solution of the problem (2.1), (2.4) if

(2.6)
(i) $u - u_0 \in V$,
(ii) for each $\varphi \in V$, it holds

$$\int_\Omega \{(a, D\varphi) + (\bar{a}, \varphi) - (f, \varphi)\} dx = \int_\Gamma (h - g, \varphi) dx' \, .$$

(Let us rewrite for once the equation (2.6) (ii) in a more detailed form:

(2.6) (ii)′

$$\int_\Omega \{a^r_i(x, u(x), Du(x)) D_i \varphi^r(x) + \bar{a}^r(x, u(x), Du(x)) \varphi^r(x) - f^r(x) \varphi^r(x)\} dx$$

$$= \int_\Gamma \{h^r(x, u(x)) - g^r(x)\} \varphi^r(x) dx' \, .$$

Let us now formulate the regularity of the problem and the Liouville conditions.

DEFINITION 2.1 (R). We say that the problem (2.1), (2.4) is regular (and denote this property by (R)) if for each weak solution u of this problem for which $Du \in L_\infty(\Omega)$, the gradient Du is locally α-Hölder continuous on $\Omega \cup \Gamma$, and for each Ω' for which $\Omega' \subset \Omega \cup \Gamma$ it holds

$$\| \nabla u \|_{C^\alpha(\overline{\Omega}')} \leqq C ,$$

where the constant C depends on $\| \nabla u \|_{L_\infty(\Omega)}$, Ω' and the data of the problem.

DEFINITION 2.2 (L). We say that the system (2.1) satisfies the Liouville condition (L) if for each $x_0 \in \Omega$ and each $\xi \in \boldsymbol{R}^m$ the solution $u \in W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}^n)$ of the equation

$$(2.7) \qquad \int_{\boldsymbol{R}^n} (a(x_0, \xi, Du), D\varphi) dx = 0 \qquad \forall \varphi \in C_0^\infty(R^n)$$

for which $Du \in L_\infty(\boldsymbol{R}^n)$ is a polynomial of at most the first degree.

DEFINITION 2.3 (L^+). Write $Z = \{\varphi \in C_0^\infty(\boldsymbol{R}^n);\ C\varphi = 0 \text{ on } \{x \in \boldsymbol{R}^n;\ x_n = 0\}\}$. We say that the problem (2.1), (2.4) satisfies the Liouville condition (L^+) if for each $x_0 \in \Gamma$; $\xi \in \boldsymbol{R}^m$; $d \in \boldsymbol{R}^m$ the solution $u \in W^{1,2}_{\mathrm{loc}}(\overline{\boldsymbol{R}^n_+})$ of the equation

$$(2.8) \qquad \int_{\boldsymbol{R}^n_+} (a(x_0, \xi, Du), D\varphi) dx = \int_{\{x \in \boldsymbol{R}^n;\ x_n = 0\}} (d, \varphi) dx' \qquad \forall \varphi \in Z$$

is the polynomial of at most the first degree, provided that $\boldsymbol{C}u$ is a polynomial of at most the first degree on $\{x \in \boldsymbol{R}^n;\ x_n = 0\}$ and $Du \in L_\infty(\boldsymbol{R}^n_+)$.

Our paper contains the proof that (roughly speaking): (2.1), (2.4) is regular iff (L) and (L^+) hold simultaneously. The necessity of the Liouville conditions is proved in §3 with the definition of the regularity being slightly changed. In §4 the proof of the implication (L) $\wedge$ (L^+) $\Rightarrow$ (R) is given.

3. The necessity of Liouville conditions. Considering the definitions 2.1–2.3 we conclude that the property (R) concerns one fixed problem (2.1), (2.4) whilst the Liouville conditions (L) and (L^+) refer to a system of problems (2.8). We do not know whether the implication (R) $\Rightarrow$ (L) $\wedge$ (L^+) holds. To obtain the implication of this type we modify at first the definition of regularity.

DEFINITION 3.1 (R'). Let for each $x_0 \in \overline{\boldsymbol{R}^n_+}$, $\xi \in \boldsymbol{R}^m$, $d \in \boldsymbol{R}^m$ and for each solution u of (2.8) for which $Du \in L_\infty(\boldsymbol{R}^n_+)$ and Cu is a polynomial of at most the first degree on $\{x \in \boldsymbol{R}^n; x_n = 0\}$ there exists $T > 0$ such that u belongs to the space $C^{1,\alpha}(\overline{B(0, T)})$ with $\alpha = \min\{1/2, 1 - n/p\}$ and

$$\|u\|_{C^{1,\alpha}} \leqq C , \tag{3.1}$$

where C and T depend only on $\|Du\|_{L_\infty}$, $|u(0)|$ and the data of the problem.

THEOREM 3.1. $(R') \Rightarrow (L^+)$.

Proof. Suppose $x_0 = 0$. The function $u_R(y) = (1/R)u(Ry)$ solves (2.8). Further $\|Du_R\|_{L_\infty} = \|Du\|_{L_\infty}$ and for $R > 1$ the values $|u_R(0)|$ and Cu_R are bounded by the same constants as the corresponding values of u. Thus u_R ($R > 1$) satisfies (3.1) with the constant independent of R. Let $x \in \boldsymbol{R}^n_+$; $TR \geqq |x|$; $Ry = x$. According to (3.1) we get

$$|D_y u_R(y) - D_y u_R(0)| \leqq C|y|^\alpha \leqq C\left|\frac{x}{R}\right|^\alpha ,$$

hence

$$|D_x u(x) - D_x u(0)| \leqq C\frac{|x|^\alpha}{R^\alpha}$$

and it implies that $D_x u(x) = D_x u(0)$ for $R \to \infty$.

Let us mention that the necessity of the condition (L) was proved in [4].

4. Sufficiency of the Liouville conditions. Put for an arbitrary vector function $f = \{f^r\}_{r=1,\cdots,s}$,

$$\begin{cases} F(x_0, R) = R^{2-n} \displaystyle\int_{B(x_0,R)} \sum_{r=1}^{s} \sum_{i=1}^{n} |D_i f^r(x)|^2 dx \quad \text{and} \\ VB(x_0, R) = \{u \in W^{1,2}(B(x_0, R)); Cu = 0 \quad \text{on} \quad \Gamma(x_0, R) \quad \text{and} \\ \qquad u = 0 \quad \text{on} \quad \partial B(x_0, R)\setminus\Gamma(x_0, R)\} . \end{cases} \tag{4.1}$$

The following notation will be used in Lemma 4.1 only:

$$B(x_0, t) = \{x \in \boldsymbol{R}^n_+;\ |x - x_0| < t\} ; \qquad t \in [0, 1] .$$

Let

$$\Gamma = \{x \in \boldsymbol{R}^n;\ |x'| < 1; x_n = 0\} .$$

With the so defined $B(x_0, t)$ the symbols $F(x_0, t)$ and $VB(x_0, t)$ have the same meaning as in (4.1).

LEMMA 4.1. *Let* $\boldsymbol{B} = \{B_{rs}^{ij}\}_{\substack{i,j=1,\cdots,n \\ r,s=1,\cdots,m}}$ *be a real matrix such that*

$$\exists \mathscr{H} > 0 \qquad \forall \eta \in R^{nm} \qquad B_{rs}^{ij}\eta_r^i\eta_s^j \geqq \mathscr{H}|\eta|^2 . \tag{4.2}$$

Then there exists a positive number K *such that for every* $x_0 = (0, \cdots, 0, q)$, $(q \in [0, 1])$ *for each* $v \in W^{1,2}(B(x_0, 1))$ *for which*

$$\boldsymbol{C}v = 0 \quad \text{on} \quad \Gamma \tag{4.3}$$

and which solves the system

$$\int_{B(x_0,1)} (\boldsymbol{B}Dv, D\varphi)dx = 0 \qquad \forall \varphi \in VB(x_0, 1) , \tag{4.4}$$

and for every $t \in]0, 1/2[$, *the inequalities*

$$V(x_0, t) \leqq Kt^2 V(x_0, 1) , \tag{4.5}$$

$$t^{-n}\int_{B(x_0,t)} |v(x) - P_t|^2 dx \leqq Kt^2 \int_{B(x_0,1)} |v(x) - Q|^2 dx \tag{4.6}$$

hold, where Q *is an arbitrary vector such that* $\boldsymbol{C}Q = 0$ *and* P_t *is either a value* $v(\tilde{x})$ *at an arbitrary point* $\tilde{x} \in \overline{B(x_0, t)}$ *or an integral mean value of* v *over any connected subset of* $\overline{B(x_0, t)}$.

Proof. Let $k \in N$ be such that $W^{2,k}(G) \subset C^1(\bar{G})$ for a bounded domain $G \subset \boldsymbol{R}^n$. Let $1 = t_0 > t_1 > \quad > t_k = 1/2$ be an equidistant subdivision of the segment $[1/2, 1]$.

Let $\Phi \in C^\infty(\overline{B(x_0, 1)})$; $\operatorname{supp} \Phi \subset B(x_0, (t_0 + t_1)/2)$; $0 \leqq \Phi \leqq 1$, $\Phi \equiv 1$ on $B(x_0, t_1)$; $|D\Phi| \leqq C/(t_0 - t_1)$.

Let $\boldsymbol{C}Q = 0$ and put

$$\varphi = \Phi^2(v - Q)$$

in (4.4). By usual calculations we obtain (denoting in what follows all the constants by C)

$$\int_{B(x_0,t_1)} |Dv|^2 dx \leqq C\int_{B(x_0,1)} |v - Q|^2 dx \tag{4.7}$$

If $\overline{B(x_0, t_1)} \subset B(x_0, 1)$ we use the fact that all the derivatives up to the order k solve the system (4.4) and we get finally the estimate

$$\int_{B(x_0,1/2)} |D^k v|^2 dx \leqq C\int_{B(x_0,1)} |v - Q|^2 dx . \tag{4.8}$$

If $B(x_0, t_1)$ reaches up to the boundary, only the tangent derivatives $D_j v$ $(j = 1, \cdots, n - 1)$ of the solution v solve again the boundary value problem. For them we get

$$\int_{B(x_0,t_2)} |D(D_j v)|^2 dx \leqq C \int_{B(x_0,t_1)} |Dv|^2 dx \,. \tag{4.8'}$$

The second normal derivative can be expressed from the equation

$$B^{ij}_{rs} D_{ij} v^s = 0 \; ; \qquad r = 1, \cdots, m \,,$$

which holds a.e. in $B(x_0, 1)$. Advancing this process up to the estimate of the derivatives of the kth order we obtain (using the Sobolev imbedding theorem)

$$\max{}^2\left\{|Dv(x)|; x \in B\left(x_0, \frac{1}{2}\right)\right\} \leqq C \int_{B(x_0,1)} |v - Q|^2 dx \,. \tag{4.9}$$

Let now $t \in]0, 1/2]$, $x, \tilde{x} \in \overline{B(x_0, tx)}$. Then

$$|v(x) - v(\tilde{x})|^2 \leqq Ct^2 \max{}^2\left\{|Dv(x)|; x \in \overline{B\left(x_0, \frac{1}{2}\right)}\right\} = C \int_{B(x_0,1)} |v - Q|^2 dx \,.$$

Let us recall that the constant C does not depend on the position of the point x_0 satisfying the assumptions of Lemma 4.1. Its value will be needed in the next text; because of an easier quotation we denote it by K. Integrating the last inequality over $B(x_0, t)$, we get (4.6) for the case $P_t = v(\tilde{x})$ with $\tilde{x} \in B(x_0, t)$. The case of P_t being an integral mean value can be reduced to the previous one by means of the integral mean value theorem.

To prove the inequality (4.5) we start with the estimate (4.8′) and applying the same method as before, we obtain

$$\left(\max \left\{|Dv(x)|; x \in \overline{B\left(x_0, \frac{1}{2}\right)}\right\}\right)^2 = C \int_{B(x_0,1)} |Dv|^2 dx \,. \tag{4.10}$$

The inequality (4.5) is an immediate consequence of (4.10).

The main result of this section is the following

THEOREM 4.2. *Let* (L) *and* (L^+) *be satisfied. Let* $u \in W^{1,2}(\Omega)$ *with the gradient* $Du \in L_\infty(\Omega)$ *be a weak solution of the problem* (2.1), (2.4). *Then* Du *is* α*-Hölder continuous on* $\Omega \cup \Gamma$ *with* $\alpha = \min(1/2, 1 - n/p)$ *and for every domain* Ω' *such that* $\bar{\Omega}' \subset \Omega \cup \Gamma$ *the inequality holds*:

$$\begin{aligned} \|u\|_{C^{1,\alpha}(\bar{\Omega}')} \leqq C(&\|Du\|_{L_\infty}, \|u_0\|_{W^{2,p}}, \|f\|_{W^{1,p/2}}, \\ &\|g\|_{L_\infty}, \operatorname{dist}(\bar{\Omega}', \boldsymbol{R}^n_+ \backslash \Omega)) \,. \end{aligned} \tag{4.11}$$

Schema of the proof of the Theorem 4.2. In Lemma 4.8 we shall prove that Du belongs to certain Morrey-Campanato space and use then embedding of this space into $C^{1,\alpha}$.

For the case $\bar{\Omega}' \subset \Omega$ it follows from the condition (L). We can prove it by the method described in [4] modifying it slightly.

For the case $\bar{\Omega}' \cap \Gamma \neq \emptyset$ more substantial modifications of the method are needed. Denoting tangent derivatives as $\omega_l^r = D_l u^r$; $r = 1, \cdots, m$; $l = 1, \cdots, n-1$, we decompose them on $B(x, R)$ as

$$\omega = v + w$$

in the following way:

(i) The function w solves the linearized equation in variations (see (4.14)) and satisfies the nonhomogeneous boundary conditions

$$Cw_l = CD_l u_0 \quad \text{on} \quad \Gamma(x, R) ,$$

$$w_l - D_l u_0 = 0 \quad \text{on} \quad \partial B(x, R) \backslash \Gamma(x, R) ,$$

$l = 1, \cdots, n-1$. The L_2-norm of Dw can be easily estimated (see Lemma 4.3).

(ii) The second component $v = \omega - w$ solves the homogeneous linearized equation (4.15) and satisfies the homogeneous boundary conditions $Cv_l = 0$ on $\Gamma(x, R)$ and nonhomogeneous boundary conditions $v_l = \omega_l - D_l u_0$ on $\partial B(x, R) \backslash \Gamma(x, R)$, $l = 1, \cdots, n-1$.

In Lemma 4.4 we shall prove that, starting with sufficiently small oscillations of v on $B(x, R)$ we can describe how they decrease on $B(x, \tau R)$, $(\tau \in (0, 1))$.

The Liouville condition (L^+) yields, for each $x_0 \in \Gamma$, the fact that

$$\liminf_{R \to 0_+} V(x_0, R) = 0 . \qquad \text{(See Theorem 4.5.)}$$

Combining this result together with the estimates of v and w, we obtain the assertion of Theorem 4.2.

First we shall describe more precisely the decomposition of ω.

Let u be a solution of (2.6) with $Du \in L_\infty(\Omega)$. Using the finite difference technique, we prove that $u \in W^{2,2}_{\text{loc}}(\Omega)$ and that each component ω_l of the tangent gradient ω solves the equation

$$\begin{aligned} \int_\Omega \Big\{ \Big(\frac{\partial a}{\partial \eta} D\omega_l + \frac{\partial a}{\partial \xi} \omega_l + \frac{\partial a}{\partial x_l}, D\varphi \Big) + \Big(\frac{\partial \bar{a}}{\partial \eta} D\omega_l \\ + \frac{\partial \bar{a}}{\partial \xi} \omega_l + \frac{\partial \bar{a}}{\partial x_l}, \varphi \Big) \Big\} dx = \int_\Omega \Big(\frac{\partial f}{\partial x_l}, \varphi \Big) dx \\ + \int_\Gamma \Big\{ \Big(\frac{\partial h}{\partial \xi} \omega_l + \frac{\partial h}{\partial x_l} - \frac{\partial g}{\partial x_l}, \varphi \Big) \Big\} dx' , \qquad \forall \varphi \in V . \end{aligned} \tag{4.12}$$

Moreover, $C(\omega_l - D_l u_0) = 0$ on Γ.

Let $x_0 \in \Omega \cup \Gamma$; $R > 0$ and $x_{0n} \leqq R$ (i.e., $\Gamma(x_0, R) \neq \emptyset$). Define $w = \{w^r\}_{r=1,\cdots,m} \in W^{1,2}(B(x_0, R))$ as a unique weak solution of the problem

$$w_l - D_l u_0 \in VB(x_0, R)\ , \tag{4.13}$$

$$\forall \varphi \in VB(x_0, R)$$

$$\begin{aligned}\int_{B(x_0,R)} &\left\{\left(\frac{\partial a}{\partial \eta}(x, u, Du)Dw_l, D\varphi\right)+\left(\frac{\partial \bar{a}}{\partial \eta}(x, u, Du)Dw_l, \varphi\right)\right\}dx \\ &= -\int_{B(x_0,R)} \left\{\left(\frac{\partial a}{\partial \xi}\omega_l + \frac{\partial a}{\partial x_l}, D\varphi\right) + \left(\frac{\partial \bar{a}}{\partial \xi}\omega_l + \frac{\partial \bar{a}}{\partial x_l}, \varphi\right)dx\right. \\ &\quad + \int_{B(x_0,R)} \left(\frac{\partial f}{\partial x_l}, \varphi\right)dx + \int_{\Gamma(x_0,R)} \left\{\frac{\partial h}{\partial \xi}\omega_l + \frac{\partial h}{\partial x_l} + \frac{\partial g}{\partial x_l}, \varphi\right\} dx'\ .\end{aligned} \tag{4.14}$$

The relations (4.12) and (4.13), (4.14) imply that (defining $v_l = \omega_l - w_l$) the component v_l solves the equation

$$\forall \varphi \in VB(x_0, R)$$

$$\int_{B(x_0,R)} \left\{\left(\frac{\partial a}{\partial \eta}(x, u, Du)Dv_l, D\varphi\right) + \left(\frac{\partial \bar{a}}{\partial \eta}(x, u, Du)Dv_l, \varphi\right)dx = 0\ ,\right. \tag{4.15}$$

and satisfies the boundary conditions

$$\begin{aligned} Cv_l &= 0 \quad \text{on} \quad \Gamma(x_0, R) \\ v_l &= \omega_l - D_l u_0 \quad \text{on} \quad \partial B(x_0, R)\backslash\Gamma(x_0, R)\ . \end{aligned} \tag{4.16}$$

The components v_l and w_l depend on the choice of x_0 and R. We shall denote them by $v = \{v_l\}_{l=1,\cdots,n-1}$, omitting to express the dependence on x_0 and R if not necessary.

Taking into account the assumptions on the coefficients, the right-hand side, the boundary conditions, and the solution u ($Du \in L_\infty(\Omega)$), we get easily that the problem (4.13), (4.14) can be rewritten as follows:

$$w - w_0 \in VB(x_0, R)\ ; \tag{4.17}$$

$$\begin{aligned}\int_{B(x_0,R)} &\{(\boldsymbol{A}Dw, D\varphi) + (\bar{\boldsymbol{A}}Dw, \varphi)\}dx \\ &= \int_{B(x_0,R)} (\boldsymbol{F}, \varphi)dx + \int_{\Gamma(x_0,R)} (\boldsymbol{H}, \varphi)dx' \qquad \forall \varphi \in VB(x_0, R)\ ,\end{aligned} \tag{4.18}$$

where

(4.19)

(1) $\boldsymbol{A} = \left\{\dfrac{\partial a_i^r}{\partial \eta_j^s}\delta_{kl}\right\} \in L_\infty(B(x_0, R))\ ,$

$(\boldsymbol{A}\eta, \eta) \geqq \mathscr{H}|\eta|^2 \qquad \forall \eta \in R^{nm(n-1)}\ ,$

(2) $\bar{\boldsymbol{A}} = \left\{\dfrac{\partial \bar{a}^r}{\partial \eta_j^s}\right\} \in L_\infty(B(x_0, R))\ ,$

(3) $w_0 = D_l u_0 \in W^{1,p}(B(x_0, R))\ ,$

(4) $\boldsymbol{F} \in L_{p/2}(B(x_0, R))\ ,$

(5) $\boldsymbol{H} \in L_\infty(\Gamma(x_0, R))\ .$

(Let us remind here that ω_l is bounded on Ω and (as it solves the system 4.12) it belongs to the space $W^{1,2}_{\mathrm{loc}}(\Omega \cup \Gamma)$; thus it has a well defined trace on Γ and, since $\|Du\|_{L_\infty} \leqq C$, we have $\|\omega_l\|_{L_\infty(\Gamma)} \leqq C$, too.)

Putting $\varphi = w - w_0$ and using the assumptions (4.19), we get

LEMMA 4.3. *There exist a positive constant C and a positive radius R_0 such that, for every $R \in (0, R_0)$ and for every solution $w \in W^{1,2}(B(x_0, R))$ of the problem (4.17), (4.18) satisfying (4.19), the inequality*

$$\|Dw\|_{L_2(B(x_0,R))} \leqq CR^{n(1/2-1/p)} \tag{4.20}$$

holds.

The local behaviour of the oscillations of the second component v is shown in the next lemma.

LEMMA 4.4. *For every Ω'; $\bar{\Omega}' \subset \Omega \cup \Gamma$, for every positive C and each $\tau \in (0, 1)$, there exist a positive ε and R_0 such that, for every solution u of the problem (2.6) with $\|Du\|_{L_\infty} \leqq C$, for every $x_0 \in \Omega'$ and $R \in\,]0, \min(R_0, 1 - |x_0|)[$, the implication*

$$V(x_0, R) < \varepsilon^2 \Longrightarrow V(x_0, \tau R) < 2K\tau^2 V(x_0, R) \tag{4.21}$$

holds.

(Here

$$V(x_0, R) = R^{2-n}\int_{B(x_0,R)} |Dv|^2 dx\,, \qquad V(x_0, \tau R) = (\tau R)^{2-n}\int_{B(x_0,\tau R)} |Dv|^2 dx$$

and, in both the expressions, v is the component of the decomposition of ω on $B(x_0, R)$). K is the maximal of the constants from Lemma 4.1, corresponding to

$$B^{rs}_{ij} = \frac{\partial a^r_i}{\partial \xi^s_j}(x_0, \zeta, \xi)\,,$$

$x_0 \in \bar{\Omega}$; $|\xi| \leqq C$ and the upper bound for ζ derived as the upper bound for weak solution of the problem (2.6) for which $\|Du\|_{L_\infty(\Omega)} \leqq C$.

Proof. Suppose that the assertion of the lemma does not hold. Then there exist $C \in (0, \infty)$, $\tau \in\,]0, 1[$, sequences $\varepsilon_\nu \searrow 0$, $R_\nu \searrow 0$, $x_\nu \to x_0 \in \Omega \cup \Gamma$ and u_ν; $\|Du_\nu\|_{L_\infty} \leqq C$, such that

$$V(x_\nu, R_\nu) = \varepsilon^2_\nu \tag{4.22}$$

and simultaneously

$$V(x_\nu, \tau R_\nu) > 2K\tau^2 V(x_\nu, R_\nu) . \tag{4.23}$$

If $x_0 \in \Omega$, all $\overline{B(x_\nu, R_\nu)} \subset \Omega$ for a sufficiently large and the proof is substantially the same as in [4]. A similar situation occurs if $x_0 \in \Gamma$ but $R_\nu < x_{\nu n}$ for infinitely many indices ν (i.e. the closed sets $\overline{B(x_\nu, R_\nu)} \subset \Omega$). In what follows, $x_{\nu j}$ will denote the jth component of the vector x_ν (i.e., $x_\nu = \{x_{\nu j}\}_{j=0}^m$). The same notation will be used for sequences u_ν, v_ν etc.

Suppose that $x_\nu \to x_0 \in \Gamma$ and $x_{\nu n} \leqq R_\nu$. Using the decomposition $\omega_{\nu l} = w_{\nu l} + v_{\nu l}$ on $B(x_\nu, R_\nu)$ and estimating Dv_ν by (4.22) and Dw_ν by Lemma 4.3 we get

$$\|D\omega_{\nu l}\|^2_{L_2(B(x_\nu, R_\nu))} \leqq c(R_\nu^{n(1-2/p)} + \varepsilon_\nu^2 R_\nu^{n-2}) , \qquad l = 1, \cdots, n-1 . \tag{4.24}$$

The second normal derivatives $\partial^2 u_\nu / \partial x_n^2$ can be expressed from the equation

$$\frac{\partial a_i^r}{\partial \eta_j^s} D_{i+j} u^s + \frac{\partial a_i^r}{\partial \xi_s} D_i u^s + \frac{\partial a_i^r}{\partial x_i} + \bar{a}^r - f^r = 0 ; \qquad r = 1, \cdots, m ,$$

which is satisfied a.e. on Ω. Thus we get

$$\|D^2 u_\nu\|^2_{L_2(B(x_\nu, R_\nu))} \leqq C(R_\nu^{n(1-2/p)} + \varepsilon_\nu^2 R_\nu^{n-2}) . \tag{4.25}$$

Put

$$\mathscr{H}_{\nu l} = \frac{1}{\text{meas}_{n-1} \Gamma(x_\nu, R_\nu)} \int_{\Gamma(x_\nu, R_\nu)} v_{\nu l}(y) dy' , \tag{4.26}$$

$$\mathscr{H}_\nu = \{\mathscr{H}_{\nu l}\}_{l=1,\cdots,n-1} ,$$

$$\psi_\nu : y \longrightarrow x , \tag{4.27}$$

where

$$x_i = x_{\nu i} + R_\nu y^i \qquad i = 1, \cdots, n-1 ,$$
$$x_n = R_\nu y_n ,$$

$$a_\nu = 1 + \frac{x_{\nu n}}{R_\nu} \in \langle 1, 2 \rangle . \tag{4.28}$$

Then the substitution ψ_ν^{-1} transforms the sets $B(x_\nu, R_\nu)$ into

$$B_\nu = \{y \in \boldsymbol{R}^n ; |y_i| < 1 \text{ for } i = 1, \cdots, n-1, \ 0 < y_n < a_\nu\} \tag{4.29}$$

and the sets $\Gamma(x_\nu, R_\nu)$ into

$$\Gamma_0 = \{y \in \boldsymbol{R}^n ; |y_i| < 1 \text{ for } i = 1, \cdots, n-1, \ y_n = 0\} . \tag{4.30}$$

Moreover, put

$$B_{\nu,\tau} = \psi_\nu^{-1}(B(x_\nu, \tau R_\nu)) .$$

Defining

$$s_\nu(y) = \frac{1}{\varepsilon_\nu}\{v_\nu(\psi_\nu(y)) - \mathscr{H}_\nu\} , \tag{4.31}$$

we get from (4.22), (4.23) that

$$S_\nu = \int_{B_\nu} |Ds_\nu|^2 dy = 1 , \tag{4.32}$$

$$S_{\nu,\tau} = \int_{B_{\nu,\tau}} |Ds_\nu|^2 dy > 2K\tau^2 . \tag{4.33}$$

Applying the following type of Poincaré's inequality to s_ν and using (4.32), we obtain

$$\|s_\nu\|_{W^{1,2}(B_\nu)} \leqq C . \tag{4.34}$$

Poincaré's inequality. There exists $C > 0$ such that for each $\nu \in N$ and for each $f \in W^{1,2}(B_\nu)$

$$\int_{B_\nu} \left[f(y) - \frac{1}{\mathrm{meas}_{n-1}\, \Gamma_0} \int_{\Gamma_0} f(z)dz'\right]^2 dy \leqq C \int_{B_\nu} |Df|^2 dy \tag{4.35}$$

holds.

In what follows, we dare to pass to a suitable subsequence without notice and without changing the notation.

We distinguish two cases

$$\begin{aligned} &\text{(a)}\quad a_\nu \searrow a_0;\ \bigcap_{\nu \in N} B_\nu \supset B_0 \\ &\qquad = \{y \in \boldsymbol{R}^n;\ |y_i| < 1;\ i = 1, \cdots, n-1,\ 0 < y_n < a_0\} ; \\ &\text{(b)}\quad a_\nu \nearrow a_0;\ \bigcup_{\nu \in N} B_\nu = B_0 . \end{aligned} \tag{4.36}$$

We shall prove that $\{s_\nu\}$ converges on B_0 to a function s solving the system with constant coefficients and such boundary conditions that Lemma 4.1 can be applied to s. Then the passage to the limit in the relations (4.32), (4.33) gives the contradiction.

From (4.34) we can conclude that there is a function $s \in W^{1,2}(B_0)$ such that

$$s_\nu \longrightarrow s \quad \text{and} \quad \varepsilon_\nu s_\nu \longrightarrow 0 \quad \text{a.e. on } B_0 \tag{4.37}$$

and

$$\begin{aligned} &\text{(a)}\quad s_\nu \longrightarrow s \text{ in } W^{1,2}(B_0) ,\quad \text{weakly} \\ &\text{(b)}\quad s_\nu \longrightarrow s \text{ in } W^{1,2}(G_0) \text{ weakly for each} \\ &\qquad\qquad \dot{G};\ \bar{G} \subset \bar{B}_0 \backslash \{y \in \boldsymbol{R}^n;\ y_n = a_0\} . \end{aligned} \tag{4.37'}$$

Taking into account the definition of s_ν (see (4.31)), we get

$$\omega_\nu(\psi_\nu(y)) = \varepsilon_\nu s_\nu(y) + \mathscr{H}_\nu + t_\nu(y) , \tag{4.38}$$

where

$$t_\nu(y) = w_\nu(\psi_\nu(y)) . \tag{4.39}$$

The boundedness of ω together with Lemma 4.3 and (4.37) yield the existence of a constant vector $\sigma = \{\sigma_l^r\}_{\substack{l=1,\cdots,n-1\\ r=1,\cdots,m}}$ such that $\mathscr{H}_{\nu l} \to \sigma_l$ and

$$\omega_{\nu l}(\psi_\nu(y)) \longrightarrow D_l u_0(x_0) + \sigma_l \qquad \text{a.e. on } B_0 . \tag{4.40}$$

A similar technique may be used for the normal derivative. Put

$$\mathscr{H}_{\nu n} = \frac{1}{\operatorname{meas}_{n-1} \Gamma(x_\nu, R_\nu)} \int_{\Gamma(x_\nu, R_\nu)} D_n u_\nu(x', x_n) dx' .$$

By Poincaré's inequality and (4.25) we get

$$\| D_n u_\nu(\psi_\nu(y)) - \mathscr{H}_{\nu n} \|^2_{L_2(B_\nu)} \leqq C(R_\nu^{2(1-n/p)} + \varepsilon_\nu^2) \longrightarrow 0 . \tag{4.41}$$

This and the boundedness of Du imply the boundedness of the sequence $\mathscr{H}_{\nu n}$ and thus the existence of such a constant vector $\xi_n = \{\xi_n^r\}_{r=1,\cdots,m}$ that

$$D_n u_\nu(\psi_\nu(y)) \longrightarrow \xi_n \qquad \text{a.e. on } B_0 . \tag{4.42}$$

Put $\xi = \{\xi_l^r\}_{\substack{r=1,\cdots,m\\ l=1,\cdots,n}}$; $\xi_l^r = D_l u_0^r(x_0) + \sigma_l^r$ for $r = 1, \cdots, m$ and $l = 1, \cdots, n-1$. Then (4.40) and (4.42) give

$$D_l u_\nu^r(\psi_\nu(y)) \longrightarrow \xi_l^r \text{ a.e. on } B_0 \text{ for } r = 1, \cdots, m, \ l = 1, \cdots, n , \tag{4.43}$$

and the norm of ξ is bounded by the same constant as the L_∞-norm of Du_ν.

Deduce now the equation for s:
Substituting $x = \psi_\nu(y)$ into (4.15) and using (4.31), we obtain

$$\int_{B_\nu} \{(MDs_\nu, D\varphi) + R_\nu(\bar{M}Ds_\nu, \varphi)\} dy = 0 \qquad \forall \nu \in N, \ \forall \varphi \in VB_\nu ; \tag{4.44}$$

where

$$M = \{m_{ij}^{rs}(\nu, y)\}_{\substack{r,s=1,\cdots,m\\ i,j=1,\cdots,n}} , \qquad m_{ij}^{rs}(\nu, y) = \frac{\partial a_i^r}{\partial \eta_j^s}(\psi_\nu(y), u_\nu(\psi_\nu(y)), Du_\nu(\psi_\nu(y))) ,$$

$$\bar{M} = \{\bar{m}_{ij}^r(\nu, y)\}_{\substack{r=1,\cdots,m\\ i,j=1,\cdots,n}} , \qquad \bar{m}_{ij}^r(\nu, y) = \frac{\partial \bar{a}^r}{\partial \eta_j^s}(\psi_\nu(y), u_\nu(\psi_\nu(y)), Du_\nu(\psi_\nu(y))) .$$

Taking into account that $\psi_\nu(y) \to x_0$ on B_0, $u_\nu(\psi_\nu(y)) \to \zeta$ on B_0 and $Du_\nu(\psi_\nu(y)) \to \xi$ a.e. on B_0, we can conclude that

$$m_{ij}^{rs}(\nu, y) \longrightarrow B_{ij}^{rs} = \frac{\partial a_i^r}{\partial \eta_j^s}(x_0, \zeta, \xi) \qquad \text{a.e. on } B_0 \tag{4.45}$$

and passing to the limit in (4.44), we get finally

$$\int_{B_0} (\boldsymbol{B}Ds, D\varphi)dy = 0 \qquad \forall \varphi \in VB_0 \, . \tag{4.46}$$

According to (4.16), $Cv_\nu = 0$ on $\Gamma(x_\nu, R_\nu)$ for each $\nu \in \boldsymbol{N}$, hence $Cs_\nu = 0$ on Γ_0 for each $\nu \in \boldsymbol{N}$ and

$$Cs = 0 \quad \text{on} \quad \Gamma_0 \, . \tag{4.47}$$

Thus the function s solves the boundary value problem of the type required in Lemma 4.1 and

$$S_\tau = \tau^{2-n} \int_{B_{0,\tau}} |Ds|^2 dy \leqq K\tau^2 \int_{B_0} |Ds|^2 dy = K\tau^2 S \, , \tag{4.48}$$

where $B_{0,\tau} = \{y \in \boldsymbol{R}^n; |y_i| < \tau \text{ for } i = 1, \cdots, n-1, |y_n - a_0 + 1| < \tau\}$, and K is the constant described in Lemma 4.4.

The weak lower semicontinuity of the functional ϑ: $s \to \int_{B_0} |Ds|^2 dy$ together with (4.37), (4.32) gives

$$S = \int_{B_0} |Ds|^2 dy \leqq 1 \, . \tag{4.49}$$

To get the contradiction it is sufficient to prove that

$$S_{\nu,\tau} \longrightarrow S_\tau = \tau^{2-n} \int_{B_{0,\tau}} |Ds|^2 dy \, .$$

We shall prove (by the choice of a test function) that $Ds_\nu \to Ds$ in $L_{2,\mathrm{loc}}(B_0)$. Let us sketch the choice for the case (a): Take $\nu_0 \in \boldsymbol{N}$ so large that

$$B_{\nu_0,\tau} \subset \left\{ y \in \boldsymbol{R}^n; y_n < a_0 - 1 + \frac{\tau + 1}{2} \right\} ;$$

let $\Phi \in C_\infty(\bar{R}_+^n)$; $\operatorname{supp} \Phi \subset B_0 \cup \Gamma_0$; $\Phi \equiv 1$ on $\bigcup_{\nu \geqq \nu_0} B_{\nu,\tau}$; Then $\varphi = (s_\nu - s)\Phi^2$ (prolonged by zero if necessary) is an admissible test function for both (4.44) and (4.46). Therefore

$$\begin{aligned} \int_{B_0} \{ & (\boldsymbol{M}Ds_\nu, D(s_\nu - s))\Phi^2 + 2\Phi(\boldsymbol{M}Ds_\nu, (s_\nu - s)D\Phi \\ & + (\bar{\boldsymbol{M}}Ds_\nu, s_\nu - s)\Phi^2\} dy = 0 \, , \end{aligned} \tag{4.50}$$

$$\int_{B_0} \{(\boldsymbol{B}Ds, D(s_\nu - s))\Phi^2 + 2\Phi(\boldsymbol{B}Ds, (s_\nu - s))D\Phi\} dy = 0 \, . \tag{4.51}$$

Finally, using the ellipticity condition

$$\begin{aligned}\int_{B_0} \Phi^2 |D(s_\nu - s)|^2 dy &\leqq \frac{1}{\mathscr{H}} \int_{B_0} \Phi^2(MD(s_\nu - s), D(s_\nu - s))dy \\ &= \frac{1}{\mathscr{H}} \left\{ \int_{B_0} \Phi^2(MDs_\nu, D(s_\nu - s))dy - \int_{B_0} \Phi^2(MDs, D(s_\nu - s))dy \right\} .\end{aligned} \tag{4.52}$$

Now we can estimate the first integral on the right hand side of (4.52) from (4.50) and the second one from (4.51) and we get

$$\int_{B_0} \Phi^2 |D(s_\nu - s)|^2 dy \longrightarrow 0 . \tag{4.53}$$

To bound the difference $S_{\nu,\tau} - S_\tau$ we write

$$\begin{aligned}|S_{\nu,\tau} - S_\tau| \leqq \tau^{2-n} \Big\{ \int_{B_0} \Phi^2(Ds_\nu^2 - Ds^2)dy &+ \int_{B_{0,\tau} \setminus B_{\nu,\tau}} |Ds|^2 dy \\ &+ \int_{B_{\nu,\tau} \setminus B_{0,\tau}} |Ds|^2 dy \Big\} .\end{aligned}$$

Here the first integral on the right hand side tends to zero by (4.53) and the second and third ones because of the uniform absolute continuity. Thus

$$S_\tau = \lim_{\nu \to \infty} S_{\nu,\tau} \geqq 2K\tau^2 \geqq 2K\tau^2 S , \tag{4.54}$$

which contradicts (4.48).

THEOREM 4.5. *Let the system* (2.1), (2.4) *satisfies the condition* (L^+), *let* u *be a weak solution of* (2.1), (2.4) *for which* $Du \in L_\infty(\Omega)$. *Then for each* $x_0 \in \Gamma$, *there exists a sequence* $R_\nu \searrow 0$ *such that*

$$\lim_{\nu \to \infty} Z(x_0, R_\nu) = 0 , \tag{4.55}$$

where $z = Du - Du_0$. (*For* $Z(x_0, R_\nu)$ *see* (4.1)).

Proof. Be $x_0 \in \Gamma$; $0 < R < \operatorname{dist}(x_0, \partial\Omega \setminus \Gamma)$. Put

$$y = y(x) = \frac{x - x_0}{R} , \tag{4.56}$$

$$u_R(y) = \frac{u(x_0 + Ry) - u(x_0)}{R} . \tag{4.57}$$

Then $y(B(x_0, R)) = B(0, 1)$. Put $0_R = y(\Omega)$. For each T, let $R(T)$ be such a positive radius that it is $B(0, T) \subset 0_R$ for $R < R(T)$.

In the following part of the proof we use the fact that for every $T > 0$ the set of second gradients $\{D^2 u_R; R < R(T)\}$ is bounded in $L_2(B(0, T))$. More precisely, it holds

LEMMA 4.6. *Let u be a solution of the problem* (2.1), (2.4) *for which $Du \in L_\infty(\Omega)$. Then for each $x_0 \in \Gamma$ and for every T, there exist $R(T)$ and C such that*

$$\|D^2 u_R\|_{L_2(B(0,T))} \leqq C \qquad \forall R < R(T) \ . \tag{4.58}$$

The value of C depends on $\|Du\|_{L_\infty}$, $\|f\|_{W^{1,p/2}}$, $\|u_0\|_{W^{2,p}}$, $\|g\|_{W^{1,\infty}}$, T, *and* dist $(x_0, \partial\Omega\backslash\Gamma)$.

Proof of the Lemma 4.6 is standard: using the finite difference technique and Nirenberg's lemma, we get the estimates for $D_{ij}u_R$, $ij \neq nn$. The bound for $D_{nn}u_R$ can be obtained by means of the equation in variations, which is valid a.e. on $B(0, T)$, and which enables us to express $D_{nn}u_R$ through the other second derivatives which we had estimated before.

Returning to the proof of Theorem 4.5, we see that the set $\{Du_R; R < R(T)\}$ is bounded in $L_\infty(\Omega)$-it follows from the assumption $Du \in L_\infty(\Omega)$ and the simple equality

$$\frac{\partial u_R(y)}{\partial y_i} = \frac{\partial u}{\partial x_i}(x_0 + Ry) \ .$$

Taking into account that $u_R(0) = 0$, we get finally the boundedness of the set $\{u_R; R < R(T)\}$ in $W^{2,2}(B(0, T))$. The compactness of the imbedding of $W^{2,2}(B(0, T))$ into $W^{1,2}(B(0, T))$ allows us to choose a sequence R_ν, $R_\nu \searrow 0$, such that $u_{R_\nu} \to p$ in $W^{1,2}(B(0, T))$, and, using the diagonal process, also

$$\begin{aligned} &\lim_{\nu\to\infty} u_{R_\nu} = p \quad \text{in} \quad W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}^n_+) \ , \\ &\lim_{\nu\to\infty} Du_{R_\nu} = Dp \quad \text{a.e. on } \boldsymbol{R}^n_+ \ . \end{aligned} \tag{4.59}$$

Deduce now the equation for the limit function p: To this end we substitute (4.56) and (4.57) into (2.6); after the passage to the limit we obtain

$$\int_{\boldsymbol{R}^n_+} (a(x_0, \xi, Dp(y)), D\varphi(y))dy = \int_{\{y \in \boldsymbol{R}^n; y_n = 0\}} (d, \varphi(y))dy' \ . \tag{4.60}$$

Using the theorem on traces and (4.59), we get

$$\lim_{\nu\to\infty} \boldsymbol{C}u_{R_\nu} = \boldsymbol{C}p \quad \text{a.e. on} \quad \{y \in \boldsymbol{R}^n; y_n = 0\} \ .$$

The transformed boundary conditions give

$$\boldsymbol{C}u_{R_\nu} = \boldsymbol{C}\Big(\frac{u_0(x_0 + R_\nu y) - u_0(x_0)}{R_\nu}\Big) \ ,$$

but $u_0 \in C_1(\bar{\Omega})$, hence

$$u_{0,R_\nu} = R_\nu^{-1}(u_0(x_0 + R_\nu y) - u_0(x_0)) \longrightarrow y_1 D_1 u_0(x_0) + \cdots + y_n D_n u_0(x_0) ,$$

so that Cp is a polynomial of at most the first degree on $\{y \in \boldsymbol{R}^n; y_n = 0\}$. The condition (L^+) implies that p is a polynomial of at most the first degree on $\boldsymbol{R}^n_+$.

Because of (4.59) and the fact that Dp is a constant vector, we have that

$$D(x_0, R_\nu) = \int_{B(0,1)} |Du_{R_\nu}(y) - (Du_{R_\nu})_{0,1}|^2 dy \longrightarrow 0$$

here $(Du)_{0,1}$ is the integral mean value of Du, i.e.

$$(Du)_{0,1} = \frac{1}{\operatorname{meas}_n B(0, 1)} \int_{B(0,1)} Du dy .$$

After an easy calculation ($u_0 \in W^{2,p}$ with $p > n$) we obtain that also

$$\widetilde{Z}(x_0, R) = \int_{B(0,1)} |D[u_{R_\nu} - u_{0,R_\nu}](y) - (D[u_{R_\nu} - u_{0,R_\nu}])_{0,1}|^2 dy \longrightarrow 0 . \tag{4.16}$$

The following lemma shows the relations between Z and $\widetilde{Z}$.

LEMMA 4.7. *Let the notation of the preceding lemma be preserved. Then there exists constants $\gamma > 0$, $\gamma_1 > 0$ such that for each point $x_0 \in \Gamma$, $R < \operatorname{dist}(x_0, \partial\Omega \backslash \Gamma)$, the estimate*

$$Z\left(x_0, \frac{R}{2}\right) \leqq \gamma \widetilde{Z}(x_0, R) + \gamma_1 \| D^2 u_0 \|^2_{L_p} R^{2(1-n/p)} \tag{4.62}$$

holds.

The proof of this lemma is similar to that of Lemma 4.1 — we insert a suitable test function of the type $\Phi^2(\omega - \omega_0 - c)$ (here $\omega_0 = \{D_i u_0\}_{i=1,\cdots,n-1}$; c is a constant vector satisfying the condition $Cc = 0$) into the equation in variations.

From (4.61) and (4.62) the assertion (4.55) of Theorem 4.5 follows.

To finish the proof of Theorem (4.2) it remains to observe that the difference between $Z(x_0, R)$ and $V(x_0, R)$ is small for small R thanks to the assumption $u_0 \in W^{2,p}(\Omega)$ and to use the same procedure as in [2], proof of Proposition 1.1 for the estimates of tangential derivatives. As for the second normal derivative, we repeat the estimates of (4.25). In such a way we get that the whole gradient belongs to the Morrey-Campanato space and thus $u \in C^{1,\alpha}(\overline{B(x_0, R_1)})$

with some R_1 sufficienty small.

REMARK. With some modification the same method can be used to prove the analogous theorems for any bounded domain with sufficiently smooth boundary.

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CHARLES UNIVERSITY
1, MALOSTRANSKE n. 25
11800 PRAHA 1, CZECHOSLOVAKIA