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FINITE HANKEL TRANSFORMS OF DISTRIBUTIONS

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Finite Hankel transforms of the second and third kind of distributions are defined and inversion theorems are established in the distributional sense. Operational transform formulae are obtained for the both transforms. These are applied to solve certain partial differential equations with distributional boundary conditions.

1. Introduction. Finite Hankel transforms of classical functions were first introduced by Sneddon [8] who applied them in solving boundary value problems for systems possessing axial symmetry. There are three kinds of finite Hankel transforms depending upon the nature of the kernel involved. These are associated to the three kinds of expansions of an arbitrary function, viz. Fourier Bessel series [6], Dini series [6] and series involving cross products of Bessel functions [4] respectively.

Finite Hankel transforms of distributions were given by Zemanian [11], Pandey and Pathak [3] as special cases of their work on general eigenfunction expansion of distributions. But, Dube [1] studied finite Hankel transform of the first kind of distributions independently. To get a deep insight it is necessary to study the other two transforms also independently. In [12] and [3] the inversion theorems are given without any consideration of the values of $H+\nu$ occurring in the definition of the transform (see (4.4)), where as the classical Dini series involves a term depending upon it. This motivated us to study independently the finite Hankel transforms of the second and third kind of distributions.

The present paper is divided into two parts. In the first part we extend the classical inversion theorem for finite Hankel transform of the second kind [6, p. 601] to a class of distributions, which gives rise to the Dini expansion of the distributions. The series converges in the weak distributional sense. We derive an operational transform formula which together with inversion formula is applied in solving certain distributional differential equations. In the second part of the paper we extend the inversion theorem for finite Hankel transform of the third kind [4] to a class of distributions. Here also the series converges in the weak distributional sense. Finally we give an application of the finite Hankel transform of the third kind.

2. The notation and terminology. We follow the notation and terminology of Schwartz [7] and Zemanian [12, 11]. Here I denotes the open interval (a, b), $0 \leqq a < b < \infty$. The letters t, x represent real variables in I. $D(I)$ is the space of infinitely differentiable functions on I with compact support contained in I. The topology of $D(I)$ is that which makes its dual $D'(I)$ of Schwartz's distributions. $E(I)$ is the space of all infinitely differentiable functions on I and $E'(I)$ is the space of distributions with compact support. $\Omega_{\nu,x}$ denotes the differential operator $D_x^2 + (1/x)D_x - (\nu^2/x^2)$.

3. The testing function space $U_{\alpha,\nu}(I)$. For any $(\alpha, \nu) \in \boldsymbol{R}^2$, we define

$$U_{\alpha,\nu}(I) = \{\varphi: I \longrightarrow \boldsymbol{C} \mid \varphi(x) \text{ is infinitely differentiable and } \varphi \text{ satisfies (3.1)}\}$$

$$\gamma_\kappa^{\alpha,\nu}(\varphi) = \sup_{a<x<b} |x^\alpha \Omega_{\nu,x}^k [x^{-1}\varphi(x)]| < \infty \quad \text{for each } k = 0, 1, 2, \cdots . \tag{3.1}$$

Clearly $U_{\alpha,\nu}(I)$ is a topological vector space. The topology of $U_{\alpha,\nu}(I)$ is generated by the collection of seminorms $\{\gamma_k^{\alpha,\nu}\}_{k=0}^\infty$. $U_{\alpha,\nu}(I)$ is a Frechet space. Its dual $U'_{\alpha,\nu}(I)$ is given the weak topology. Members of $U'_{\alpha,\nu}(I)$ will be referred as distributions.

Note. (i) $D(I) \subset U_{\alpha,\nu}(I)$ and topology of $D(I)$ is stronger than that induced on it by $U_{\alpha,\nu}(I) \Rightarrow f \in U'_{\alpha,\nu}(I)$ then $f|_D \in D'(I)$.

(ii) $E'(I)$ can be identified as a subspace of $U'_{\alpha,\nu}(I)$.

(iii) Given $f \in U'_{\alpha,\nu}(I)$ there exists $r \in \boldsymbol{N}^+$ and a positive constant C s.t.

$$|\langle f, \varphi\rangle| \leqq C \max_{0\leqq k\leqq r} \gamma_k^{\alpha,\nu}(\varphi) \quad \forall \varphi \in U_{\alpha,\nu}(I)$$

since f is bounded.

PART I

In this part we take $I = (0, 1)$ and study finite Hankel transform of the second kind of distributions.

4. Dini series. The Dini expansion associated with $f(t)$ is

$$B_0(t) + \sum_{m=1}^\infty b_m J_\nu(\lambda_m t) \tag{4.1}$$

where $J_\nu(t)$ is a Bessel function of first kind and λ_m, $m = 1, 2, 3, \cdots$, are the positive roots (arranged in ascending order of magnitude) of the transcendental equation

$$zJ'_\nu(z) + HJ_\nu(z) = 0 \tag{4.2}$$
$$\nu \geqq -\frac{1}{2}$$

b_m, $m = 1, 2, 3, \cdots$; are given by

$$b_m = \frac{2\lambda_m^2 \int_0^1 tf(t)J_\nu(\lambda_m t)dt}{(\lambda_m^2 - \nu^2)J_\nu^2(\lambda_m) + \lambda_m^2 J_\nu'^2(\lambda_m)} \tag{4.3}$$

and

$$B_0(t) = \begin{cases} 0 \quad \text{if} \quad H + \nu > 0 \\ 2(\nu + 1)t^\nu \int_0^1 x^{\nu+1} f(x)dx \quad \text{if} \quad H + \nu = 0 \\ \dfrac{2\lambda_0^2 I_\nu(\lambda_0 t)}{(\lambda_0^2 + \nu^2)I_\nu^2(\lambda_0) - \lambda_0^2 I_\nu'^2(\lambda_0)} \int_0^1 xf(x)I_0(\lambda_0 x)dx \\ \qquad \text{if} \quad H + \nu < 0 \,. \end{cases} \tag{4.4}$$

The condition of validity of (4.1) are given in the following theorem [6, p. 601].

THEOREM 4.1. *Let $f(t)$ be a function defined over the interval $(0, 1)$, and let $\int_0^1 t^{1/2} f(t)dt$ exist and (if it is improper integral) let it be absolutely convergent. If $f(t)$ has limited total fluctuation in (a, b) where $0 \leqq a < b \leqq 1$ then the series* (4.1) *converges to the sum $1/2[f(t + 0) + f(t - 0)]$ at all points t* s.t. *$a + \Delta \leqq t \leqq b - \Delta$ where Δ is arbitrarily small; and the convergence is uniform if $f(x)$ is continuous in (a, b).*

If, instead of the coefficients b_m, we introduce the finite Hankel transform of second kind of the function $f(x)$, deonted by $H_2(m)$, and defined by equation

$$H_2(m) = \int_0^1 tf(t)J_\nu(\lambda_m t)dt, \; m = 1, 2, 3, \cdots , \tag{4.5}$$

the above theorem on Dini series yields the inversion formula

$$f(t) = B_0(t) + \sum_{m=1}^{\infty} \frac{2\lambda_m^2 H_2(m)}{(\lambda_m^2 - \nu^2)J_\nu^2(\lambda_m) + \lambda_m^2 J_\nu'^2(\lambda_m)} \,. \tag{4.6}$$

The Theorem 4.1 will be extended to a class of generalized functions.

5. The generalized finite Hankel transform of the second

kind. Throughout this part we always have $\alpha \geqq 1/2$ and $\nu \geqq -1/2$.

DEFINITION. For $f \in U'_{\alpha,\nu}(I)$ we define its finite Hankel transform $\mathscr{H}_{2,\nu}(f)$ of second kind as:

$$(5.1) \qquad (\mathscr{H}_{2,\nu}f)(m) = F_2(m) = \langle f(x), xJ\nu(\lambda_m x)\rangle \quad m = 1, 2, 3, \cdots .$$

(5.1) is well defined since $xJ_\nu(\lambda_m x) \in U_{\alpha,\nu}(I)$, $m = 1, 2, 3, \cdots$, for $\alpha + \nu \geqq 0$. Note that for a function $f(x)$ defined on I such that $\int_0^1 x^{1-\alpha}|f(x)|dx$ exists for $\alpha \geqq 1/2$, we get a regular distribution T_f, which we identify with f, defined as

$$\langle T_f, \varphi\rangle = \int_0^1 f(x)\varphi(x)dx\ , \quad \varphi \in U_{\alpha,\nu}(I)\ .$$

Let us define

$$T_N(t, x; H) = A_0(x, t) + \sum_{m=1}^{N} \frac{2\lambda_m^2}{\eta_m^2} J_\nu(\lambda_m x)J_\nu(\lambda_m t)$$

where

$$\eta_m^2 = (\lambda_m^2 - \nu^2)J_\nu^2(\lambda_m) + \lambda_m^2 J_m'^2(\lambda_m)$$

and

$$A_0(x, t) = \begin{cases} 0 \quad \text{if} \quad H + \nu > 0 \\ 2(\nu + 1)x^\nu t^\nu \quad \text{if} \quad H + \nu = 0 \\ \dfrac{2\lambda_0^2 I_\nu(\lambda_0 x)I_\nu(\lambda_0 t)}{(\lambda_0^2 + \nu^2)I_\nu^2(\lambda_0) - \lambda_0^2 I_\nu'^2(\lambda_0)} \quad \text{if} \quad H + \nu < 0\ . \end{cases}$$

Notice that $xA_0(x, t) \in U_{\alpha,\nu}(I)$ if $\alpha + \nu \geqq 0$. Therefore

$$xT_N(t, x; H) \in U_{\alpha,\nu}(I) \quad \text{when} \quad \eta_m^2 \neq 0\ .$$

6. The inversion of (5.1). The following theorem provides an inversion formula for the distributional transform (5.1) which in turn gives a Dini series representation for $f \in U'_{\alpha,\nu}(I)$.

THEOREM 6.1. (*Inversion*). *Let $f \in U'_{\alpha,\nu}(I)$, $\alpha \geqq 1/2$, $\nu \geqq -1/2$. Let $F_2(m)$ be the finite Hankel transform of f. Then*

$$(6.1) \qquad f(t) = \lim_{N\to\infty} \sum_{m=1}^{N} \frac{2\lambda_m^2}{\eta_m^2} F_2(m)J_\nu(\lambda_m t) + \langle f(x), xA_0(x, t)\rangle$$

in the sense of convergence in $D'(I)$.

Proof. We have to prove that

$$(6.2)\qquad \left\langle \sum_{m=1}^{N} \frac{2\lambda_m^2}{\eta_m^2} F_2(m) J_\nu(\lambda_m x),\ \varphi(x) \right\rangle + \langle B_0(x), \varphi(x)\rangle \longrightarrow \langle f(t), \varphi(t)\rangle \quad \text{as} \quad N \longrightarrow \infty$$

for any $\varphi \in D(I)$, where $B_0(x)$ denotes $\langle f(t), tA_0(x, t)\rangle$. Now $\varphi(x) \in D(I)$ if and only if $x\varphi(x) \in D(I)$. So that (6.2) is equivalent to showing that

$$(6.3)\qquad \left\langle \sum_{m=1}^{N} \frac{2\lambda_m^2}{\eta_m^2} F_2(m) J_\nu(\lambda_m x),\ x\varphi(x) \right\rangle + \langle B_0(x), x\varphi(x)\rangle \longrightarrow \langle f(t), t\varphi(t)\rangle \quad \text{as} \quad N \longrightarrow \infty .$$

Suppose that support $(\varphi) \subset (a, b) \subsetneqq [0, 1]$. Now left hand side of (6.3)

$$(6.4)\qquad = \int_a^b \sum_{m=1}^{N} \left[\frac{2\lambda_m^2}{\eta_m^2} F_2(m) J_\nu(\lambda_m x) + B_0(x) \right] x\varphi(x) dx .$$

Since $B_0(x) = \langle f(t), tA_0(x, t)\rangle$ and $(2\lambda_m^2/\eta_m^2)F_2(m)J_\nu(\lambda_m x)$ are locally integrable over $(0, 1)$ and supp $(\varphi) \subset [a, b]$; (6.4) can be written as

$$\int_a^b \left[\sum_{m=1}^{N} \frac{2\lambda_m^2}{\eta_m^2} \langle f(t), tJ_\nu(\lambda_m t)\rangle J_\nu(\lambda_m x) + \langle f(t), tA_0(x, t)\rangle \right] x\varphi(x) dx$$

$$= \int_a^b \left\langle f(t),\ t\left[\sum_{m=1}^{N} \frac{2\lambda_m^2}{\eta_m^2} J_\nu(\lambda_m t) J_\nu(\lambda_m x) + A_0(x, t) \right] \right\rangle x\varphi(x) dx$$

$$(6.5)\qquad = \int_a^b \langle f(t), tT_N(t, x; H)\rangle x\varphi(x) dx$$

$$(6.6)\qquad = \left\langle f(t),\ t\int_a^b T_N(t, x; H) x\varphi(x) dx \right\rangle$$

$$(6.7)\qquad \longrightarrow \langle f(t), t\varphi(t)\rangle \quad \text{as} \quad N \longrightarrow \infty .$$

Once we prove the equality of (6.5), (6.6) and (6.7) our proof of the theorem will be complete. We prove the above by the following series of lemmas.

LEMMA 6.2. *Let $f \in U'_{\alpha,\nu}(I)$. Then for any $N \in N^+$ and $\varphi \in D(I)$ we have*

$$(6.8)\qquad \int_0^1 \langle f(t), tT_N(t, x; H)\rangle x\varphi(x) dx = \left\langle f(t), \int_0^1 tT_N(t, x; H) x\varphi(x) dx \right\rangle .$$

Proof. Since $tT_N(t, x; H) \in U_{\alpha,\nu}(I)$ for fixed x, the left hand side of (6.8) makes sense. Also since

$$\int_0^1 tT_N(t, x; H)x\varphi(x)dx = t\int_0^1\left[\sum_{m=1}^N \frac{2\lambda_m^2}{\eta_m^2}J_\nu(\lambda_m x)J_\nu(\lambda_m t) + A_0(x, t)\right]x\varphi(x)dx$$

and since $\int_0^1 J_\nu(\lambda_m x)x\varphi(x)dx < \infty$ and $\int_0^1 A_0(x, t)x\varphi(x)dx < \infty$, we have $\int_0^1 tT_N(t, x; H)x\varphi(x)dx \in U_{\alpha,\nu}(I)$; the right hand side of (6.8) makes sense.

Now left hand side of (6.8)

$$\begin{aligned}
&= \int_0^1 \langle f(t), tT_N(t, x; H)\rangle x\varphi(x)dx \\
&= \int_0^1 \left\langle f(t), t\left[\sum_1^N \frac{2\lambda_m^2}{\eta_m^2}J_\nu(\lambda_m x)J_\nu(\lambda_m t) + A_0(x, t)\right]\right\rangle x\varphi(x)dx \\
&= \sum_1^N \frac{2\lambda_m^2}{\eta_m^2}\langle f(t), tJ_\nu(\lambda_m t)\rangle \int_0^1 J_\nu(\lambda_m x)x\varphi(x)dx \\
&\quad + \left\langle f(t), t\int_0^1 A_0(x, t)x\varphi(x)dx\right\rangle \\
&= \sum_1^N \left\langle f(t), \frac{2\lambda_m^2}{\eta_m^2}t\int_0^1 J_\nu(\lambda_m x)J_\nu(\lambda_m t)x\varphi(x)dx\right\rangle \\
&\quad + \left\langle f(t), t\int_0^1 A_0(x, t)x\varphi(x)dx\right\rangle \\
&= \left\langle f(t), t\int_0^1 T_N(t, x; H)x\varphi(x)dx\right\rangle .
\end{aligned}$$

This proves the lemma.

LEMMA 6.3. *Let $a, b \in \mathbf{R}$ such that $0 < a < b < 1$. Then*

$$\lim_{N\to\infty}\int_a^b T_N(t, x; H)xdx = 1 , \quad a < t < b .$$

Proof. We have

$$\int_a^b T_N(t, x; H)xdx = \int_a^b x[T_N(t, x) - S_N(t, x; H)]dx .$$

Now for $a < t < b$, $\int_a^b xT_N(t, x)dx \to 1$ as $N \to \infty$ [1, p. 368]. And $\int_a^b S_N(t, x; H)xdx \to 0$ as $N \to \infty$ by the analogue of Riemann Lebesgue lemma [6, p. 599].

$T_N(t, x)$ is defined as:

$$T_N(t, x) = \sum_{m=1}^N \frac{2J_\nu(j_m x)J_\nu(j_m t)}{J_{\nu+1}^2(j_m)}$$

where j_m, $m = 1, 2, 3\cdots$, are the positive zeros of $J_\nu(z)$ arranged in ascending order. $S_N(t, x; H)$ is defined as $T_N(t, x) - T_N(t, x; H)$.

LEMMA 6.4. *Let $\psi(x) \in D(I)$. Then for $\alpha \geqq 1/2$ and $\nu \geqq -1/2$ $t^\alpha \int_a^b T_N(t, x; H)[\psi(x) - \psi(t)]x dx \to 0$ as $N \to \infty$ uniformly for all $t \in (0, 1)$ where* supp. $(\psi) \subset [a, b] \subsetneqq (0, 1)$.

Proof. We have $(0, 1) = [(0, a) \cup (b, 1)] \cup [a, b]$. For $t \in (0, a) \cup (b, 1)$; $\psi(t) = 0$. Therefore

$$\int_a^b T_N(t, x; H)[\psi(x) - \psi(t)]x dx = \int_a^b x T_N(t, x; H)\psi(x) dx$$
$$= \int_a^b x\psi(x) T_N(t, x) dx - \int_a^b x\psi(x) S_N(t, x; H) dx .$$

In view of the analogue of Riemann-Lebesgue lemma [6, p. 590 and p. 600], given $\varepsilon > 0$ there exists N_0 such that for $N \geqq N_0$ we have

$$\left|\int_a^b x\psi(x) T_N(t, x) dx\right| \leqq \frac{8C_1^2\varepsilon}{\pi C_2^2(2 - t - b)\sqrt{t}} < \frac{8C_1^2\varepsilon}{\pi C_2^2(1 - b)\sqrt{t}}$$

and

$$\left|\int_a^b x\psi(x) S_N(t, x; H) dx\right| < \frac{2C_3\varepsilon}{(2 - t - b)\sqrt{t}} < \frac{2C_3}{(1 - b)\sqrt{t}}\varepsilon$$

where C_1, C_2 and C_3 are some constants. Thus

$$t^\alpha \left|\int_a^b T_N(t, x; H) x\psi(x) dx\right| < \frac{C}{(1 - b)} t^{\alpha - 1/2}\varepsilon < \frac{C}{(1 - b)}\varepsilon$$

since $\alpha \geqq 1/2$, where C is a constant. So that

$$t^\alpha \int_a^b T_n(t, x; H)[\psi(x) - \psi(t)]x dx \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty \tag{6.9}$$

uniformly for all $t \in (0, a) \cup (b, 1)$.

Next we have to deal with the case $t \in [a, b]$. From [1] we know that

$$t^\alpha \int_a^b T_N(t, x)[\psi(x) - \psi(t)]x dx \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty \tag{6.10}$$

uniformly for all $t \in [a, b]$. Hence we have merely to show that

$$t^\alpha \int_a^b S_N(t, x; H)[\psi(x) - \psi(t)]x dx \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty$$

uniformly for all $t \in [a, b]$.

Let $F(t, x) = x^{-\nu}[\psi(x) - \psi(t)]$ for $0 < x < 1, 0 < t < 1$. Clearly $F(t, x)$ is continuous for $0 < x < 1, 0 < t < 1$. So that

$$\int_a^b S_N(t, x; H)[\psi(x) - \psi(t)]x dx = \int_a^b x^{\nu+1} F(t, x) S_N(t, x; H) dx .$$

Divide $[a, b]$ into p equal parts by means of the points $a = x_0, x_1, \cdots, x_p = b$. For $\varepsilon > 0$ arbitrary, choose p large enough so that

$$\sum_{m=1}^{p} (U_m - L_m)(x_m - x_{m-1}) < \varepsilon$$

where U_m and L_m are upper and lower bound of $F(t, x)$ in (x_{m-1}, x_m) for $a \leqq t \leqq b$.

Let $F(t, x) = F(t, x_{m-1}) + W_m(t, x)$. Then $|W_m(t, x)| \leqq U_m - L_m$. Therefore for all $N \geqq N_1$(depending on ε)

$$\begin{aligned} &\int_a^b S_N(t, x; H)[\psi(x) - \psi(t)]x dx \\ &\quad = \sum_{m=1}^{p} F(t, x_{m-1}) \int_{x_{m-1}}^{x_m} x^{\nu+1} S_N(t, x; H) dx \\ &\quad\quad + \sum_{m=1}^{p} \int_{x_{m-1}}^{x_m} x^{\nu+1} W_m(t, x) S_N(t, x; H) dx \\ &\quad < \frac{2C_3}{(1-b)\sqrt{t}} \varepsilon \quad [6, \text{ p. } 600] . \end{aligned}$$

Therefore

$$t^\alpha \int_a^b S_N(t, x; H)[\psi(x) - \psi(t)]x dx < \frac{2C_3}{(1-b)} t^{\alpha - 1/2} \varepsilon < \frac{2C_3}{(1-b)} \varepsilon .$$

This together with (6.9) and (6.10) proves the lemma.

LEMMA 6.5. *Let* $\varphi \in D(I)$ *with* supp $(\varphi) \subset [a, b]$. *Then for* $\alpha \geqq 1/2$ *and* $\nu \geqq -1/2$

$$t^\alpha \Omega_{\nu,t}^k \left[\int_a^b T_N(t, x; H) x \varphi(x) dx - \varphi(t) \right] \longrightarrow 0$$

as $N \to \infty$ *uniformly for all* $t \in (0, 1)$.

Proof. It is easily seen that

$$\Omega_{\nu,x}[T_N(t, x; H)] = \Omega_{\nu,t}[T_N(t, x; H)] .$$

Therefore by integration by parts,

$$\begin{aligned} &\Omega_{\nu,t}^k \int_a^b T_N(t, x; H) x \varphi(x) dx \\ &\quad = \int_a^b T_N(t, x; H) \Omega_{\nu,x}^k [\varphi(x)] x dx . \end{aligned}$$

Using Lemma 6.3 we get, as $N \to \infty$,

$$\Omega^k_{\nu,t}\left[\int_a^b T_N(t, x; H)\varphi(x)xdx - \varphi(t)\right]$$
$$= \int_a^b T_N(t, x; H)[\Omega^k_{\nu,x}[\varphi(x)] - \Omega^k_{\nu,t}[\varphi(t)]]xdx$$
$$= \int_a^b T_N(t, x; H)[\psi(x) - \psi(t)]xdx$$

where $\psi(x) = \Omega^k_{\nu,x}\varphi(x) \in D(I)$ and supp $(\psi) \subset [a, b]$. Now an application of Lemma 6.4 proves the Lemma 6.5.

THEOREM 6.6. (*The uniqueness theorem.*) *Let* $f, g \in U'_{\alpha,\nu}(I)$. *If* $F_2(m) = G_2(m)$ *for each* $m = 1, 2, \cdots$; *and*

$$\langle f(x), xA_0(x, t)\rangle = \langle g(x), xA_0(x, t)\rangle .$$

Then

$$f = g \quad \textit{in the sense of equality in } D'(I) .$$

The proof is trivial.

7. Illustration of the inversion theorem by means of a numerical example. For $0 < k < 1$, $\delta(t - k) \in E'(I) \subset U'_{\alpha,\nu}(I)$. The finite Hankel transform of $\delta(t - k)$ *is*

$$\mathscr{H}_{2,\nu}(\delta(t - k))(m) = \langle \delta(t - k), tJ_\nu(\lambda_m t)\rangle$$
$$= kJ_\nu(\lambda_m k) \quad m = 1, 2, \cdots ,$$

and

$$\langle \delta(t - k), tA_0(x, t)\rangle = kA_0(x, k) .$$

For any $\varphi \in D(I)$

$$\left\langle \sum_1^N \frac{2\lambda_m^2}{\eta_m^2} kJ_\nu(\lambda_m k)J_\nu(\lambda_m x), x\varphi(x)\right\rangle + \langle kA_0(x, k), x\varphi(x)\rangle$$
$$= k\int_0^1 \sum_{m=1}^N \frac{2\lambda_m^2}{\eta_m^2} J_\nu(\lambda_m k)J_\nu(\lambda_m x)x\varphi(x)dx$$
$$+ k\int_0^1 A_0(x, k)x\varphi(x)dx$$
$$= k\int_0^1\left[\sum_{m=1}^N \frac{2\lambda_m^2}{\eta_m^2} J_\nu(\lambda_m k)J_\nu(\lambda_m x) + A_0(x, k)\right]x\varphi(x)dx$$
$$= k\int_0^1 T_N(t, x; H)x\varphi(x)dx \longrightarrow k\varphi(k) \quad \text{as} \quad N \longrightarrow \infty .$$

But $\langle \delta(t - k), t\varphi(t)\rangle = k\varphi(k)$. Therefore the inversion theorem is illustrated.

This also yields Dini series expansion for $\delta(t - k)$ as

$$\delta(t-k)=\lim_{N\to\infty}\sum_{m=1}^{N}2(\lambda_m/\eta_m)^2kJ_\nu(\lambda_m k)J_\nu(\lambda_m t)+kA_0(k,t)$$

in the sense of convergence in $D'(I)$.

8. Applications. Now we obtain an operation transform formula which together with inversion theorem is useful in solving certain distributional differential equations.

Now for $\varphi\in U_{\alpha,\nu}(I)$ and $f\in U'_{\alpha,\nu}(I)$

$$\langle\Omega^*_{x,\nu}f(x),\varphi(x)\rangle=\langle f(x),x\Omega_{x,\nu}x^{-1}\varphi(x)\rangle\ . \tag{8.1}$$

(8.1) defines a generalized operator $\Omega^*_{x,\nu}$ on $U'_{\alpha,\nu}$ (I) adjoint of the operator $x\Omega_{x,\nu}x^{-1}$ on $U_{\alpha,\nu}(I)$.

$$\varphi(x)\in U_{\alpha,\nu}(I)\Longrightarrow x\Omega_{x,\nu}x^{-1}\varphi(x)\in U_{\alpha,\nu}(I)\ .$$

Therefore $\Omega^*_{x,\nu}$ is well defined by (8.1).

Now since $\varphi(x)\to x\Omega_{x,\nu}x^{-1}\varphi(x)$ is a linear continuous map on $U_{\alpha,\nu}(I)$, $\Omega^*_{x,\nu}$ is linear and continuous on $U'_{\alpha,\nu}(I)$. By induction on k we get

$$\langle\Omega^{*k}_{x,\nu}f(x),\varphi(x)\rangle=\langle f(x),x\Omega^k_{\nu,x}x^{-1}\varphi(x)\rangle \tag{8.2}$$

and $\Omega^{*k}_{x,\nu}$ is linear and continuous on $U'_{\alpha,\nu}(I)$. So that

$$\begin{aligned}\langle\Omega^{*k}_{x,\nu}f(x),xJ_\nu(\lambda_m x)\rangle&=\langle f(x),x\Omega^k_{x,\nu}J_\nu(\lambda_m x)\rangle\\&=(-1)^k\lambda_m^{2k}\langle f(x),xJ_\nu(\lambda_m x)\rangle\ .\end{aligned}$$

Thus

$$\begin{aligned}\mathscr{H}_{2,\nu}[\Omega^{*k}_{x,\nu}f](m)&=\langle\Omega^{*k}_{x,\nu}f(x),xJ_\nu(\lambda_m x)\rangle\\&=(-1)^k\lambda_m^{2k}\mathscr{H}_{2,\nu}(f)(m)\ ,\quad m=1,2,\cdots\ .\end{aligned} \tag{8.3}$$

For f a regular distribution in $U'_{\alpha,\nu}(I)$ generated by elements of $D(I)$, we get

$$\Omega^*_{x,\nu}f=\Omega_{x,\nu}f\qquad\text{(integration by parts)}.$$

Also for f a regular distribution in $U'_{\alpha,\nu}(I)$, if we put some suitable condition on it so that the limit terms in integration by parts in (8.1) vanish, we get

$$\Omega^*_{x,\nu}f=\Omega_{x,\nu}f\ .$$

Now consider the operational equation

$$P(\Omega^*_{x,\nu})u=g\ ,\quad 0<x<1\ . \tag{8.4}$$

We wish to solve (8.4) for P a polynomial such that $P(-\lambda_m^2)\neq0$, $m=1,2,\cdots$; and $g\in U'_{\alpha,\nu}(I)$ is given; $u\in U'_{\alpha,\nu}(I)$ is unknown to be found.

Apply generalized finite Hankel transform to (8.4) to get

$$\langle P(\Omega^*_{x,\nu})u(x),\, xJ_\nu(\lambda_m x)\rangle = G_2(m)\ .$$

Now left hand side $= P(-\lambda_m^2)U_2(m)$ by (8.3), hence

$$U_2(m) = \frac{G_2(m)}{P(-\lambda_m^2)}\ . \tag{8.5}$$

Applying the inversion theorem to (8.5) we get

$$u(x) = \lim_{N\to\infty}\sum_{m=1}^{N}\frac{2\lambda_m^2}{\eta_m^2}\,\frac{G_2(m)}{P(-\lambda_m^2)}J_\nu(\lambda_m x) + \langle u(t),\, tA_0(x,\, t)\rangle\ . \tag{8.6}$$

We have to find $\langle u(t),\, tA_0(x,\, t)\rangle$.

Case (1): When $H + \nu > 0$, $A_0(x, t) = 0$. Therefore

$$\langle u(t),\, tA_0(x,\, t)\rangle = 0\ .$$

Case (ii): When $H + \nu = 0$, $A_0(x, t) = 2(\nu + 1)x^\nu t^\nu$.
Now, let us assume that

$$P(x) = \sum_{r=0}^{n} a_r x^r\ ,\quad a_0 \neq 0\ .$$

Then

$$\begin{aligned}\langle g(x),\, xA_0(x,\, t)\rangle &= \langle P(\Omega^*_{x,\nu})u(x),\, xA_0(x,\, t)\rangle\\ &= \left\langle \sum_{r=0}^{n} a_r(\Omega^{*r}_{x,\nu})u(x),\, xA_0(x,\, t)\right\rangle\\ &= \left\langle u(x),\, \sum_{r=0}^{n} a_r x(\Omega^r_{x,\nu})(x^{-1}xA_0(x,\, t))\right\rangle\\ &= a_0\langle u(x),\, xA_0(x,\, t)\rangle\ .\end{aligned}$$

Thus

$$\langle u(t),\, tA_0(x,\, t) > \; = \frac{1}{a_0}\langle g(t),\, tA_0(x,\, t)\rangle\ .$$

Case (iii) When $H + \nu < 0$,

$$A_0(x,\, t) = \frac{2\lambda_0^2 I_\nu(\lambda_0 x)I_\nu(\lambda_0 t)}{\eta_0^2}$$

where

$$\eta_0^2 = (\lambda_0^2 + \nu^2)I_\nu^2(\lambda_0) - \lambda_0^2 I_\nu'^2(\lambda_0)\ .$$

Now using the fact that

$$\Omega^k_{x,\nu}A_0(x,\, t) = \lambda_0^{2k}A_0(x,\, t)\ ,$$

we have

$$\langle \Omega_{x,\nu}^{*k} u(x),\, xA_0(x,\, t)\rangle = \lambda_0^{2k}\langle u(x),\, xA_0(x,\, t)\rangle\ .$$

So that

$$\begin{aligned}\langle g(x),\, xA_0(x,\, t)\rangle &= \langle P(\Omega_{x,\nu}^{*})u(x),\, xA_0(x,\, t)\rangle \\ &= P(\lambda_0^2)\langle u(x),\, xA_0(x,\, t)\rangle\ .\end{aligned}$$

This gives

$$\langle u(t),\, tA_0(x,\, t)\rangle = \frac{1}{P(\lambda_0^2)}\langle g(t),\, tA_0(x,\, t)\rangle$$

provided that $P(\lambda_0^2) \neq 0$.

Thus finally from (8.6) we have

$$u(x) = \lim_{N\to 1}\sum_{m=1}^{N}\frac{2\lambda_m^2}{\eta_m^2}\frac{G_2(m)}{P(-\lambda_m^2)}J_\nu(\lambda_m x),\ (H+\nu>0) \tag{8.7}$$

$$u(x) = \lim_{N\to\infty}\sum_{m=1}^{N}\frac{2\lambda_m^2}{\eta_m^2}\frac{G_2(m)}{P(-\lambda_m^2)}J_\nu(\lambda_m x) + \frac{1}{a_0}\langle g(t),\, tA_0(x,\, t)\rangle \quad (H+\nu=0,\ a_0\neq 0) \tag{8.8}$$

$$u(x) = \lim_{N\to\infty}\sum_{m=1}^{N}\frac{2\lambda_m^2}{\eta_m^2}\frac{G_2(m)}{P(-\lambda_m^2)}J_\nu(\lambda_m x) + \frac{1}{P(\lambda_0^2)}\langle g(t),\, tA_0(x,\, t)\rangle \quad (H+\nu<0,\ P(\lambda_0^2)\neq 0)\ . \tag{8.9}$$

$u(x)$ given by (8.7)-(8.9) gives the solution of (8.4) with equality in the sense of $D'(I)$. This solution is in fact a restriction of $u \in U'_{\alpha,\nu}(I)$ to $D(I)$, and is unique in view of Theorem 6.6.

It can easily be shown that u given by (8.7) – (8.9) is also a solution of

$$P(\Omega_{x,\nu})u = g\ . \tag{8.10}$$

Now for

$$P(x) = (x - a_1^2)\cdots(x - a_n^2)$$

where a_i's are distinct real numbers, the general solution of (8.10) in $D'(I)$ is given by

$$\begin{aligned}u(x) &= \lim_{N\to\infty}\sum_{m=1}^{N}\frac{2\lambda_m^2 G_2(m)}{\eta_m^2 P(-\lambda_m^2)}J_\nu(\lambda_m x) \\ &\quad + \sum_{k=1}^{n} J_\nu(a_k x i)\Big[c_k\int_{1/2}^{x}[tJ_\nu^2(a_k t i)]^{-1}dt + d_k\Big]\end{aligned}$$

where C_k, d_k are arbitrary constants, when $H+\nu>0$. Similarly when $H+\nu=0$ and $H+\nu<0$, the solution in $D'(I)$ is obtained by adding

$$\sum_{k=1}^{n} J_\nu(a_k xi)\left[C_k\int_{1/2}^{x}\frac{dt}{tJ_\nu^2(a_k ti)} + d_k\right]$$

to the right hand sides of (8.8) and (8.9) respectively.

9. Application of the finite Hankel transform of the second kind. (Heat flow in an infinite cylinder with a radiation condition.)

We wish to solve the heat equation in cylindrical coordinates inside an infinitely long cylinder of radius unity, by using the theory of the finite Hankel transform of second kind developed in the preceding pages. We seek a conventional function $u(r, t)$; where r is radius and t is time, (u does not depend on θ and z) satisfying the differential equation

$$D_r^2 u + \frac{1}{r}D_r u = D_t u \quad (0 < r < 1, 0 < t < \infty) \tag{9.1}$$

and the following initial and boundary conditions:

(i) As $t \to 0^+$, $u(r, t) \to f(r) \in U'_{\alpha,\nu}(I)$ in the sense of convergence in $D'(I)$.

(ii) As $r \to 1^-$, $D_r u + H_u \to 0$ in $D'(I)$ for each fixed $t > 0$, where $H > 0$.

When u denotes the temperature within the cylinder, $H > 0$ means that the heat is being radiated away from the surface of the cylinder.

The differential equation for u can be written as

$$\Omega_{0,r} u = \frac{\partial u}{\partial t} . \tag{9.2}$$

Let us apply the generalized finite Hankel transform $\mathscr{H}_{2,0}$ to (9.1) to get

$$-\lambda_m^2 U_2(m, t) = \frac{\partial}{\partial t} U_2(m, t)$$

where

$$U_2(m, t) = \mathscr{H}_{2,0}[u(r, t)] = \langle u(r, t), rJ_0(\lambda_m r)\rangle ,$$

so that

$$U_2(m, t) = A(m)e^{-\lambda_m^2 t} .$$

The initial condition determines the constant $A(m)$. Thus

$$A(m) = F_2(m) = \langle f(r), rJ_0(\lambda_m r)\rangle .$$

Hence

$$U_2(m, t) = F_2(m)e^{-\lambda_m^2 t}\,.$$

Therefore, by inversion Theorem 6.1 we have

$$u(r, t) = \lim_{N\to\infty}\sum_{m=1}^{N} \frac{2F_2(m)e^{-\lambda_m^2 t}J_0(\lambda_m r)}{J_0^2(\lambda_m) + J_1^2(\lambda_m)} \tag{9.3}$$

in $D'(I)$, since $H > 0$. Here $\lambda_m'^s$ are the roots of the equation

$$\lambda J_0'(\lambda) + HJ_0(\lambda) = 0\,.$$

We want to prove that $u(r, t)$ given by (9.3) is truly a solution of (9.1) that satisfies the given initial and boundary conditions. Using the boundedness property of generalized functions we have

$$|F_2(m)| \leqq C \max_{0\leqq k\leqq n} \gamma_k^{\alpha,\nu}(rJ_0(r\lambda_m))\,,$$

so that

$$F_2(m) = O(\lambda_m^{2n-1/2}) \quad \text{as} \quad m \longrightarrow \infty$$

for some nonnegative integer n. Also

$$\lambda_m \sim \pi\left(m + \frac{1}{4}\right) \quad \text{as} \quad m \longrightarrow \infty$$

$$J_0^2(\lambda_m) + J_1^2(\lambda_m) \sim \frac{2}{\pi\lambda_m} \quad \text{as} \quad m \longrightarrow \infty\,.$$

Hence

$$[J_0^2(\lambda_m) + J_1^2(\lambda_m)]^{-1} = O(m) \quad \text{as} \quad m \longrightarrow \infty\,.$$

Using the above facts we see that the series (9.3) and series obtained by applying $\Omega_{0,r}$ and D_t separately under the summation sign of (9.3) converges uniformly on $0 < r < 1$ and $t > 0$. So by applying $\Omega_{0,r} - D_t$ and using the fact $\Omega_{0,t}[J_0(\lambda_m r)] = -\lambda_m^2 J_0(\lambda_m r)$ we see that (9.3) satisfies the differential equation (9.1).

Let us verify the boundary condition (ii). We have

$$\lim_{r\to 1^-}[D_r u + Hu] = \lim_{r\to 1^-}\left[\sum_1^\infty D_r\left\{\frac{2F_2(m)e^{-\lambda_m^2 t}J_0(\lambda_m r)}{J_0^2(\lambda_m) + J_1^2(\lambda_m)}\right\} + H\sum_1^\infty \frac{2F_2(m)e^{-\lambda_m^2 t}J_0(\lambda_m r)}{J_0^2(\lambda_m) + J_1^2(\lambda_m)}\right]$$

and since the convergence is uniform, we can take the limit $r \to 1^-$ inside the summation sign and arrive at the conclusion.

Next we wish to verify the initial condition (i). For any $\varphi \in D(I)$, we have

$$\lim_{t\to 0+}\left\langle \sum_{m=1}^{\infty}\frac{2F_2(m)e^{-\lambda_m^2 t}J_0(\lambda_m r)}{J_0^2(\lambda_m)+J_1^2(\lambda_m)},\ \varphi(r)\right\rangle$$

$$=\lim_{t\to 0+}\int_0^1\sum_{m=1}^{\infty}\frac{2F_2(m)e^{-\lambda_m^2 t}}{J_0^2(\lambda_m)+J_1^2(\lambda_m)}J_0(\lambda_m r)\varphi(r)dr$$

$$=\int_0^1\sum_{m=1}^{\infty}\frac{2F_2(m)}{J_0^2(\lambda_m)+J_1^2(\lambda_m)}J_0(\lambda_m r)\varphi(r)dr$$

(since the convergence is uniform on $0 < r < 1$ and $t > 0$)

$$=\langle f(r),\ \varphi(r)\rangle \quad \text{by Theorem 6.1.}$$

PART II

10. Finite Hankel transform of the third kind. The finite Hankel transform of the third kind of an arbitrary function $f(x)$, defined on $0 < a < x < b$ is defined by

$$H_3(m)=\int_a^b tf(t)C_\nu(\gamma_m t,\ \gamma_m b)dt\ ,\quad m=1,2,3,\cdots; \tag{10.1}$$

where

$$C_\nu(\alpha,\ \beta)=J_\nu(\alpha)Y_\nu(\beta)-Y_\nu(\alpha)J_\nu(\beta)$$

and γ_m is the mth positive root of the equation

$$C_\nu(az,\ bz)=0\ .$$

The following theorem [4] provides an inversion formula for the transform (10.1).

THEOREM 10.1. *If $f(t)$ is summable over (a, b) and of bounded variation in the neighborhood of the point $t = x$, then the series*

$$\sum_{m=1}^{\infty}a_m C_\nu(\gamma_m x,\ \gamma_m b)\ , \tag{10.2}$$

where

$$a_m=\frac{\pi^2\gamma_m^2 J_\nu^2(\gamma_m a)}{2[J_\nu^2(\gamma_m a)-J_\nu^2(\gamma_m b)]}H_3(m) \tag{10.3}$$

and ν is any real number, converges to the sum $1/2[f(x+0)+f(x-0)]$.

We extend the above theorem of Titchmarsh for $f\in U'_{0,\nu}(I)$, where ν is any real number, $I=(a,b)$, $0 < a < b < \infty$. For convenience we shall write $U'_\nu(I)$ in place $U'_{0,\nu}(I)$.

DEFINITION. For $f \in U'_\nu(I)$, $\nu \in \boldsymbol{R}$, we define the distributional finite Hankel transform of f of third kind by

(10.4) $(\mathscr{H}_{3,\nu}f)(m) = F_3(m) = \langle f(t), tC_\nu(\gamma_m t, \gamma_m b)\rangle \quad m = 1, 2, \cdots .$

Throughout the second part of the paper we always have $\nu \in \boldsymbol{R}$. Also we set

$$\bar{\eta}_m^2 = \frac{\pi^2}{2}\gamma_m^2 \frac{J_\nu^2(\gamma_m a)}{J_\nu^2(\gamma_m a) - J_\nu^2(\gamma_m b)}$$

and

$$R_N(t, x) = \sum_{m=1}^{N} \bar{\eta}_m^2 C_\nu(\gamma_m x, \gamma_m b) C_\nu(\gamma_m t, \gamma_m b) . \tag{10.5}$$

11. The inversion of (10.4). We will now prove the following inversion theorem for our generalized finite Hankel transform.

THEOREM 11.1. (*Inversion*). *Let f be an arbitrary distribution in the space $U'_\nu(I)$, $\nu \in \boldsymbol{R}$ and let $F_3(m)$ be the finite Hankel transform of the third kind of f. Then in the sense of convergence in $D'(I)$,*

$$f(t) = \lim_{N\to\infty} \sum_{m=1}^{N} \bar{\eta}_m^2 F_3(m) C_\nu(\gamma_m t, \gamma_m b) . \tag{11.1}$$

Proof. Since $\varphi(x) \in D(I) \Leftrightarrow x\varphi(x) \in D(I)$, it suffices to prove that

$$\left\langle \sum_{m=1}^{N} \bar{\eta}_m^2 F_3(m) C_\nu(\gamma_m t, \gamma_m b),\ x\varphi(x) \right\rangle \longrightarrow \langle f(t), t\varphi(t)\rangle \quad \text{as} \quad N \longrightarrow \infty .$$

The proof is similar to the proof of the Theorem 6.1. To complete the proof we have to prove similar lemmas as needed in the proof of the Theorem 6.1. We do it next.

LEMMA 11.2. *Let $f \in U'_\nu(I)$. Then for any $N \in \boldsymbol{N}^+$ and $\varphi \in D(I)$ we have*

$$\int_a^b \left\langle f(t), tR_N(t, x) \right\rangle x\varphi(x)dx = \left\langle f(t), \int_a^b tR_N(t, x)x\varphi(x)dx \right\rangle . \tag{11.2}$$

Proof is similar to that of Lemma 6.2.

LEMMA 11.3. *For $0 < a < b < \infty$ we have*

$$\lim_{N\to\infty} \int_a^b R_N(t, x)xdx = 1 \quad (0 < a < t < b) . \tag{11.3}$$

Proof. See [4, equation (7)].

LEMMA 11.4. *Let $f(t)$ be a bounded function in (a, b), $0 < a < b$. Then $\varepsilon > 0$, there exists $N_0 \in N^+$ such that for all $N \geqq N_0$,*

$$\int_A^B t f(t) R_N(t, x) dt < \frac{2K\varepsilon}{\rho_x} \tag{11.4}$$

where $a < A < B < b$, and x lies out side the interval $[A, B]$, K is a constant, and

$$\rho_x = \operatorname{dist}(x, [A, B]) > 0 .$$

Proof. Let $F(t) = t^{-\nu} f(t)$. Then following the same line of proof as in Lemma 6.4, we get

$$\begin{aligned}\int_A^B t f(t) R_N(t, x) dt = \sum_{m=1}^{p} F(t_{m-1}) \int_{t_{m-1}}^{t_m} t^{\nu+1} R_N(t, x) dt \\ + \sum_{m=1}^{p} \int_{t_{m-1}}^{t_m} t^{\nu+1} R_N(t, x) W_m(t) dt .\end{aligned} \tag{11.5}$$

Let $M = \max\{A^{\nu+1}, B^{\nu+1}\}$.

Choose the partition p so fine that

$$\sum_{m=1}^{p} (U_m - L_m)(t_m - t_{m-1}) M < \varepsilon .$$

Let K_1 be the upper bound of $F(t)$ in $[A, B]$. From [4], we know that

$$|R_N(t, x)| < \frac{k}{|t - x|}$$

$$\left|\int_t^b t^{\nu+1} R_N(t, x) dt\right| < \frac{k}{A_N(b - x)} + \frac{k}{A_N(t - x)} \qquad (x < t)$$

$$\left|\int_a^t t^{\nu+1} R_N(t, x) dt\right| < \frac{k}{A_N(x - t)} + \frac{k}{A_N(x - a)} \qquad (x > t)$$

where

$$\gamma_N < A_N < \gamma_{N+1} .$$

Using the above relations in (11.5) we get

$$\left|\int_A^B t f(t) R_N(t, x) dt\right| \leqq \frac{k}{\rho_x}\left[\frac{2k_1}{A_N} p + \varepsilon\right] < \frac{2k}{\rho_x}\varepsilon .$$

This proves the lemma.

LEMMA 11.5. *Let $\psi(x) \in D(I)$. Then*

$$\int_A^B R_N(t, x)[\psi(x) - \psi(t)] x dx \longrightarrow 0 \tag{11.6}$$

as $N \to \infty$ *uniformly for all* $t \in (a, b)$, *where* $\operatorname{supp}(\psi) \subset [A, B] \subset (a, b)$.

Proof. The proof can be given by using Lemma 11.4 and following the pattern of proof of Lemma 6.4.

LEMMA 11.6.

$$\Omega^k_{\nu,t}\left[\int_A^B R_N(t, x)x\varphi(x)dx - \varphi(t)\right] \longrightarrow 0$$

as $N \to \infty$, *uniformly for* $t \in (a, b)$ *and for each* $k = 0, 1, 2, \cdots$.

The proof is similar to that of Lemma 6.5.

THEOREM 11.7. (*The uniqueness theorem*). *Let* $f, g \in U'_\nu(I)$. *If* $F_3(m) = G_3(m)$ *for each* $m = 1, 2, \cdots$, *then* $f = g$ *in the sense of equality in* $D'(I)$.

The proof is trivial.

We verify our inversion Theorem 11.1 by means of an easy example. $\delta(t-k) \in U'_\nu(I)$.

$$(\mathscr{H}_{3,\nu}(\delta(t-k)))(m) = kC_\nu(\gamma_m k, \gamma_m b) \qquad m = 1, 2, \cdots .$$

For $\varphi \in D(I)$,

$$\left\langle \sum_{m=1}^N \bar{\eta}_m^2 kC_\nu(\gamma_m k, \gamma_m b)C_\nu(\gamma_m x, \gamma_m b), x\varphi(x)\right\rangle$$
$$\longrightarrow k\varphi(k)\ , \quad \text{as} \quad N \longrightarrow \infty$$
$$= \langle \delta(t-k), t\varphi(t)\rangle\ .$$

We also get

$$\delta(t-k) = \lim_{N\to\infty} \sum_{m=1}^N \bar{\eta}_m^2 kC_\nu(\gamma_m k, \gamma_m b)C_\nu(\gamma_m t, \gamma_m b)\ .$$

12. Application of the finite Hankel transform of the third kind. Consider again the operator equation

$$P(\Omega^*_{x,\nu})u = g\ . \tag{12.1}$$

By applying finite Hankel transform of the third kind we get

$$U_3(m) = \frac{G_3(m)}{P(-\lambda_m^2)}\ , \quad m = 1, 2, \cdots ; \quad (P(-\lambda_m^2) \neq 0)\ . \tag{12.2}$$

Then using inversion Theorem 11.1, the solution can be written as

$$u(t) = \lim_{N\to\infty} \sum_{m=1}^N \bar{\eta}_m^2 \frac{G_3(m)}{P(-\lambda_m^2)} C_\nu(\gamma_m t, \gamma_m b)\ . \tag{12.3}$$

Now we solve diffusion equation in an infinitely long hollow cylinder bounded by $r = a$, $r = b(b > a)$ when these surfaces are kept at zero temperature. Let us determine a function $u(r, t)$ which satisfies the differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t} \qquad a \leqq r \leqq b,\ t > 0 \tag{12.4}$$

and the following boundary and initial conditions:

(i) $u(a, t) = 0$
$u(b, t) = 0$

(ii) $\lim_{t\to 0+} u(r, t) = f(r) \in U_\nu'(a, b)$

where equality is in the sense of $D'(a, b)$.

Applying the finite Hankel transform of the third kind we can reduce the above differential equation to

$$-\gamma_m^2 U_3(m, t) = \frac{\partial}{\partial t} U_3(m, t) , \tag{12.5}$$

where

$$U_3(m, t) = \mathscr{H}_{3,0}(u(r, t)) = \langle u(r, t), rC_0(\gamma_m r, \gamma_m b)\rangle .$$

The solution of (12.5) is given by

$$U_3(m, t) = A(m)e^{-\gamma_m^2 t}$$

where $A(m)$ is an arbitrary constant. Applying initial condition we find that

$$A(m) = F_3(m) .$$

Hence

$$U_3(m) = F_3(m)e^{-\gamma_m^2 t} .$$

Therefore, by inversion theorem

$$u(r, t) = \lim_{N\to\infty} \sum_{m=1}^{N} \bar{\eta}_m^2 F_3(m)e^{-\gamma_m^2 t} C_0(\gamma_m r, \gamma_m b) ,$$

equality in the sense of $D'(a, b)$.

Formally we take

$$u(r, t) = \sum_{m=1}^{\infty} \bar{\eta}_m^2 F_3(m)e^{-\gamma_m^2 t} C_0(\gamma_m r, \gamma_m b) . \tag{12.6}$$

Various steps involved above and that (12.6) is a solution to (12.4) can be justified as in §9.

REFERENCES

1. L. S. Dube, *On finite Hankel transformation of generalized functions*, Pacific J. Math., **62** No. 2 (1976), 365-378.

2. L. S. Dube and J. N. Pandey, *On the Hankel transform of distributions*, Tôhoku Math. J., **27** (1975), 337-354.

3. J. N. Pandey and R. S. Pathak, *Eigenfunction expansion of generalized functions*, Nagoya Math. J., **72** (1978), 1-25.

4. E. C. Titchmarsh, *A class of expansions in series of Bessel functions*, Proc. London Math. Soc., Second Sr. **22** (1924), Records, xiii-xvi.

5. I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, vol. **3**, Academic Press, New York, 1967.

6. G. N. Watson, *Theory of Bessel Functions*, 2nd ed., Cambridge University Press, 1958.

7. L. Schwartz, *Theorie des Distributions*, Hermann, Paris, 1978.

8. I. N. Sneddon, *On finite Hankel transforms*, Phil. Mag., ser. 7, **17** (1946), 17-25.

9. I. N. Sneddon, *The Use of Integral Transforms*, McGraw Hill Book Company, 1972.

10. J. N. Pandey and A. H. Zemanian, *Complex inversion for the generalized convolution transformation*, Pacific. J. Math., **25** (I) (1968), 147-157.

11. A. H. Zemanian, *Orthonormal series expansions of distributions and certain distributional transform calculus*, J. Math. Anal. and Appl., **14** (1966), 263-275.

12. ———, *Generalized Integral Transformation*, Interscience, New York, 1968.

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