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HOMOMORPHISMS OF MONO-UNARY ALGEBRAS

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Novotny has presented what amounts to a necessary and sufficient condition for the existence of certain homomorphisms between mono-unary algebras. In this paper, an example is presented to show that Novotny's condition is not sufficient, and a slightly stronger condition is shown to be both necessary and sufficient. The techniques of the proof are essentially the same as those used by Novotny.

Before proceeding, a brief summary is given of the relevant definitions.

A mono-unary algebra ("algebra" for short) is a pair (M, f) where f is any self-map of the set M; a homomorphism from (M, f) to (N, g) is a map $F: M \to N$ such that $F \circ f = g \circ F$.

Given such an algebra (M, f), let $f^0(x) = x$ for all $x \in M$, and $f^{n+1}(x) = f(f^n(x))$ for all $n \in \omega$; then $[x] = \{f^n(x) \mid n \in \omega\}$ is the subalgebra generated by x. For $x, y \in M$, define $x \rho y$ iff $[x] \cap [y] \neq \varnothing$ (equivalently, $x \rho y$ iff $f^m(x) = f^n(y)$ for some $m, n \in \omega$). This defines a congruence ρ on the algebra (M, f), the blocks of which are called the connected components of (M, f); if there is only one such component then the algebra is called connected. The connected components of M are each subalgebras of M, and a map from M into any algebra is a homomorphism iff its restriction to each connected component of M is a homomorphism. For this reason we need only consider homomorphisms from connected algebras M.

For each $x \in M$, either $f^m(x) = f^n(x)$ for some $m \neq n$, in which case $[x]$ is finite, or $[x]$ is infinite. If $[x]$ is finite, let $L(x)$ be the smallest natural number m with $f^m(x) = f^{m+k}(x)$ for some $k \neq 0$, and let $R(x)$ be the smallest natural number $k \neq 0$ with $f^{L(x)}(x) = f^{L(x)+k}(x)$. ($R(x)$ is the "rank" as defined by Novotny.) Then $f^m(x) = f^n(x)$ for $m < n$ implies $L(x) \leqq m$ and $R(x) \mid n - m$. If $[x]$ is infinite, define $L(x) = \infty$ and $R(x) = 0$. (Here, and in the remainder of the paper, ∞ is defined to be greater than every ordinal number, and we will use the convention that 0 is divisible by *every* natural number.)

Now (as in Novotny) define sets $M_\alpha \subseteqq M$ for ordinals α inductively as follows:

$$M_0 = \{x \in M \mid f^{-1}(x) = \varnothing\}$$

$$M_\alpha = \left\{x \in M - \bigcup_{\lambda<\alpha} M_\lambda \,\middle|\, f^{-1}(x) \subseteqq \bigcup_{\lambda<\alpha} M_\lambda\right\} .$$

Then the M_α are all pairwise disjoint, and for all $x \in M$, either

$x \in M_\alpha$ for some α, or there exist elements $x_n \in M$ for $n \in w$ with $x_0 = x$ and $f(x_{n+1}) = x_n$ for all $x \in \omega$.

Define

$$S(x) = \begin{cases} \alpha & \text{if} \quad x \in M_\alpha \\ \infty & \text{if} \quad x \notin \cup M_\alpha . \end{cases}$$

Note that $S(f(x)) \geqq S(x) + 1$ (provided we define $\infty + 1 = \infty$).

Now suppose (N, g) is also a mono-unary algebra, and that the sets $N_\alpha \subseteqq N$ have been defined analogously to the M_α, along with the corresponding function S.

THEOREM. *For any $a \in M$, $b \in N$, if M is connected then there is a homomorphism $F: M \to N$ with $F(a) = b$ iff $L(b) \leqq L(a)$, $R(b) \mid R(a)$, and $S(f^n(a)) \leqq S(g^n(b))$ for all $n \in \omega$.*

Proof. If F is a homomorphism from (M, f) to (N, g) with $F(a) = b$, and if $f^m(a) = f^n(a)$ then $g^m(b) = g^m(F(a)) = F(f^m(a)) = F(f^n(a)) = g^n(F(a)) = g^n(b)$, and hence $L(b) \leqq L(a)$ and $R(b) \mid R(a)$. A straightforward induction on α shows that for all $x \in M$, if $F(x) \in \bigcup_{\lambda<\alpha} N_\lambda$ then $x \in \bigcup_{\lambda<\alpha} M_\lambda$, and hence $S(x) \leqq S(F(x))$ for all $x \in M$, which yields $S(f^n(a)) \leqq S(g^n(b))$ for all n.

Conversely, suppose $L(b) \leqq L(a)$, $R(b) \mid R(a)$ and $S(f^n(a)) \leqq S(g^n(b))$ for all $n \in \omega$. Define sets $X_n \subseteqq M$ for $n \in \omega$ as follows:

$$X_0 = [a]$$

$$X_{n+1} = \left\{ x \in M - \bigcup_{i<n} X_i \,\middle|\, f(x) \in X_n \right\} .$$

Then the X_n are pairwise disjoint, and since M is connected, $\bigcup_{n \in \omega} X_n = M$. We define $F(x)$ for $x \in X_n$ by induction on n, in such a way that $S(x) \leqq S(F(x))$.

If $x \in X_0$ then $x = f^m(a)$ for some $m \in \omega$. If in addition $x = f^n(a)$ for $n > m$ then $L(a) \leqq m$ and $R(a) \mid n - m$, and so by hypothesis $L(b) \leqq m$ and $R(b) \mid n - m$, which implies $g^m(b) = g^n(b)$. Define $F(x) = g^m(b)$; the preceding sentence shows that this is well-defined. Moreover, $S(x) = S(f^m(a)) \leqq S(g^m(b)) = S(F(x))$, as required.

If $x \in X_{n+1}$ then $f(x) \in X_n$ and so by inductive hypothesis, $F(f(x))$ is defined and $S(F(f(x))) \geqq S(f(x))$. Since $S(f(x)) \geqq S(x) + 1$, this implies that $F(f(x)) \notin N_\lambda$ for any $\lambda \leqq S(x)$, and hence $g^{-1}(F(f(x))) \nsubseteq \bigcup_{\lambda<S(x)} N_\lambda$. For each $x \in X_{n+1}$, let $F(x)$ be any element of $g^{-1}(F(f(x))) - \bigcup_{\lambda<S(x)} N_\lambda$; then $g(F(x)) = F(f(x))$ and $S(F(x)) \geqq S(x)$.

Thus this yields a homomorphism $F: (M, f) \to (N, g)$ with $F(a) = b$, as required. □

COROLLARY. *If M is connected and $a \in M$, $b \in N$ such that*

$R(b) \mid R(a)$ and $S(f^n(a)) \leqq S(g^n(b))$ for all $u \in \omega$ then there is a homomorphism $F: M \to N$ with $F(a) \in [b]$.

Proof. Let $m = L(b)$; then $L(g^m(b)) = 0$ and $R(g^m(b)) = R(b)$ and for all $n \in \omega$, $S(g^n(b)) \leqq S(g^{n+m}(b)) = S(g^n(g^m(b)))$, and so by the theorem there is a homomorphism $F: (M, f) \to (Ng)$ with $F(a) = g^m(b)$. □

Novotny's Hauptsatz 2.14 claims that the above theorem is true without the restriction $L(b) \leqq L(a)$, but the following example shows that this is not the case. Let $M = \{a\}$ (so $f(a) = a$) and let $N = \omega$, $g(0) = 0$, $g(n + 1) = n$ for all n; then $R(a) = 1$ and $S(f^n(a)) = \infty$ for all n. Also, for $b = 1 \in N$, we have $R(b) = 1$, and $S(g^n(b) = \infty$ for each n. Clearly the map $F: M \to N$ with $F(a) = b$ is not a homomorphism, since $F(f(a)) = b \neq 0 = g(F(a))$.

Reference

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