RADICAL CLASSES OF REGULAR RINGS WITH ARTINIAN PRIMITIVE IMAGES

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This paper deals with radical classes consisting of regular rings, all of whose primitive homomorphic images are artinian (such rings will be called PA-regular). Noteworthy examples of such radical classes include, for each n, the class of regular rings satisfying the condition

$$a \text{ nilpotent} \Longrightarrow a^n = 0 ,$$

and thus, in particular, the class of all strongly regular rings. It is shown that every radical class $\mathscr{R}$ consisting of PA-regular rings is hereditary, and is the lower radical class defined by those of its members which are isomorphic to matrix rings of strongly regular rings with identities.

Moreover, for each m, the class of strongly regular rings for which the $m \times m$ matrix ring is in $\mathscr{R}$ is a radical class. This fact establishes a very important rôle for the radical classes of strongly regular rings, and accordingly a section is devoted to the latter. In the final section some attention is given to the question of closure of radical classes under direct products.

It is probably safe to say that more is known about the radical classes which contain all nilpotent rings than about those which contain none. Certainly—with the (important) exception of the semi-simple radical classes—the radical classes which have been most studied are towards the supernilpotent end of the spectrum.

The present paper has three main aims:

(1) to present some examples of (hereditary) subidempotent radical classes;

(2) to examine radical theory in a relatively tractable but non-trivial class of rings—the regular rings whose primitive homomorphic images are all artinian;

(3) to say something about radical classes which are closed under directs.

Investigations akin to objective (2), wherein radical theory is studied "in microcosm", may be regarded as compromise substitutes for the (probably unrealistic) aim of describing all radical classes.

The question of dirct product closure has been around for a long time. It is easy to see that a hereditary radical class with this closure property must be either supernilpotent of subidempotent, and we shall here be concerned with the latter, providing a few examples and counterexamples and answering, in the negative, Richard Wiegandt's

question [22]: "Are hereditary, direct product-closed radical classes equationally definable?"

We shall use the following terminology and notation. A regular ring with all its primitive images artinian will be called PA-regular; $M_n(A)$ will denote the ring of $n \times n$ matrices over A; $\lhd$ will indicate an ideal; the upper and lower radical classes defined by a class $\mathscr{A}$ will be denoted by $U(\mathscr{A})$ and $L(\mathscr{A})$ respectively; Q is the field of rational numbers, Z the ring of integers, Z^+ the set of positive integers. All rings are associative. It never really matters, but for the sake of definiteness, "artinian" and "primitive" are to be interpreted as on the left. Finally, all classes of rings are assumed to be isomorphically closed.

1. Some radical classes of PA-regular rings. We begin by presenting some examples of radical classes consisting of PA-regular rings.

Let $\mathscr{P}$ be a class of artinian primitive rings, $\mathscr{P}^c$ the class of primitive rings (artinian or not) not in $\mathscr{P}$, and let $\mathscr{R}[\mathscr{P}]$ be the class of regular rings, all of whose primitive homomorphic images are in $\mathscr{P}$.

PROPOSITION 1.1. *For any class $\mathscr{P}$ of artinian primitive rings, $\mathscr{R}[\mathscr{P}]$ is the class of regular rings in $U(\mathscr{P}^c)$.*

Proof. If $0 \neq I \lhd A \in \mathscr{P}^c$, then I is primitive. If I is in $\mathscr{P}$, it has an identity, and thus $A = I \oplus J$ for some ideal J. But then $J = 0$, since A is primitive, so $I = A \in \mathscr{P}^c$. Thus every non-zero ideal of a ring in $\mathscr{P}^c$ is itself in $\mathscr{P}^c$. It follows that $U(\mathscr{P}^c)$ is the class of rings without homomorphic images in $\mathscr{P}^c$, whence $\mathscr{R}[\mathscr{P}] \subseteq U(\mathscr{P}^c)$. The reverse inclusion is clear. $\square$

COROLLARY 1.2. *For every class $\mathscr{P}$ of artinian primitve rings, $\mathscr{R}[\mathscr{P}]$ is a radical class.* $\square$

COROLLARY 1.3. *The class of all PA-regular rings is radical.* $\square$

COROLLARY 1.4. *Let $\mathscr{X}$ be a class of PA-regular rings. Then $L(\mathscr{X})$ consists of PA-regular rings.* $\square$

Some further examples, important for later work, are obtained from a consideration of the indices (of nilpotence) of nilpotent elements.

PROPOSITION 1.5. *The following conditions are equivalent for a*

regular ring R and a positive integer n.

(i) $a^n = 0$ *for every nilpotent element a of R.*

(ii) *Every primitive homomorphic image of R is isomorphic to $M_r(\Delta)$ for some division ring Δ and some $r \leq n$.*

(iii) *R is a subdirect product of rings $M_{r_\lambda}(\Delta_\lambda)$, where the Δ_λ are division rings and the $r_\lambda \leq n$.*

Proof. (i) $\Rightarrow$ (ii): See Kaplansky [14], Theorem 2.3.

(ii) $\Rightarrow$ (iii): R is regular, and therefore semi-primitive.

(iii) $\Rightarrow$ (i): If b is a nilpotent element of $M_{r_\lambda}(\Delta_\lambda)$, where $r_\lambda \leq n$ and Δ_λ is a division ring, then $b^{r_\lambda} = 0$, so $b^n = 0$. Thus $a^n = 0$ for any nilpotent element a of (any subring of) $\prod M_{r_\lambda}(\Delta_\lambda)$. $\qquad\square$

From now no we shall denote by $\mathscr{R}_n$ the class of regular rings satisfying the conditions of Proposition 1.5.

THEOREM 1.6. *For each n, $\mathscr{R}_n$ is a radical class which is closed under regular subrings and direct products.*

Proof. If $A \in \mathscr{R}_n$ and $A/I \neq 0$, then since every primitive homomorphic image of A/I is a homomorphic image of A, we have $A/I \in \mathscr{R}_n$, so $\mathscr{R}_n$ is homomorphically closed. If $J \lhd R$, and $J, R/J \in \mathscr{R}_n$, then certainly R is regular. Both J and R/J are subdirect products of rings of the form $M_r(\Delta)$ where Δ is a division ring and $r \leq n$. By the theorem of [20], R also has such a subdirect product representation and accordingly R is in $\mathscr{R}_n$, so the latter is closed under extensions. Finally, if $\{I_\lambda \mid \lambda \in \Lambda\}$ is a chain of $\mathscr{R}_n$-ideals of some ring, then if a is a nilpotent element of $\bigcup I_\lambda$, there is a $\lambda_0 \in \Lambda$ for which $a \in I_{\lambda_0} \in \mathscr{R}_n$ and thus $a^n = 0$. Since $\bigcup I_\lambda$ is regular, it is in $\mathscr{R}_n$. Thus $\mathscr{R}_n$ is a radical class.

Since the defining condition on nilpotent elements is preserved by subrings and direct products and direct products of regular rings are regular, the other assertions of the theorem follow. $\qquad\square$

COROLLARY 1.7. *The class of strongly regular rings is a radical class.*

Proof. A regular ring is strongly regular if and only if it has no nilpotent elements (see, e.g., [6]) so the class of strongly regular rings is $\mathscr{R}_1$. $\qquad\square$

Corollary 1.7 was previously proved by Spircu [21] and announced by Osondu [17].

In a sense the examples of radical classes provided by Theorem 1.6 are special cases of those provided by Corollary 1.2: $\mathscr{R}_n = \mathscr{R}[\mathscr{P}]$ where $\mathscr{P}$ is the class of $r \times r$ matrix rings over division rings, $r \leq n$. Note however that in Theorem 1.6 we require only that rings have representations as subdirect products of certain primitive rings: all primitive homomorphic images then turn out to be of this kind.

2. Lower radical representations. We turn now to the problem of finding subclasses $\mathscr{M}$, as small as possible, of radical classes $\mathscr{R}$ of PA-regular rings, such that $L(\mathscr{M}) = \mathscr{R}$. It turns out that classes of matrix rings over strongly regular rings are what we seek.

THEOREM 2.1. *Let $\mathscr{R}$ be a radical class consisting of* PA-*regular rings, and let $\mathscr{M}$ be the subclass of $\mathscr{R}$ consisting of rings isomorphic to complete matrix rings over strongly regular rings with identities. Then*
  (i) $\mathscr{R} = L(\mathscr{M})$ *and*
  (ii) $\mathscr{R}$ *is hereditary.*

Proof. (i) Let R be a non-zero ring in $\mathscr{R}$. By Theorem 3, p. 239, of Jacobson [12], each non-zero ideal of R, and, in particular, R itself, has an ideal I isomorphic to $M_n(A)$ for some ring A without nilpotent elements and some n. Furthermore, I has an identity u which is a central idempotent of R. In the proof of the cited theorem, it is established that $A \cong eRe$ for some idempotent $e \in R$. If $a \in eRe$, then $eae = a$, while since a is in R, there is an element $b \in R$ for which $a = aba$. Then

$$a(ebe)a = eae^2be^2ae = (eae)b(eae) = aba = a .$$

Thus eRe is regular, so A is. Since it has no nilpotent elements, A is strongly regular. Let (u_{ij}) be the identity of $M_n(A)$. Then for $a \in A$, we have

$$(u_{ij})\begin{pmatrix} a & & \\ & a & 0 \\ & & \ddots \\ & 0 & & a \end{pmatrix} = \begin{pmatrix} a & & \\ & a & 0 \\ & & \ddots \\ & 0 & & a \end{pmatrix} = \begin{pmatrix} a & & \\ & a & 0 \\ & & \ddots \\ & 0 & & a \end{pmatrix}(u_{ij}) ,$$

whence, comparing $(1, 1)$ entries, we see that u_{11} is an identity for A.

Since I has an identity u, we have $R = I \oplus K$ for some $K \lhd R$, so, as a homomorphic image of R, $M_n(A) \cong I$ is in $\mathscr{R}$. This means that $I \cong M_n(A) \in \mathscr{M}$. Every non-zero homomorphic image of R is again in $\mathscr{R}$, so it has a non-zero ideal in $\mathscr{M}$ as above. Thus $\mathscr{R} = L(\mathscr{M})$.

(ii) Let $J \lhd A \in \mathscr{R}$. Then $\mathscr{R}(J) \lhd R$, so $J/\mathscr{R}(J) \lhd R/\mathscr{R}(J) \in \mathscr{R}$. If $J/\mathscr{R}(J) \neq 0$, then by the theorem from [12] quoted in the proof of (i), $J/\mathscr{R}(J)$ has a non-zero ideal in $\mathscr{M} \subseteq \mathscr{R}$. This is impossible, so $J = \mathscr{R}(J)$, and $\mathscr{R}$ is hereditary. $\square$

COROLLARY 2.2. *Every radical class of strongly rings is hereditary.* $\square$

A special case of Corollary 2.2.—all radical classes of boolean rings are hereditary—was proved in Galay's thesis [7].

It should be pointed out, however, that radical classes of regular rings need not be hereditary.

EXAMPLE 2.3. Let R be the ring of all linear transformations of an $\aleph_0$-dimensional vector space, R_0 the ring of linear transformations of finite rank. Then R_0 is a simple ring without identity, so R_0 has no non-zero accessible subring which is a homomorphic image of R. Thus $R_0 \lhd R$ but $R_0 \notin L(\{R\})$, so $L(\{R\})$ is not hereditary.

Theorem 2.1 suggests an investigation of the converse question: if $\mathscr{C}$ is a class of matrix rings over strongly regular rings (with identities), what does $L(\mathscr{C})$ look like? We shall now investigate this. The following result shows that $\mathscr{C}$ may, without loss, be assumed to be a radical class.

THEOREM 2.4. *Let $\mathscr{R}$ be a radical class consisting of PA-regular rings. Then there are radical classes $\mathscr{U}_n$, $n \in Z^+$, consisting of strongly regular rings such that*

$$\mathscr{R} = L(\bigcup_{n \in Z^+} \{M_n(A) \mid A \in \mathscr{U}_n\}) .$$

Proof. By Theorem 2.1, $\mathscr{R} = L(\bar{\mathscr{M}})$ where $\bar{\mathscr{M}}$ is the class of rings in $\mathscr{R}$ isomorphic to matrix rings over strongly regular rings (not necessarily with identities). For each n, let $\mathscr{U}_n = \{A \mid A$ is strongly regular and $M_n(A) \in \mathscr{R}\}$. Since the strongly regular rings form a radical class (Corollary 1.7) and the class $\{A \mid M_n(A) \in \mathscr{R}\}$ is radical (see [10] or [16]), each $\mathscr{U}_n$ is a radical class. Clearly

$$\mathscr{R} = L(\bigcup_{n \in Z^+} \{M_n(A) \mid A \in \mathscr{U}_n\}) . \qquad \square$$

To finish off this section, we present some results relating the possible primitive homomorphic images of rings in a radical class $\mathscr{R}$ consisting of PA-regular rings to the classes $\mathscr{U}_n$ of Theorem 2.4.

PROPOSITION 2.5. *Let N be a set of natural numbers, and for*

each $n \in N$ let $\mathscr{U}_n$ be a radical class consisting of strongly regular rings. If R is in

$$L(\{M_n(A) \mid n \in N,\ A \in \mathscr{U}_n\})\ ,$$

then R is regular, and every non-zero homomorphic image of R is a subdirect product of rings isomorphic to $M_n(\varDelta)$, $n \in N$, where $\varDelta$ is a division ring in $\mathscr{U}_n$.

Proof. If $A \in \mathscr{U}_n$ and $J \lhd M_n(A)$, then $J = M_n(I)$ for some $I \lhd A$ (see, e.g., de la Rosa [19]), so $M_n(A)/J \cong M_n(A/I)$, where $A/I \in \mathscr{U}_n$. Thus each $\{M_n(A) \mid A \in \mathscr{U}_n\}$ is homomorphically closed. Since all rings are idempotent, every non-zero ring R in $L(\{M_n(A) \mid n \in N,\ A \in \mathscr{U}_n\})$ has a non-zero ideal isomorphic to $M_n(B)$ for some $n \in N$, $B \in \mathscr{U}_n$. By Proposition 1.6 and Corollary 1.3 (since $M_n(B)$ is in $\mathscr{R}_n$), R is PA-regular. Thus R is a subdirect product of matrix rings $M_{r_\mu}(\varDelta_\mu)$ over division rings $\varDelta_\mu$. But then each such $M_{r_\mu}(\varDelta_\mu)$ has a non-zero ideal isomorphic to $M_n(C)$, $n \in N$, $C \in \mathscr{U}_n$. This means that $M_{r_\mu}(\varDelta_\mu) \cong M_n(C)$, so $M_n(C)$, and hence also C, is simple. But C is strongly regular, and therefore a division ring, so $r_\mu = n$ and $\varDelta_\mu = C \in \mathscr{U}_n$ (see, e.g., Kertész [15], p. 180).

Thus R is a subdirect product of matrix rings $M_n(\varDelta)$, $n \in N$, over division rings $\varDelta \in \mathscr{U}_n$. R is also regular, and all the above applies equally well to any (non-zero) homomorphic image. $\square$

The next result provides an alternative formulation of the previous one (cf. Proposition 1.5).

PROPOSITION 2.6. *The following conditions are equivalent for a regular ring R and a class $\mathscr{P}$ of matrix rings of bounded order over division rings.*

(i) *Every primitive homomorphic image of R is in $\mathscr{P}$.*

(ii) *Every homomorphic image of R is a subdirect product of rings in $\mathscr{P}$.*

Proof. (i) $\Rightarrow$ (ii): Each homomorphic image R/I of R is regular, and thus a subdirect product of primitive rings. The latter, being primitive images of R, are in $\mathscr{P}$.

(ii) $\Rightarrow$ (i): Let S be a primitive homomorphic image of R. Then S is a subdirect product of rings in $\mathscr{P}$, so for every nilpotent element a of S we have $a^n = 0$, where n is an upper bound for the orders of the matrix rings in $\mathscr{P}$. This forces S to be simple artinian, and hence in $\mathscr{P}$ (see Proposition 1.5). $\square$

COROLLARY 2.7. *Let N be a set of natural numbers, and for*

each $n \in N$ *let* $\mathscr{U}_n$ *be a radical class consisting of strongly regular rings. If* $R \in L(\{M_n(A) \mid n \in N, A \in \mathscr{U}_n\})$ *then every primitive homomorphic image of* R *is isomorphic to a matrix ring* $M_n(\varDelta)$, *where* $n \in N$ *and* $\varDelta$ *is a division ring in* $\mathscr{U}_n$. □

3. Radical theory for strongly regular rings. In a strongly regular ring, each element a has associated with it a unique element a' such that

$$a = a^2 a' = aa'a = a'a^2 \quad \text{and} \quad a' = (a')^2 a = a'aa' = a(a')^2 .$$

Thus the strongly regular rings are equipped with an extra unary operation, $'$, and if this, as well as the ring operations, is taken into account, then the class of strongly regular rings is a variety.

This variety can serve as a universal class for radical theory. Since as a class of rings, the class of strongly regular rings is hereditary and homomorphically closed, it can serve as a universal class of rings also. Moreover, the notions of homomorphism, kernel and image are the same whichever way one looks at things, so we get the same radical classes either way, and can accordingly afford to blur the distinction.

Now when a hereditary radical class $\mathscr{U}$ of rings is used as a universal class, every radical class in $\mathscr{U}$ is a radical class of rings. This was first noted (for topological rings) by Arnautov and Vodincear [2]; for a proof, see also [10]. Thus radical classes of strongly regular rings (in any sense) are radical classes of rings. Moreover, in view of the representation in Theorem 2.4, they assume a crucial rôle in the wider context of radical theory for PA-regular rings.

Varieties of regular rings (where account is taken of $'$ as well as the rings operations) provide examples.

THEOREM 3.1. *Let* $\mathscr{V}$ *be a variety of strongly regular rings. Then*
  (i) $\mathscr{V}$ *is generated by the division rings it contains, and*
  (ii) $\mathscr{V}$ *is closed under extensions.*

Proof. (i) Every $A \in \mathscr{V}$ is a subdirect product (*qua* ring and *qua* strongly regular ring) of division rings, and the latter are necessarily in $\mathscr{V}$.

(ii) Let A be a strongly regular ring with an ideal I such that I and A/I are in $\mathscr{V}$. Then A has a family $\{J_\lambda \mid \lambda \in \varLambda\}$ of ideals such that each A/J_λ is a division ring and $\bigcap_\varLambda J_\lambda = 0$. If, for some λ, $I \subseteq J_\lambda$, then A/J_λ, as a homomorphic image of A/I, is in $\mathscr{V}$, while if $I \nsubseteq J_\lambda$, then by maximality, $I + J_\lambda = A$, so $A/J_\lambda \cong I/I \cap J_\lambda \in \mathscr{V}$. Hence A is in $\mathscr{V}$. □

COROLLARY 3.2. *Every variety of strongly regular rings is a radical class.*

Proof. As for Theorem 1.4 of [8]. $\qquad\square$

One can obtain examples of varieties of strongly rings by intersecting the class of strongly regular rings with varieties of rings. There are others, however. For instance, whereas the variety of *rings* generated by Q is the class of commutative rings, the variety of *strongly regular rings* generated by Q consists of rings with characteristic 0 and thus excludes, for example, all finite fields. Another non-trivial example of a variety is the class of all strongly regular rings of characteristic 0.

A variety of strongly regular rings is generated by its subclass of division rings; it's also defined by a set of identities which involve $'$ as well as the ring operations. But for division rings, we have $a' = a^{-1}$ if $a \neq 0$ while $0' = 0$. Thus, for example, the variety of strongly regular rings of characteristic 0 is defined by the set of identities

$$\{pa^2(pa)' = a \,|\, p \text{ prime}\} \,.$$

The structure of the set of idempotents provides further examples of radical classes of strongly regular rings. If A is strongly regular, then all idempotents in A are central, and the set $E(A)$ of all idempotents is a boolean ring with respect to the multiplication of A and the new addition given by

$$e \dotplus f = e + f - 2ef \,.$$

THEOREM 3.3. *Let $\mathscr{R}$ be a radical class of boolean rings, $\mathscr{R}^*$ the class of strongly regular rings A for which $E(A) \in \mathscr{R}$. Then $\mathscr{R}^*$ is a radical class.*

Proof. If $A \in \mathscr{R}^*$ and $I \lhd A$, then for every $e \in E(A/I)$ there is an $\bar{e} \in E(A)$ such that $\bar{e} + I = e$ (see [11], p. 52: Lallement's Lemma). The natural homomorphism from A to A/I therefore induces a surjective homomorphism from $E(A)$ to $E(A/I)$. Accordingly, $E(A/I) \in \mathscr{R}$, so that $A/I \in \mathscr{R}^*$ and it follows that $\mathscr{R}^*$ is homomorphically closed.

Now let B be a strongly regular ring with an ideal J such that J and B/J are in $\mathscr{R}^*$. The natural map $B \to B/J$ induces a surjective homomorphism $E(B) \to E(B/J)$, with kernel $\{e \in E(B) \,|\, e \in J\} = E(J)$. Since $E(J)$ and $E(B/J)$ are in $\mathscr{R}$, so is $E(B)$, i.e., B is in $\mathscr{R}^*$, and the latter is closed under extensions.

If $\{L_\lambda \,|\, \lambda \in \varLambda\}$ is a chain of $\mathscr{R}^*$-ideals of a strongly regular ring R and $R = \bigcup_\lambda L_\lambda$, then $E(R) = \bigcup_\lambda E(L_\lambda) \in \mathscr{R}$, since $E(L_\lambda) \lhd E(R)$ for each λ, so $R \in \mathscr{R}^*$. This completes the proof. $\qquad\square$

Some non-obvious examples of radical class of boolean rings can be obtained as follows from some results of Cramer [5]. For each cardinal number α, let $\mathscr{F}_\alpha$ be the class of boolean rings A such that for $a \in A$, the principal ideal of A generated by a has, when viewed as a boolean algebra, no free subalgebras of rank α. Details of this and other examples are given in Galay's thesis [7].

4. Radical classes obtained from varieties. We return to the question of the production of radical classes consisting of PA-regular rings from radical classes of strongly regular rings. In this section we shall look at the case of varieties of strongly regular rings from this viewpoint, and direct some attention to the problem of deciding when the resultant radical classes of PA-regular rings are closed under direct products.

THEOREM 4.1. *Let $\mathscr{U}$ be a variety of regular rings, N a set of positive integers. Then a ring R is in $L(\{M_n(A) \,|\, A \in \mathscr{U}\})$ if and only if R is regular and every primitive homomorphic image of R is isomorphic to $M_n(\varDelta)$ for some division ring $\varDelta$ in $\mathscr{U}$.*

Proof. "Only if" follows from Corollary 2.7. (By Corollary 3.2, $\mathscr{U}$ is a radical class.)

For the converse, let R be a regular ring for which every primitive homomorphic image is isomorphic to $M_n(\varDelta)$, $n \in N$, $\varDelta \in \mathscr{U}$. Then R is PA-regular, so by the proof of Theorem 2.1, R has a non-zero ideal $L \cong M_r(Y)$, for some r, where Y is a strongly regular ring with identity. Then L is a homomorphic image of R, so all of its primitive images are isomorphic to rings $M_n(\varDelta)$, $n \in N$, $\varDelta \in \mathscr{U}$. Let Y (being strongly regular) have ideals J_λ such that Y/J_λ is a division ring, for each λ, and $\bigcap J_\lambda = 0$. Then for each λ, $M_r(Y/J_\lambda)$ is a primitive homomorphic image of $M_r(Y) \cong L$, so $r \in N$ and $Y/J_\lambda \in \mathscr{U}$ for each λ. But since $\mathscr{U}$ is closed under subdirect products, we then have $Y \in \mathscr{U}$ and so L is isomorphic to a ring in $\{M_n(A) \,|\, n \in N, A \in \mathscr{U}\}$ whence $R \in L(\{M_n(A) \,|\, n \in N, A \in \mathscr{U}\})$. $\qquad\square$

By Proposition 2.6, the last result says that for any variety $\mathscr{U}$ of strongly regular rings and set N of natural numbers, $L(\{M_n(A) \,|\, A \in \mathscr{U}, n \in N\})$ is the class of regular, strongly $U(\{M_n(\varDelta) \,|\, \varDelta$ is a division ring in $\mathscr{U}, n \in N\})$-semi-simple rings. For (at least) some classes $\mathscr{M}$

of simple artinian rings, all strongly $U(\mathscr{M})$-semi-simple rings are regular. This is certainly so when there is a (ring) polynomial identity satisfied by each $R \in \mathscr{M}$, since Armendariz and Fisher [2] have shown that hereditarily idempotent rings satisfying polynomial identities are regular. From these considerations we get

COROLLARY 4.2. *Let $\mathscr{V}$ be a variety of rings, $\mathscr{U}$ the class of strongly regular rings in $\mathscr{V}$, N a finite set of natural numbers. Then*

$$L(\{M_n(A) \mid A \in \mathscr{U}, n \in N\})$$
$$= \{R \mid Every \; primitive \; homomorphic \; image \; of$$
$$R \; is \; isomorphic \; to \; a \; ring \; M_n(\varDelta) \; where \; n \in N$$
$$and \; \varDelta \; is \; a \; division \; ring \; in \; \mathscr{U}\} \, .$$

Proof. Every ring A in $\mathscr{U}$ satisfies a fixed polynomial identity α. Let $\bar{n} = \max(N)$. Then there is an identity β satisfied by each $M_{\bar{n}}(A)$ and therefore by each $M_n(A)$, $n \in N$.   □

COROLLARY 4.3. *Let $\mathscr{V}$ be a variety of rings, N a finite set of natural numbers,*

$$\mathscr{M} = \{M_n(\varDelta) \mid n \in N, \varDelta \in \mathscr{V}, \varDelta \; is \; a \; division \; ring\} \, .$$

Then $L(\{M_n(A) \mid n \in N, A \in \mathscr{V}, A \; is \; strongly \; regular\})$ is the class of strongly $U(\mathscr{M})$-semi-simple rings.   □

For the rest of this section we shall consider closure under direct products for radical classes consisting of PA-regular rings.

We do not know whether the radical class of Theorem 4.1 is closed under direct products when N is finite (nor, in fact that of Corollary 4.2). However, direct product closure does require an upper bound on the indices of nilpotence of elements of rings in a radical class.

THEOREM 4.4. *Let $\mathscr{R}$ be a direct product-closed radical class consisting of PA-regular rings. Then $\mathscr{R} \subseteq \mathscr{R}_n$ for some n.*

Proof. Suppose $\mathscr{R}$ contains infinitely many rings $M_{n_1}(\varDelta_1)$, $M_{n_2}(\varDelta_2), \cdots$ where $\varDelta_1, \varDelta_2, \cdots$ are division rings and $n_1 < n_2 < \cdots$. Then $\prod M_{n_i}(\varDelta_i) \in \mathscr{R}$. Let $\varPhi$ be a non-principal ultrafilter of subsets of $\{n_1, n_2, \cdots\}$. Then $\prod M_{n_i}(\varDelta_i)/\varPhi$ is primitive (Amitsur [1], Theorem 2A) and belongs to $\mathscr{R}$. Hence $\prod M_{n_i}(\varDelta_i)/\varPhi$ is isomorphic to $M_m(\varGamma)$ for some division ring $\varGamma$ and some $m \in Z^+$. But then, by Theorem 2B of [1], $\{i \mid n_i = m\} \in \varPhi$, while $|\{i \mid n_i = m\}| = 0$ or 1. But this can't

happen since Φ is non-principal. $\qquad\qquad\square$

(For a discussion of ultrafilters, see, e.g. [4], Ch. VII).

The classes $\mathscr{R}_n$ themselves are examples of direct product closed radical classes consisting of PA-regular rings. In [9] it was shown that for each n the class $\mathscr{C}_n$ of hereditarily idempotent (and hence regular [2]) rings satisfying the standard identity of degree n is a radical class; clearly the $\mathscr{C}_n$ are closed under direct products. We now present a generalization of this.

THEOREM 4.5. *Let $\mathscr{V}$ be a variety of rings, $\mathscr{R}[\mathscr{V}]$ the class of regular rings in $\mathscr{V}$. Then $\mathscr{R}[\mathscr{V}]$ is a hereditary radical class which is closed under direct products.*

Proof. Every primitive homomorphic image of every ring in $\mathscr{R}[\mathscr{V}]$ satisfies a polynomial identity and is therefore artinian [13]. Let $M_n(\varDelta)$ be such a ring, with $\varDelta$ a division ring. Then $M_n(Z(\varDelta))$ is in $\mathscr{V}$, so there is an upper bound on possible values of n. If k is such an upper bound, then $\mathscr{R}[\mathscr{V}] \subseteq \mathscr{R}_k$, so $\mathscr{R}[\mathscr{V}] \subseteq \mathscr{R}_k \cap \mathscr{V}$. But $\mathscr{R}_k \cap \mathscr{V} \subseteq \{A \in \mathscr{V} \,|\, A \text{ is regular}\} = \mathscr{R}[\mathscr{V}]$, so $\mathscr{R}[\mathscr{V}] = \mathscr{R}_k \cap \mathscr{V}$. Thus $\mathscr{R}[\mathscr{V}]$ is hereditary, homomorphically closed and closed under direct products. If $\{I_\lambda \,|\, \lambda \in \varLambda\}$ is a chain of ideals of a ring R and each I_λ is in $\mathscr{R}[\mathscr{V}]$, then $\bigcup_{\lambda \in \varLambda} I_\lambda$ is regular and satisfies all the identities of the I_λ, so it is in $\mathscr{V}$. All that is now needed is to prove that $\mathscr{R}[\mathscr{V}]$ is closed under extensions. If $I \lhd T$ and if $I, T/I \in \mathscr{R}[\mathscr{V}]$, then certainly T is regular. Each of $I, T/I$ is a subdirect product of simple rings in $\mathscr{R}_k \cap \mathscr{V}$. By the theorem of Snider [20] used previously, so is T. This proves the theorem. $\qquad\square$

Richard Wiegandt [22] asks whether every hereditary, direct product-closed radical class is a variety. (Of course by "variety" here we mean variety of algebras possibly with other operations besides the ring operations; e.g. the Jacobson radical class is a variety when account is also taken of the circle operation and the class of strongly regular rings is a variety as we have already noted.) The answer to this question is negative. Raphael [18] notes that the class of regular rings is not a variety since it fails to be closed under equalizers; he cites the following example, attributed to Paré: $M_2(Q)$ has an equalizer subring which is not regular. In fact Q can be replaced by any division ring $\varDelta$ in this example. We now extend this to $M_n(\varDelta)$ for all n.

PROPOSITION 4.6. *Let $\varDelta$ be a division ring. Define $f: M_n(\varDelta) \to M_n(\varDelta)$ $(n > 1)$ by*

$f((a_{ij}))$

$$= \begin{pmatrix} a_{11}-a_{12}+a_{13}-\cdots+(-1)^{n-1}a_{1n} & a_{12}-a_{13}+\cdots+(-1)^{n-2}a_{1n}\cdots & a_{1n} \\ (a_{11}+a_{21})-(a_{12}+a_{22})+\cdots+(-1)^{n-1}(a_{1n}+a_{2n}) & \cdots & (a_{1n}+a_{2n}) \\ (a_{21}+a_{31})-(a_{22}+a_{32})+\cdots+(-1)^{n-1}(a_{2n}+a_{3n}) & \cdots & (a_{2n}+a_{3n}) \\ \vdots & & \vdots \\ (a_{n-1,1}+a_{n1})-(a_{n-1,2}+a_{n2})+\cdots+(-1)^{n-1}(a_{n-1,n}+a_{nn}) & \cdots & (a_{n-1,n}+a_{nn}) \end{pmatrix}$$

Then f is a ring homomorphism. Every matrix in the equalizer of f and the identity has the form

$$\begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * \end{pmatrix}.$$

Proof. It is clear that f preserves addition. The proof of preservation of multiplication is tedious; suffice it to say that when one is calculating the (i, j) entry of a product

$$(\bar{a}_{ij})(\bar{b}_{ij}) = f((a_{ij}))f((b_{ij})) ,$$

there is a fair amount of cancelling between $\bar{a}_{ij}\bar{b}_{kj}$ and $\bar{a}_{i,k+1}\bar{b}_{k+1,j}$. If (a_{ij}) is in the equalizer, then

$$a_{11} = a_{11} - a_{12} + a_{13} - \cdots + (-1)^{n-1}a_{1n} ,$$

so

$$0 = a_{12} - a_{13} + \cdots + (-1)^{n-2}a_{1n} = a_{12} ,$$
$$0 = a_{13} - a_{14} + \cdots + (-1)^{n-3}a_{1n} = a_{13} ,$$
$$\vdots$$
$$0 = a_{1n} . \qquad\qquad \square$$

THEOREM 4.7. *Let $\mathscr{R}$ be a radical class consisting of PA-regular rings. If $\mathscr{R}$ is a variety, then $\mathscr{R}$ contains only strongly regular rings.*

Proof. If $\mathscr{R}$ contains a ring which is not strongly regular, then $M_n(\Delta) \in \mathscr{R}$ for some division ring Δ and some $n > 1$. Let S be the equalizer of the identity map of $M_n(\Delta)$ and the map f of Proposition 4.6. Let $[1]_{n1}$ be the matrix whose $(n, 1)$ entry is 1 and whose other entries are 0. Then $[1]_{n1} \in S$, while for $A \in S$, we have

$$[1]_{n1}A[1]_{n1} = [1]_{n1}\begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * \end{pmatrix}[1]_{n1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a & 0 & 0 & \cdots & 0 \end{pmatrix}[1]_{n1} = 0 .$$

Thus there is no $A \in S$ for which $[1]_{n1}A[1]_{n1} = [1]_{n1}$, whence S is not regular, so certainly $S \notin \mathscr{R}$. But if $\mathscr{R}$ were a variety, it would be closed under equalizers. $\qquad\square$

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