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FINITELY GENERATED PROJECTIVE EXTENSIONS OF UNIFORM ALGEBRAS

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Let A and B be uniform algebras and suppose that B is an extension of A, finitely generated and projective as an A-module. Let π denote the natural projection from the maximal ideal space of B onto the maximal ideal space of A. We show that K is a generalized peak interpolation set for B if and only if $\pi(K)$ is a generalized peak interpolation set for A. Then we give a topological description of the maximals sets of antisymmetry of B in terms of those of A. Finally, we prove that if B is strongly separable over A, then the algebra of B-holomorphic functions is strongly separable over the algebra of A-holomorphic functions.

1. Introduction. The main motivation for this work comes from a series of results discovered over the last twenty years concerning the structure of certain types of integral extensions of (commutative complex unital) Banach algebras. More precisely, the results which we are refering to group roughly in two classes. On one hand, we have the theory of the so-called algebraic or Arens-Hoffman extensions [1, 2, 7, 8]. These are extensions of the form $A[x]/(\alpha(x))$, where $\alpha(x)$ is a monic polynomial with coefficients in the base algebra A. Moreover, part of this theory was recently extended [12] to the case where the extension is finitely generated and projective as an A-module. On the other hand, we have results coming from the study of strongly separable extensions [4, 9], i.e., extensions which are finitely generated and projective as A-modules and separable as A-algebras.

Still a word on method. As we showed in [12], given a finitely generated projective extension B of a Banach algebra A and an element $b \in B$, one can pick out, among all monic polynomials $\alpha(x) \in A[x]$ such that $\alpha(b) = 0$, a canonical one, which enjoys some useful properties (see Lemma 1 below for a precise statement). Then, our method consists in obtaining information about B from information about A (and conversely) by means of these canonical integrity equations.

We fix now some notation. $M_{(\ )}$ and $\partial_{(\ )}$ denote, respectively, the character spectrum and the Shilov boundary operators on Banach algebras; $\hat{f}$ is the Gelfand transform of f. A will denote a fixed uniform algebra on a compact space X, and B a finitely generated projective extension of A. It is known that B can be endowed with a canonical Banach algebra structure [9, Th. 4, p. 138 and 12, § 3].

However, for our purposes, it will not be necessary to handle any specific Banach algebra norm on B and only the existence of such a norm will be required. We will also assume, except in Theorem 1, that B is a uniform algebra.

We write π for the projection from M_B onto M_A defined $\pi(\psi) = \psi|_A$, $\psi \in M_B$.

If $\alpha(x) = \sum_{i=0}^{n} \alpha_i x^i$ is a polynomial with coefficients in A and $\phi \in M_A$, then we write

$$\alpha_\phi(x) = \sum_{i=0}^{n} \phi(\alpha_i) x^i \in \boldsymbol{C}[x]$$

and

$$Z(\alpha_\phi) = \{\lambda \in \boldsymbol{C}: \alpha_\phi(\lambda) = 0\} .$$

Our arguments are based on the following result from [12]:

LEMMA 1. *Assume that B has a well defined rank over A, say n. Then for each $b \in B$ there exists a monic polynomial $\alpha(x) \in A[x]$ of degree n such that $\alpha(b) = 0$ and*

$$Z(\alpha_\phi) = \hat{b}(\pi^{-1}(\phi)) , \quad \phi \in M_A . \tag{1}$$

The notation and terminology we use are standard (see [6] and [11]). For basic facts about projective modules and (algebraically) separable algebras the reader is referred to [5].

2. Peak interpolation sets.

Before stating our first result we recall some well known definitions.

Let $\mathfrak{B}$ be a Banach algebra, Z a closed boundary for $\mathfrak{B}$ and K a closed subset of Z. Then K is said to be a peak set for $\mathfrak{B}$ on Z if there exists $f \in \mathfrak{B}$ such that $\hat{f} = 1$ on K and $|\hat{f}| < 1$ on $Z \backslash K$. If K is an intersection of peak sets for $\mathfrak{B}$ on Z, we say that K is a generalized peak set for $\mathfrak{B}$ on Z. We call K an interpolation set for $\mathfrak{B}$ if, given any $h \in C(K)$, there exists $f \in \mathfrak{B}$ such that $\hat{f} = h$ on K. If K is both a (generalized) peak set for $\mathfrak{B}$ on Z and an interpolation set for $\mathfrak{B}$, then it is called a (generalized) peak interpolation set for $\mathfrak{B}$ on Z. When $\mathfrak{B}$ is a uniform algebra on Z, the explicit reference to Z is usually dropped, so that one simply speaks of (generalized) peak (resp. peak interpolation) sets for $\mathfrak{B}$.

As we said before, in the following theorem we just assume that B is a finitely generated projective extension of A, endowed with some Banach algebra structure. The corollary in [12, § 2] implies that $Y = \pi^{-1}(X)$ is a closed boundary for B.

THEOREM 1. *If $K \subset Y$ is a generalized interpolation set for B on Y, then $\pi(K)$ is a generalized peak interpolation set for A.*

COROLLARY. *If $K \subset Y$ is a peak interpolation set for B on Y, then $\pi(K)$ is a peak interpolation set for A.*

This corollary follows immediately from the theorem and from the fact that π preserves G_δ-sets.

Proof of Theorem 1. Without loss of generality, we may assume B to have a well defined rank over A. This is so because there are mutually orthogonal idempotents $e_1, \cdots, e_p$ in A such that $e_1 + \cdots + e_p = 1$ and, for each i, $e_i B$ is a finitely generated projective extension of $e_i A$, with a well defined rank over $e_i A$ [5, 4.11, p. 31].

Let B_0 be the uniform closure in $C(M_B)$ of $\hat{B}$. Then $M_{B_0} = M_B$ and $\partial_{B_0} = \partial_B$. Moreover, as it is easily seen, B_0 satisfies the conclusion of Lemma 1. We shall regard B_0 as a uniform algebra on Y and we shall prove the theorem under the weaker hypothesis that K is a generalized peak interpolation set for B_0.

If $H = \pi(K)$, then, by the Bishop-Glicksberg characterization of generalized peak interpolation sets [11, 20.10, p. 210], we have to show that $|\mu|(H) = 0$ for any complex regular Borel measure μ on X orthogonal to A. In order to see this, it is clearly sufficient to prove the following:

(2) for each $\phi \in H$ there exists a closed neighborhood (in X) U_0 of ϕ such that $|\mu|(U_0 \cap H) = 0$.

Fix $\phi \in H$ and let $\psi_1, \cdots, \psi_m$ be the different points in $\pi^{-1}(\phi)$. Apply the structure theorem for π [12, Th. 1] to find mutually disjoint open neighborhoods (in Y) $V_1, \cdots, V_m$ of $\psi_1, \cdots, \psi_m$ and an open neighborhood (in X) U of ϕ such that

$$\pi(V_i) = U \quad \text{for each } i , \quad \pi^{-1}(U) = \bigcup_i V_i$$

and

$$m(\psi_i) = \sum_{\psi \in \pi^{-1}(\theta) \cap V_i} m(\psi) , \quad \theta \in U , \quad 1 \leqq i \leqq m \tag{3}$$

where $m(\cdot)$ is the multiplicity function defined in § 1 of [12].

Let U_0 be a compact neighborhood of ϕ, contained in U. We have $U_0 \cap H = \bigcup_i \pi(K \cap \pi^{-1}(U_0) \cap V_i)$, and so, it order to prove (2), we can assume that K is included in a V_i, say V_1.

We claim now that there exists $g \in B_0$ satisfying

$$g(\pi^{-1}(H) \cap V_1) = \{1\} \quad \text{and} \quad g(\pi^{-1}(H) \cap V_i) = \{0\} , \quad i > 1 . \tag{4}$$

To prove the claim, write $I = \{f \in A: f(H) = \{0\}\}$, $J = \{f \in B_0: f(K) = \{0\}\}$, and observe that B_0/J is an integral extension of A/I.

Hence, each character in $M_{A/I}$ is the restriction to A/I of a character of B_0/J [14, p. 259]. On the other hand, our hypothesis on K implies that $B_0/J \cong C(K)$, and thus

$$H = \pi(K) = \pi(M_{B_0/J}) = M_{A/I} .$$

Now, B/IB is a finitely generated projective extension of A/I and so it is a Banach algebra under some norm. Since $M_{B/IB} = \pi^{-1}(H)$ (because $H = M_{A/I}$) and $\pi^{-1}(H)$ is the disjoint union of the closed subsets $\pi^{-1}(H) \cap V_i$, the Shilov idempotent theorem [10, 8.9, p. 73] can be applied to get $g \in \hat{B}$ satisfying (4).

To end the proof we still need some auxiliary tools. For $\psi \in M_B$ and for each neighborhood V (in M_B) of ψ, we define

$$c_\psi(V) = \max_{\theta \in \pi(V)} \operatorname{card}(\pi^{-1}(\theta) \cap V)$$

and

$$r(\psi) = \min_V c_\psi(V)$$

where V ranges over all neighborhoods of ψ. The number $r(\psi)$ may be interpreted as a ramification index for π at ψ.

For $\theta \in H$ we define

$$N(\theta) = \max_{\psi \in \pi^{-1}(\theta) \cap K} r(\psi) .$$

Now we can complete the proof of (2). We will proceed by induction on $N(\phi)$.

Suppose $N(\phi) = 1$. In this case, we will prove directly that H is a generalized peak interpolation set for A. As $r(\psi_1) = 1$, shrinking V_1 and U if necessary, we can also assume that $\pi | V_1$ is an homeomorphism onto U. Given an open neighborhood W of H, $W \subset U$, and given $\varepsilon > 0$, then, by hypothesis, there is a peak set K' for B_0 such that $K \subset K' \subset \pi^{-1}(W)$. Choose $f \in B_0$ which peaks on K' and satisfies

$$|f(\psi)| < \varepsilon/n \|g\|_Y , \quad \psi \in Y \backslash \pi^{-1}(W)$$

and then consider the function h defined by

$$h(\theta) = \sum_{\psi \in \pi^{-1}(\theta)} m(\psi)(fg)(\psi) , \quad \theta \in X .$$

If $\alpha(x) = \sum_{i=0}^n \alpha_i x^i$ is a polynomial related to $b = fg$ as in Lemma 1, then $h = -\alpha_{n-1} \in A$ by (1). Writing $k = h/m(\psi_1)$, we obtain

$$k(\theta) = 1 , \quad \theta \in H$$
$$|k(\theta)| \leqq \varepsilon , \quad \theta \in X \backslash W$$
$$\|k\|_X \leqq n \|g\|_Y .$$

Thus H is a generalized peak set for A.

Let $h_0 \in C(H)$. If $f \in B_0$ is such that $f(\psi) = h_0(\pi(\psi))$, $\psi \in K$, and k is constructed from f and g as above, then $k|_H = h_0$ and $k \in A$.

Now, assume that $N(\phi) > 1$ and that (2) holds for those $\theta \in H$ with $N(\theta) < N(\phi)$. Since $N(\phi) = r(\psi_1)$, we may suppose that $N(\phi) = c_{\psi_1}(V_1)$. If we put

$$H_1 = \{\theta \in H: r(\psi) < r(\psi_1) \text{ for each } \psi \in \pi^{-1}(\theta) \cap K\}$$

and

$$H_0 = \{\theta \in H: \pi^{-1}(\theta) \cap V_1 \text{ is a singleton}\}\ ,$$

then $|\mu|(H_1) = 0$ by the inductive hypothesis. Moreover, H_0 is closed and $H = H_0 \cup H_1$. We claim now that H_0 is a generalized peak interpolation set for A. To see this, observe that $K_0 = \pi^{-1}(H_0) \cap K$ is a generalized peak interpolation set for B_0 and that π is an homeomorphism from K_0 onto H_0. Then, the same argument as above can be used to deduce the desired conclusion. Thus, $|\mu|(H_0) = 0$ and the proof is complete.

REMARKS. (a) The projectivity hypothesis on B cannot be relaxed, as shown by the following example.

EXAMPLE 1. Let B the disk algebra, and put $A = \{f \in B: f(0) = f(1)\}$. Then B is an extension of A, finitely generated as an A-module, but 1 is not a peak point for A, although it is a peak point for B.

(b) It would be interesting to find out whether Theorem 1 is true when A is a semisimple Banach algebra, and also whether an analogous statement for peak sets holds. In both cases, the main difficulty seems to be the reduction to a local statement.

From now on, B will be assumed to be a uniform algebra on Y. Recall [12, Th. 3] that this is the case if and only if $m(\psi) = 1$, $\psi \in \partial_B$, and then, in particular, ∂_B is a covering space of ∂_A with projection π.

THEOREM 2. *Let K be a subset* (resp. *a G_δ subset*) *of Y. Then, K is a generalized peak interpolation* (resp. *a peak interpolation*) *set for B if and only if $\pi(K)$ is a generalized peak interpolation* (resp. *a peak interpolation*) *set for A.*

Proof. The "only if part" is contained in Theorem 1.

Assume $H = \pi(K)$ to be a generalized peak interpolation set for A. If $I = \{f \in A: f(H) = \{0\}\}$, then B/IB is a finitely generated projective extension of $A/I = C(H)$. Since $m(\psi) = 1$, $\psi \in \partial_B$ and $H \subset \partial_A$, we can write

$$m(\psi) = 1\ , \quad \psi \in \pi^{-1}(H) = M_{B/IB}\ .$$

But this means that B/IB is separable as $C(H)$-algebra (use [5, Th. 7.1, (c) $\rightarrow$ (a), p. 72]), and hence, applying [3, Th. 2, p. 30], we conclude that $B/IB \cong C(\pi^{-1}(H))$. As $\pi^{-1}(H)$ is obviously a generalized peak set for B, the proof is complete.

COROLLARY. *If γ_A and γ_B denote the Choquet boundaries of A and B respectively, then $\gamma_B = \pi^{-1}(\gamma_A)$.*

REMARKS. (a) The "if part" of Theorem 2 is not true without uniformity assumptions on B. For example, if $A = C[0, 1]$ and $B = A[x]/(x^2 - f)$, where $f(t) = t$ for each $t \in [0, 1]$, then B is not a uniform algebra according to [12, Th. 3]. Therefore, Y itself is not an interpolation set for B.

(b) The statement in the "only if part" of Theorem 2 is not true for interpolation sets as shown by the following example.

EXAMPLE 2. Consider a uniform algebra A with the following property (for example the disc algebra): there exist interpolation sets K_0, K_1 for A such that $K_0 \cup K_1$ is not an interpolation set for A. Put $B = A[x]/(x^2 - 1)$, so that we may identify M_B with $M_A \times \{0, 1\}$. Now, $K = \bigcup_{j=0}^{1} K_j \times \{j\}$ is an interpolation set for B, but $\pi(K) = K_0 \cup K_1$ is not an interpolation set for A.

(c) The arguments used in the proof of Theorem 2 can be applied to deal with some examples arising from the theory of several complex variables.

EXAMPLE 3. Write

$$D_n = \{z \in C^p \colon |z_1|^2 + \cdots + |z_{p-1}|^2 + |z_p|^{2n} \leqq 1\}\ ,$$

n a positive integer. Then

$$\begin{aligned} D_n &\xrightarrow{\ \pi\ } D_1 \\ z &\longrightarrow (z_1, \cdots, z_{p-1}, z_p^n) \end{aligned}$$

is an n-sheeted covering map, ramified along $\{z \colon z_p = 0\}$. Let A (resp. B) be the algebra of continuous functions on D_1 (resp. D_n) which are holomorphic in the interior of D_1 (resp. D_n). We claim that Theorem 2 is true in this case. To prove the claim, take first a peak interpolation set K for B. Then K is a zero set for B, that is, there is $f \in B$ with $K = f^{-1}(0)$. Since Lemma 1 also works in this case, $\pi(K)$

is a zero set for A, and hence a peak interpolation set for A [13, Th. 1.1, p. 484]. Assume now that K is a peak interpolation set for A. Since π is a local homeomorphism on $D_n\backslash\{z: z_p = 0\}$, one can prove that $\pi^{-1}(H)$ is a peak interpolation set for B for each compact subset H of $K\backslash\{z: z_p = 0\}$. But on $K \cap \{z: z_p = 0\}$ B and A are the same algebra, thus $\pi^{-1}(K)$ is a peak interpolation set for B.

In particular, we have shown that on ∂B zero sets, peak sets and peak interpolation set are the same, although, for $n > 1$ and $p > 1$, the interior of D_n is not strictly pseudoconvex.

3. Antisymmetric decompositions.

LEMMA 2. *If A is an antisymmetric algebra, then the maximal sets of antisymmetry for B on M_B are the connected components of M_B.*

Proof. We show first that if M_B is connected, then B is antisymmetric. To see this, let $f \in B$ and assume $\hat{f}$ to be a real function. Since A is antisymmetric, there are not nontrivial idempotents in A, and thus B has a well defined rank over A [5, 4.12, p. 32], say n. If $\alpha(x) = \sum_{i=0}^{n} \alpha_i x^i$ is a polynomial obtained from f by applying Lemma 1, then each $\hat{\alpha}_i$ is a real function (by (1)), so constant. Therefore, $\hat{f}(M_B)$ is finite, but since M_B is connected, $\hat{f}(M_B)$ is in fact a point.

Consider now m different connected components of M_B, say $C_1, \cdots, C_m$, and separate them by means of mutually disjoint open and closed sets $U_1, \cdots, U_m$. As M_A is connected, one has $\pi(U_i) = M_A$ for each i, and this implies that $m \leqq n$. Thus, M_B has only a finite number of connected components $C_1, \cdots, C_m$, which are open and closed sets. Let e_i be the idempotent in B whose Gelfand transform is the characteristic function of C_i. The first part of the proof, applied to e_iB, tells us that C_i is a set of antisymmetry for B. If $C_i \subsetneqq S$, then $\hat{e}_i$ is real and nonconstant on S, so C_i is, in fact, a maximal set of antisymmetry for B on M_B.

THEOREM 3. *Let $(K_i)_{i \in I}$ be the family of maximal sets of antisymmetry for A on X. Then the family of maximal sets of antisymmetry for B on Y is $(C_{ij} \cap Y)_{i,j}$, where, for each $i \in I$, $(C_{ij})_{1 \leq j \leq p_i}$ is the collection of connected components of $\pi^{-1}(\hat{K}_i)$, $\hat{K}_i$ being the A-convex hull of K_i. Moreover, $\sup_{i \in I} p_i < \infty$.*

Proof. Assume first that $X = M_A$, so that $K_i = \hat{K}_i$, $i \in I$. For each $i \in I$, $A_i = \{f|_{K_i}: f \in A\}$ is an antisymmetric uniform algebra, $M_{A_i} = K_i$ and $\partial_{A_i} \subset K_i \cap \partial_A$ [6, Th. 11 (c), p. 167]. If we set $I_i =$

$\{f \in A: f(K_i) = \{0\}\}$, then $B_i = B/I_iB$ is a finitely generated projective extension of $A/I_i \cong A_i$ and $M_{Bi} = \pi^{-1}(K_i)$. By the corollary in § 2 of [12], we have

$$\partial_{Bi} = \pi^{-1}(\partial_{Ai}) \subset \pi^{-1}(\partial_A) = \partial_B .$$

Now, since B is a uniform algebra, we get [12, Th. 3]

$$m(\psi) = 1 , \quad \psi \in \partial_B$$

and thus, in particular, $m(\psi) = 1$, $\psi \in \partial_{B_i}$. Again appealing to [12, Th. 3], we conclude that B_i is a uniform algebra, hence that $B_i = \{f \mid \pi - 1_{(K_i)}: f \in B\}$.

By Lemma 2, for each $j \in \{1, 2, \cdots, p_i\}$, C_{ij} is a set of antisymmetry for B. Let S be the maximal set of antisymmetry for B on M_B containing C_{ij}. If $\pi(S) \neq K_i$, then there would exist an $a \in A$ which would be real and nonconstant on $\pi(S)$. But then $a \in B$ would be real and nonconstant on S, which is impossible. So $\pi(S) = K_i$. From this we obtain $C_{ij} \subset S \subset \pi^{-1}(K_i)$, and, since S is connected, we get $S = C_{ij}$.

In order to prove the general case, notice that $(\hat{K}_i)_{i \in I}$ is the family of maximal sets of antisymmetry for A on M_A [6, Th. 15, p. 171]. But the maximal sets of antisymmetry for B on Y are the intersection with Y of these for B on M_B [6, Th. 14, p. 171], and thus from the first part of the proof we can draw out the desired conclusion.

In the following theorem we write E_A (resp. E_B) for the essential set of A (resp. B).

THEOREM 4. *We have* $E_B = \pi^{-1}(E_A)$. *In particular, B is essential if and only if A is essential.*

Proof. One can prove, using an elementary argument based on the structure theorem for π, that

$$(5) \qquad \pi^{-1}(\mathrm{Cl}\, F) = \mathrm{Cl}\, \pi^{-1}(F) , \quad \text{for each} \quad F \subset X ,$$

where Cl denotes topological closure.

Let P_A (resp. P_B) be the union of all one point maximal sets of antisymmetry for A (resp. B), so that, by Theorem 3, $P_B = \pi^{-1}(P_A)$. Using (5) and [6, Corollary 2, p. 65] we obtain

$$E_B = \mathrm{Cl}\,(\pi^{-1}(X \backslash P_A)) = \pi^{-1}(\mathrm{Cl}\,(X \backslash P_A)) = \pi^{-1}(E_A) .$$

4. *B*-holomorphic functions. If $\mathfrak{B}$ is a uniform algebra and U an open subset of $M_{\mathfrak{B}}$, then a complex function on U is called a locally $\mathfrak{B}$-approximable function if each point in U has a neighborhood on which f is uniformly approximable by functions in $\mathfrak{B}$. We write $L_{\mathfrak{B}}$ for the algebra of the locally $\mathfrak{B}$-approximable functions defined on all $M_{\mathfrak{B}}$, and we denote by $H_{\mathfrak{B}}$ the smallest subalgebra of $C(M_{\mathfrak{B}})$ which contains $\mathfrak{B}$ and is closed under local uniform approximation.

At this point, some remarks on the relation between $L_{\mathfrak{B}}$ and $H_{\mathfrak{B}}$ are in order. A simple inductive argument [10, 8.1, p. 19] shows the following:

There exists an ordinal μ such that to each ordinal $\nu \leqq \mu$ there corresponds a subalgebra $H_{\mathfrak{B}}^{\nu}$ of $C(M_{\mathfrak{B}})$ with the following properties:

(i) $H_{\mathfrak{B}}^{0} = \mathfrak{B}$, $H_{\mathfrak{B}}^{\mu} = H_{\mathfrak{B}}$, and $H_{\mathfrak{B}}^{\alpha} \subsetneqq H_{\mathfrak{B}}^{\beta}$ for $0 \leqq \alpha < \beta \leqq \mu$.

(ii) If $0 < \nu \leqq \mu$ then $H_{\mathfrak{B}}^{\nu} = L_{\mathfrak{B}_{\nu}}$, where $\mathfrak{B}_{\nu}$ is the uniform closure in $C(M_{\mathfrak{B}})$ of the algebra $\bigcup_{\alpha<\nu} H_{\mathfrak{B}}^{\alpha}$.

Let us observe that, by a theorem of Rickart [10, 40.3, p. 116], we have $M_{\mathfrak{B}_{\nu}} = M_{\mathfrak{B}}$ for each ν, so that, according to (ii) and to our notational conventions, $H_{\mathfrak{B}}^{\nu}$ is a subalgebra of $C(M_{\mathfrak{B}})$.

Functions in $H_{\mathfrak{B}}$ (resp. $H_{\mathfrak{B}}^{\nu}$) are called $\mathfrak{B}$-holomorphic functions (resp. $\mathfrak{B}$-holomorphic functions of class ν) by Rickart [10, § 17].

THEOREM 5. *If B is a strongly separable A-algebra, then*

(a) H_B (resp. L_B) *is a strongly separable* H_A (resp. L_A)*-algebra.*

(b) *If* $A = H_A$ (resp. $A = L_A$), *then* $B = H_B$ (resp. $B = L_B$).

We divide the proof of Theorem 5 into three lemmas.

Let S be an extension of a commutative unital ring R. If S is finitely generated and projective as an R-module, then one can define a distinguished R-module homomorphism from S into R, called the trace map. Then one proves [5, 2.1, p. 92]:

LEMMA 3. *The extension S of the commutative unial ring R is strongly separable over R if and only if there is an R-module homomorphism t from S into R and elements of S $x_1, \cdots, x_m$; $y_1, \cdots, y_m$ with*

(i) $\sum x_j y_j = 1$

(ii) $x = \sum_j t(xy_j)x_j$, $x \in S$.

Moreover the map t is always the trace map from S to R.

LEMMA 4. *If B is a strongly separable A-algebra and has a well defined rank over A, then for each $\phi \in M_A$ there exists a monic polinomial $\alpha(x) \in A[x]$ and a $d \in A$ with $d(\phi) \neq 0$, such that B_d is isomorphic, as an A_d-algebra, to $A_d[x]/(\alpha(x))$ (here A_d and B_d stand for the* *quotient rings of the multiplicative system of the powers of* d).

Proof. Given $\phi \in M_A$, consider an element b_0 of B which separates the points of $\pi^{-1}(\phi)$, and a polynomial $\alpha(x) \in A[x]$ given by Lemma 1 applied to b_0. Define d as the discriminant of $\alpha(x)$, so that $d(\phi) \neq 0$. Let φ be the unique A_d-algebra homomorphism from $A_d[x]/(\alpha(x))$ into B_d satisfying $\varphi([x]) = b_0/1$, where $[x]$ is the class of the polynomial x.

Suppose that, for certain nonnegative integers m_i,

$$\sum_{i=0}^{n-1} (a_i/d^{m_i})(b_0^i/1) = 0 \ , \quad a_i \in A \ , \quad n = \operatorname{rank}_A B \ .$$

Then, for a large nonnegative integer m,

$$\sum_{i=0}^{n-1} d^m a_i b_0^i = 0 \ .$$

If $d(\omega) \neq 0$, $\omega \in M_A$, then the complex polynomial

$$\sum_{i=0}^{n-1} d^m(\omega) a_i(\omega) x^i$$

is annihilated by the n elements of $b_0(\pi^{-1}(\omega))$. From this we obtain $da_i = 0$, $0 \leqq i \leqq n-1$. Thus

$$a_i/d^{m_i} = 0 \quad \text{in} \quad A_d \ , \quad 0 \leqq i \leqq n-1 \ ,$$

that is, φ is injective.

Write $B_0 = \operatorname{Im} \varphi$. Then B_0 is a strongly separable A_d-algebra [5, Problem 8, p. 85] with a well defined rank n over A_d. Since B_d is a finitely generated projective A_d-module, we conclude, owing to the lifting property of projective modules over separable algebras [5, 2.3, p. 48], that B_d is a finitely generated projective B_0-module. But the rank of B_d over A_d is n, so the rank of B_d over B_0 is 1, and thus $B_d = B_0$. Therefore φ is an isomorphism.

LEMMA 5. *With the same hypothesis on* B *as in Lemma* 4, *the following holds*:

If V, U *are open set in* M_B *and* M_A *respectively, and if* $\pi | V$ *is an homeomorphism onto* U, *then, for each* $b \in B$, $\beta = b \circ (\pi | V)^{-1}$ *is locally* A-*approximable on* U.

Proof. Consider $\phi \in U$ and let b_0, $\alpha(x)$ and d be as in the proof of Lemma 4. Shrinking U and V we may assume that $d(\omega) \neq 0$ for each $\omega \in U$.

By Lemma 4, we have

$$\beta = \sum_{i=0}^{n-1} (a_i/d^{m_i})\beta_0^i$$

where $\beta_0 = b_0 \circ (\pi|V)^{-1}$, $a_i \in A$ and m_i is a nonnegative integer for each i.

A standard argument shows that d^{-1} is a locally A-approximable function on U. That β_0 is also locally A-approximable on U follows from the fact that, d never being zero on U, β_0 may be locally expressed as a uniformly convergent power series in the coefficients of $\alpha(x)$.

Proof of Theorem 5. According to Lemma 3 there exist elements of B $x_1, \cdots, x_m$; $y_1, \cdots, y_m$ such that $\sum x_i y_i = 1$ and

$$t(f) = \sum_i t(fy_i)x_i\,, \quad f \in B\,. \tag{6}$$

We may assume B to have a well defined rank n over A (use the argument in the first paragraph of the proof of Theorem 1). This assumption and the strong separability of B imply that M_B is an n-sheeted covering space of M_A with projection π [9, Th. 5, p. 138]. In this context, the trace map is given by

$$t(f)(\phi) = \sum_{\psi \in \pi^{-1}(\phi)} f(\psi)\,, \quad f \in B\,, \quad \phi \in M_A\,. \tag{7}$$

But $C(M_B)$ is a strongly separable extension of $C(M_A)$ [3, Th. 2, p. 30] and, consequently, its trace map is given, for $f \in C(M_B)$, by (7). Moreover, relation (6) is true for $f \in C(M_B)$. Therefore, by Lemma 3, we only have to show that

$$t(f) \in H_A \quad \text{if} \quad f \in H_B\,. \tag{8}$$

We will prove, by transfinite induction, that

$$t(f) \in H_A^\nu \quad \text{if} \quad f \in H_B^\nu\,, \quad \text{for each } \nu\,. \tag{9}$$

If $\nu > 0$ is an ordinal such that (9) is true for all $\alpha < \nu$, then clearly we have

$$t(f) \in A_\nu \quad \text{if} \quad f \in B_\nu\,.$$

Therefore, replacing A_ν and B_ν by A and B, we are led to prove that

$$t(f) \in L_A \quad \text{if} \quad f \in L_B\,.$$

Given $f \in L_B$, fix $\phi \in M_A$ and consider mutually disjoint open neighborhoods V_i of the points in $\pi^{-1}(\phi)$, and an open neighborhood U of ϕ such that

(i) $\pi|V_i$ is an homeomorphism onto U for each i.

(ii) f is uniformly approximable on each V_i by functions in B.

Given $\varepsilon > 0$, consider $b_i \in B$ with $\|f - b_i\|_{V_i} < \varepsilon$, $1 \leqq i \leqq n$. Thus

$$\left\|t(f) - \sum_{i=1}^{n} \beta_i\right\|_U < n\varepsilon ,$$

where $\beta_i = b_i \circ (\pi|V_i)^{-1}$, $1 \leqq i \leqq n$. Now Lemma 5 says that $t(f) \in L_A$. This completes the induction and so the proof of (8).

REMARK. The above proof shows that a slightly more general statement is true. In fact, we have proved that H_B^ν is a strongly separable H_A^ν-algebra and that $A = H_A^\nu$ implies $B = H_B^\nu$ for each ordinal ν.

Finally, we point out two problems whose solution we do not know.

PROBLEM 1. In the standard hypothesis of this paper, that is, $A \subset B$, A and B uniform algebras, B finitely generated and projective as an A-module, is H_B (resp. L_B) finitely generated and projective as an H_A (resp. L_A)-module?

PROBLEM 2. With the same hypothesis on A and B, does $A = H_A$ (resp. $A = L_A$) imply $B = H_B$ (resp. $B = L_B$)?.

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