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SHIFTS ON INDEFINITE INNER PRODUCT SPACES II

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This paper continues the study of isometries on indefinite inner product spaces by means of their wandering subspaces. In the author's earlier paper of the same title (Pacific J. Math., 81 (1979), 113–130), it was shown that the subspace on which an isometry acts as a shift need not be regular and that vectors in this subspace need not be recoverable from their Fourier coefficients by summation. We present here necessary and sufficient conditions for this situation not to occur, and also show that these conditions are sufficient (but not necessary) for the isometry to have a Wold decomposition.

1. Introduction. Throughout this paper we will be using the notation and assuming the results of the paper [4]. Our attention will be restricted to isometries on *Krein* spaces $\mathscr{K}$ (see [1, Chapter V]), where the indefinite inner product $[\cdot, \cdot]$ is related to a Hilbert space inner product $(\cdot, \cdot)$ on $\mathscr{K}$ by means of a *fundamental symmetry* J:

$$[x, y] = (Jx, y) , \quad J = J^* = J^{-1} .$$

Except in § 2, where we prove a lemma on projections in Hilbert space, we will be using the indefinite inner product $[\cdot, \cdot]$ to define properties of operators and subspaces. In particular, an isometry V preserves the indefinite inner product, and the concepts of adjoint and orthogonality use this inner product. Thus if $\mathscr{L}$ is a subspace of $\mathscr{K}$, then

$$\mathscr{L}^\perp = \{h \in \mathscr{K} : [h, k] = 0 \text{ for all } k \in \mathscr{L}\} .$$

If $\mathscr{L} \oplus \mathscr{L}^\perp = \mathscr{K}$, then $\mathscr{L}$ is called *regular*. A projection P satisfies $P = P^2 = P^*$, i.e., self-adjoint with respect to the indefinite inner product, and the regular subspaces are those that are the ranges of projections. In § 2, where an indefinite inner product will not be used, we will use Q to denote an orthogonal projection in Hilbert space and P to denote any other projection.

Suppose V is an isometry on a Krein space $\mathscr{K}$, and let $\mathscr{L} = (VK)^\perp$. Then $\mathscr{L}$ is wandering for V, i.e., $V^p\mathscr{L} \perp V^q\mathscr{L}$ for all nonnegative integers $p \neq q$. Since V is an isometry, VV^* is the projection onto $V\mathscr{K}$. Thus the projection P onto $\mathscr{L}$ is given by $P = I - VV^*$, and so $\mathscr{L}$ is regular.

We make the definition $M_+(\mathscr{L}) = \bigvee_{n=0}^\infty V^n\mathscr{L}$. Every vector

$h \in M_+(\mathscr{L})$ has associated with it its sequence $\{l_n\}_{n\geq 0}$ of Fourier coefficients, where $l_n = PV^{*n}h$. Unlike the situation in Hilbert space, it is not necessarily true that

$$h = \sum_{n=0}^{\infty} V^n l_n \tag{1.1}$$

(see [4, Example 7.3]), and indeed the Fourier coefficients need not determine the vector h uniquely [4, Theorem 7.2]. The theorem below gives necessary and sufficient conditions for $M_+(\mathscr{L})$ to be regular and (1.1) to be true.

THEOREM. *Let V be an isometry on a Krein space $\mathscr{K}$, and let $L = (V\mathscr{K})^\perp$. Then the following are equivalent:*

(i) $\sup\{\| V^n V^{*n} \|: n = 0, 1, 2, \cdots\} < \infty$;

(ii) $\lim_{n\to\infty} V^n V^{*n}$ *exists;*

(iii) $M_+(\mathscr{L})$ *is regular and every vector* $h \in M_+(\mathscr{L})$ *can be written as*

$$h = \sum_{n=0}^{\infty} V^n l_n ,$$

where $\{l_n\}_{n\geq 0}$ is the sequence of Fourier coefficients of h.

These conditions are sufficient, but not necessary, for V to have a Wold decomposition. In particular, if $\lim_{n\to\infty} V^n V^{*n} = 0$ *then V is a unilateral shift.*

If conditions (i), (ii), *and* (iii) *are satisfied, then we have, for each $h, k \in M_+(\mathscr{L})$,*

$$[h, k] = \sum_{n=0}^{\infty} [l_n, l'_n] ,$$

where $\{l_n\}_{n\geq 0}$ and $\{l'_n\}_{n\geq 0}$ are the sequences of Fourier coefficients of h and k, respectively. □

The limit in (ii) (and throughout this paper) is in the strong operator topology on $\mathscr{K}$.

The Wold decomposition referred to above is a decomposition of the space $\mathscr{K}$ into orthogonal subspaces $\mathscr{K} = \mathscr{K}_0 \oplus \mathscr{K}_1$ which reduce V such that $V|\mathscr{K}_0$ is a unilateral shift and $V|\mathscr{K}_1$ is unitary [4, § 8]. The indefinite inner product is used in defining the terms isometry, orthogonal, shift, and unitary, and so this is a different type of Wold decomposition from that discussed in [2] and [3] for isometries on Banach spaces.

Condition (i) of the theorem is implied by the uniform boundedness of the operators V^n, $n = 0, 1, 2, \cdots$, but the converse is not true. This is shown in § 4, where an isometry V is constructed which satisfies the conditions of the theorem but which has $\lim_{n\to\infty} \| V^n \| = \infty$. In § 4 we also show that the requirement that $M_+(\mathscr{L})$ be regular can not be dropped from condition (iii) of the theorem.

2. Projections in Hilbert space. The proof of the theorem rests upon the following lemma, which will be proved in the context of a Hilbert space $\mathscr{H}$, without reference to an indefinite inner product. Similar results for projections in a Banach space appear in [2, Theorems 1 and 3]. (See also [3, Lemma 3].)

LEMMA. *Let $\{P_n\}_{n\geqq 0}$ be an increasing sequence of projections (not necessarily self-adjoint) on a Hilbert space $\mathscr{H}$, and let $\mathscr{M}_n = P_n\mathscr{H}$, $\mathscr{N}_n = (I - P_n)\mathscr{H}$, $\mathscr{M} = \bigvee_{n=0}^{\infty} \mathscr{M}_n$, $\mathscr{N} = \bigcap_{n=0}^{\infty} \mathscr{N}_n$. Then*

$$\sup \{\| P_n \|: n = 0, 1, 2, \cdots\} < \infty$$

if and only if the strong limit $\lim_{n\to\infty} P_n$ exists. In this case we have $\mathscr{M} \cap \mathscr{N} = \{0\}$, $\mathscr{M} + \mathscr{N} = \mathscr{H}$, and $P = \lim_{n\to\infty} P_n$ is the projection with range $\mathscr{M}$ and null space $\mathscr{N}$.

Proof. If $\lim_{n\to\infty} P_n$ exists, then an application of the uniform boundedness principle shows that

$$\sup \{\| P_n \|: n = 0, 1, 2, \cdots\} < \infty .$$

Conversely, assume that the sequence $\{P_n\}_{n\geqq 0}$ is uniformly bounded, and let Q_n denote the orthogonal (self-adjoint) projection onto $\mathscr{M}_n$. Since $\{\mathscr{M}_n\}_{n\geqq 0}$ is an increasing sequence, it is clear that $\lim_{n\to\infty} Q_n$ exists and that $P_nP_m = P_n$ for $m > n$. Also, the fact that P_n and Q_n are projections with the same range $\mathscr{M}_n$ implies that $Q_nP_n = P_n$ and $I - P_n = (I - P_n)(I - Q_n)$. Thus we have, for $m > n$,

$$P_m - P_n = (I - P_n)P_m = (I - P_n)(I - Q_n)P_m = (I - P_n)(Q_m - Q_n)P_m .$$

Since $\lim_{n\to\infty} Q_n$ exists and the sequence $\{P_n\}_{n\geqq 0}$ is uniformly bounded, it follows that $\lim_{n\to\infty} P_n$ exists.

If $P = \lim_{n\to\infty} P_n$, then P is obviously a projection with range contained in $\mathscr{M}$ and containing each $\mathscr{M}_n$. Thus, the range of P is $\mathscr{M}$. Also, since $(I - P_n)h$ is in $\mathscr{N}_n$ for all h, and $\lim_{n\to\infty} (I-P_n)h = (I - P)h$, it readily follows that the null space of P is $\mathscr{N}$. Thus $\mathscr{M} \cap \mathscr{N} = \{0\}$ and $\mathscr{M} + \mathscr{N} = \mathscr{H}$. □

3. Proof of the theorem. Let V be an isometry on a Krein space $\mathscr{K}$, and let $\mathscr{L} = (V\mathscr{K})^{\perp}$. Then $\mathscr{L}$ is regular with the projection P onto $\mathscr{L}$ being given by $P = I - VV^*$. The projection onto $V^n\mathscr{L}$ is then $V^nPV^{*n} = V^n(I - VV^*)V^{*n}$ [4, Theorem 5.2]. If we define

$$\mathscr{M}_n = \bigvee_{k=0}^{n-1} V^k\mathscr{L}, \quad \text{and} \quad \mathscr{N}_n = \mathscr{M}_n^{\perp} ,$$

then the projection onto $\mathscr{M}_n$, with null space $\mathscr{N}_n$, is

$$P_n = \sum_{k=0}^{n-1} V^k(I - VV^*)V^{*k} = I - V^nV^{*n} .$$

By the lemma, this sequence of projections converges strongly if and only if it is uniformly bounded, and therefore $\lim_{n\to\infty} V^nV^{*n}$ exists if and only if $\sup \| V^nV^{*n} \| < \infty$. Furthermore, since $\bigvee_{n=0}^{\infty} \mathscr{M}_n = M_+(\mathscr{L})$, the lemma shows that in this case $M_+(\mathscr{L})$ is regular, and thus V has a Wold decomposition [4, Theorem 8.2]. If $\lim_{n\to\infty} V^nV^{*n} = 0$, then $\lim_{n\to\infty} P_n = I$, and so $M_+(\mathscr{L}) = \mathscr{K}$, i.e., V is a unilateral shift.

Let h be a vector in $M_+(\mathscr{L})$ and let $\{l_n\}_{n\geqq 0}$ be its sequence of Fourier coefficients:

$$l_n = PV^{*n}h = (I - VV^*)V^{*n}h \qquad (n = 0, 1, 2, \cdots) .$$

Then we have

$$\sum_{k=0}^{n-1} V^kl_k = \sum_{k=0}^{n-1} V^k(I - VV^*)V^{*k}h = (I - V^nV^{*n})h = P_nh . \tag{3.1}$$

If $\lim_{n\to\infty} V^nV^{*n}$ exists, then $\lim_{n\to\infty} P_n = \lim_{n\to\infty} (I - V^nV^{*n})$ exists and equals the projection onto $M_+(\mathscr{L})$. Thus

$$h = \lim_{n\to\infty} P_nh = \sum_{k=0}^{\infty} V^kl_k . \tag{3.2}$$

Now assume that $M_+(\mathscr{L})$ is regular and that (3.2) holds for $h \in M_+(\mathscr{L})$. By (3.1), this implies that $\lim_{n\to\infty} V^nV^{*n}h$ exists for all $h \in M_+(\mathscr{L})$. If $k \perp M_+(\mathscr{L})$, then $k \perp \mathscr{M}_n$ for each n and so $P_nk = 0$. Consequently, $V^nV^{*n}k = k$ for each n. We have assumed that $M_+(\mathscr{L})$ is regular, and thus every vector in $\mathscr{K}$ is of the form $h + k$, with $h \in M_+(\mathscr{L})$ and $k \perp M_+(\mathscr{L})$. Thus the strong limit $\lim_{n\to\infty} V^nV^{*n}$ exists.

We have now shown that the conditions (i), (iii), and (iii) of the theorem are equivalent. Suppose that these conditions are satisfied, and let $h, k \in M_+(\mathscr{L})$. Then

$$h = \sum_{n=0}^{\infty} V^nl_n \text{ and } k = \sum_{n=0}^{\infty} V^nl'_n ,$$

where $\{l_n\}_{n\geqq 0}$ and $\{l'_n\}_{n\geqq 0}$ are the sequences of Fourier coefficients of h and k, respectively. $\mathscr{L}$ is a wandering subspace for V, and thus

$V^p l_p \perp V^q l_q$ for $p \neq q$. Consequently, since V is an isometry and the inner product is continuous, we have

$$[h, k] = \sum_{n=0}^{\infty} [l_n, l'_n] .$$

The only assertion from the theorem that remains to be proved is the claim that the equivalent conditions (i), (ii), and (iii) are not necessary for V to have a Wold decomposition. This is demonstrated by [4, Example 7.3] in which V is a shift (and thus has a trivial Wold decomposition), but (1.1) is not satisfied. □

4. Two examples.

The first example in this section shows that the uniform boundedness of $V^n V^{*n}$ is not equivalent to the uniform boundedness of V^n, i.e., it is possible for V to satisfy the conditions of the theorem and yet still have

$$\sup\{\|V^n\|: n = 0, 1, 2, \cdots\} = \infty .$$

EXAMPLE 1. Let $\mathscr{K}_0$ be the two dimensional complex space of vectors $x = \{x_0, x_1\}$ with Hilbert space inner product

$$(x, y) = x_0\bar{y}_0 + x_1\bar{y}_1$$

and indefinite inner product

$$[x, y] = x_0\bar{y}_0 - x_1\bar{y}_1 .$$

Define U on $\mathscr{K}_0$ by $Ux = 1/4\{5x_0 + 3x_1, 3x_0 + 5x_1\}$. Then U preserves the indefinite inner product $[\cdot, \cdot]$ but has $\|U^n\| \to \infty$ as $n \to \infty$.

Let $\mathscr{K}$ be the space consisting of square summable sequences of vectors $h = \{h_n\}_{n\geq 0}$ with $h_n \in \mathscr{K}_0$, $n = 0, 1, 2, \cdots$. The inner products on $\mathscr{K}$ are defined by

$$(h, k) = \sum_{n=0}^{\infty} (h_n, k_n) \quad \text{and} \quad [h, k] = \sum_{n=0}^{\infty} [h_n, k_n] .$$

We define V on $\mathscr{K}$ by setting $Vh = k$, where $k_0 = 0$ and $k_n = Uh_{n-1}$ for $n = 1, 2, \cdots$. It is clear that V is an isometry (in fact a shift), and that $\|V^n\| \to \infty$ as $n \to \infty$. But the operator $V^n V^{*n}$ is the Hilbert space (norm one) projection onto the subspace of $\mathscr{K}$ consisting of all sequences h for which $h_0 = h_1 = \cdots = h_{n-1} = 0$. Thus, $V^n V^{*n}$ is uniformly bounded for $n = 0, 1, 2, \cdots$, and V satisfies the conditions of the theorem. □

The second example shows that it is possible for (1.1) to be satisfied for each $h \in M_+(\mathscr{L})$, and yet for $M_+(\mathscr{L})$ not to be regular. Consequently, in condition (iii) of the theorem we can not omit the requirement that $M_+(\mathscr{L})$ be regular.

EXAMPLE 2. Let $\mathscr{K}_0$ be the two dimensional space described in Example 1, and let $\mathscr{K}$ be the space of square summable sequences $h = \{h_n\}_{n=-\infty}^{\infty}$ with h_n a complex number for $n \leqq -1$, and $h_n \in \mathscr{K}_0$ for $n \geqq 0$. The inner products on $\mathscr{K}$ are defined by

$$(h, k) = \sum_{n=-\infty}^{-1} h_n \bar{k}_n + \sum_{n=0}^{\infty} (h_n, k_n)$$

and

$$[h, k] = -\sum_{n=-\infty}^{-1} h_n \bar{k}_n + \sum_{n=0}^{\infty} [h_n, k_n] .$$

Consider the sequence of numbers $\{\alpha_n\}_{n \geqq 0}$ given by $\alpha_n = 1/2 \arccos 4^{-n}$. Then $0 = \alpha_0 < \alpha_n < \alpha_{n+1} < \pi/4$ and $\alpha_n \to \pi/4$ as $n \to \infty$. For $n \geqq 0$, let x_n and y_n be the sequences in $\mathscr{K}$ which are zero except at position n, where x_n has the value $2^n\{\cos \alpha_n, \sin \alpha_n\} \in \mathscr{K}_0$ and y_n has the value $2^n\{\sin \alpha_n, \cos \alpha_n\} \in \mathscr{K}_0$. Then for $n \geqq 0$ we have

$$[x_n, x_n] = -[y_n, y_n] = 1 \text{ and } [x_n, y_n] = 0 .$$

For $n \leqq -1$, we denote by y_n the sequence in $\mathscr{K}$ which is zero except at position n, where it has the value 1. Then every sequence $h = \{h_n\}_{n=-\infty}^{\infty} \in \mathscr{K}$ can be written

$$h = \sum_{n=-\infty}^{-1} h_n y_n + \sum_{n=0}^{\infty} (a_n x_n + b_n y_n) , \tag{4.1}$$

where $\sum_{n=-\infty}^{-1} |h_n|^2 < \infty$ and $\sum_{n=0}^{\infty} \| a_n x_n + b_n y_n \|^2 < \infty$.

Note that

$$\| a_n x_n + b_n y_n \|^2 = 4^n(|a_n|^2 + |b_n|^2 + 2 \operatorname{Re} a_n b_n \sin 2\alpha_n)$$

and

$$\| a_n x_{n+1} + b_n y_{n+1} \|^2 = 4^{n+1}(|a_n|^2 + |b_n|^2 + 2 \operatorname{Re} a_n b_n \sin 2\alpha_{n+1}) .$$

Since $\sin 2\alpha_{n+1} > \sin 2\alpha_n$, we can deduce that

$$\begin{aligned} \| a_n x_{n+1} + b_n y_{n+1} \|^2 &\leqq 4(1 + \sin 2\alpha_{n+1})(1 + \sin 2\alpha_n)^{-1} \| a_n x_n + b_n y_n \|^2 \\ &\leqq 8 \| a_n x_n + b_n y_n \|^2 . \end{aligned}$$

Consequently, we can define a bounded operator V on $\mathscr{K}$ by

$$Vh = \sum_{n=-\infty}^{-1} h_n y_{n+1} + \sum_{n=0}^{\infty} (a_n x_{n+1} + b_n y_{n+1}) .$$

It is easily seen that $[Vh, Vk] = [h, k]$ for each $h, k \in \mathscr{K}$ and that $\mathscr{L} = (V\mathscr{K})^\perp$ is spanned by the single vector x_0. Consequently, $M_+(\mathscr{L})$ is the subspace of all vectors of the form

$$h = \sum_{n=0}^{\infty} a_n x_n \quad \text{with} \quad \sum_{n=0}^{\infty} \| a_n x_n \|^2 < \infty \ . \tag{4.2}$$

Since (4.2) can also be written as

$$h = \sum_{n=0}^{\infty} V^n(a_n x_0) \ ,$$

it follows that the sequence of Fourier coefficients of h is $\{a_n x_0\}_{n\geq 0}$ and that (1.1) is therefore satisfied for all $h \in M_+(\mathscr{L})$.

$M_+(\mathscr{L})^\perp$ is the subspace of all vectors of the form

$$k = \sum_{n=-\infty}^{\infty} b_n y_n \quad \text{with}$$

$$\sum_{n=-\infty}^{-1} |b_n|^2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \| b_n y_n \|^2 < \infty \ .$$

Consider the vector h given by (4.1) with $h_n = 0$ for $n \leqq -1$, and $a_n = -b_n = 2^{-n}$. Note that the square summability condition is satisfied:

$$\begin{aligned} \sum_{n=0}^{\infty} \| a_n x_n + b_n y_n \|^2 &= \sum_{n=0}^{\infty} 2(1 - \sin 2\alpha_n) \\ &= \sum_{n=0}^{\infty} 2(1 - (1 - 16^{-n})^{1/2}) < \infty \ . \end{aligned}$$

However, $\| a_n x_n \| = \| b_n y_n \| = 1$, so that $\sum_{n=0}^{\infty} \| a_n x_n \|^2$ and $\sum_{n=0}^{\infty} \|b_n y_n\|^2$ both diverge. Thus h can not be written as a sum of vectors in $M_+(\mathscr{L})$ and $M_+(\mathscr{L})^\perp$, and so $M_+(\mathscr{L})$ is not regular. □

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