INTERPOLATION, CONTINUATION, AND QUADRATIC INEQUALITIES

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1. C. H. FitzGerald considered conditions for interpolating the values of an analytic function in terms of quadratic inequalities. In addition, he used them to obtain an interesting result concerning analytic continuation of functions of two variables and related them to Pick-Nevanlinna interpolation.

We shall show that these theorems follow directly from well-known principles of functional analysis. Since these principles are not limited to interpolating values of an analytic function, we shall obtain applications to interpolating functional values of both analytic and harmonic functions of several variables. In addition, we obtain analogous applications to analytic and harmonic continuation and to Pick-Nevanlinna interpolation.

2. **Abstract framework.** Let X be a reflexive topological vector space over the real or complex field and X^* its topological dual space. Assume that $p(x^*)$ is a continuous semi-norm defined on X^*. A classical theorem of Helly and Hahn (see G. Köthe [4, p. 113]) is the following. Since the proof is brief, we include it for completeness.

THEOREM 2.1 (*Helly-Hahn*). *Let* $\{c_i\}_{i \in I}$ *be a set of scalars and* $\{x_i^*\}_{i \in I}$ *a corresponding set of functionals in* X^*. *Then there exists an* $x \in X$ *such that*

(i) $x_i^*(x) = c_i$ *for all* $i \in I$, *and*

(ii) $|x^*(x)| \leqq p(x^*)$ *for all* $x^* \in X^*$,

if and only if

(1)
$$\left| \sum_{k=1}^{N} \alpha_{i_k} c_{i_k} \right| \leqq p\left(\sum_{k=1}^{N} \alpha_{i_k} x_{i_k}^* \right)$$

for every finite collection of scalars $\alpha_{i_1}, \cdots, \alpha_{i_N}$.

Proof. The necessity is trivial. For the sufficiency, let A be the linear span of $\{x_i^*\}_{i \in I}$. Define a linear functional x on A by setting $x(\sum_{k=1}^{N} \alpha_{i_k} x_{i_k}^*) = \sum_{k=1}^{N} \alpha_{i_k} c_{i_k}$. Then x is well defined since if $\sum \alpha_{i_k} x_{i_k}^* = \sum \beta_{i_k} x_{i_k}^*$, then (1) implies

$$|\sum \alpha_{i_k} c_{i_k} - \sum \beta_{i_k} c_{i_k}| \leqq p(\sum \alpha_{i_k} x_{i_k}^* - \sum \beta_{i_k} x_{i_k}^*) = p(0) = 0.$$

Obviously,

$$(2) \qquad\qquad |x(x^*)| \leqq p(x^*)$$

for each $x^* \in A$. By the Hahn-Banach theorem we may extend x to all of X^* and (2) persists. Since X is reflexive, we may identify x with a point of X.

We shall be concerned with applications of the Helly-Hahn theorem. One important case occurs when the semi-norm p is generated by a positive, continuous, conjugate-linear operator $T: X^* \to X$. Positive means that $x^*(Tx^*) \geqq 0$ for every $x^* \in X^*$, and conjugate-linear means that $T(\alpha x^*) = \bar{\alpha} T x^*$ when the scalar field is complex. In this case $p(x^*) = \sqrt{x^*(Tx^*)}$ is a continuous semi-norm on X^* and the Helly-Hahn theorem takes the following form.

COROLLARY 2.1. *Let $\{c_i\}_{i \in I}$ be a set of scalars and $\{x_i^*\}_{i \in I}$ a corresponding set of functionals in X^*. Suppose that $T: X^* \to X$ is a positive, continuous, conjugate-linear operator. Then there exists an $x \in X$ such that*
  (i) *$x_i^*(x) = c_i$ for all $i \in I$ and*
  (ii) *$|x^*(x)|^2 \leqq x^*(Tx^*)$ for all $x^* \in X^*$*
if and only if

$$(3) \qquad\qquad \left| \sum_{k=1}^{N} \alpha_{i_k} c_{i_k} \right|^2 \leqq \sum_{k,\kappa=1}^{N} \alpha_{i_k} \bar{\alpha}_{i_\kappa} x_{i_k}^*(Tx_{i_\kappa}^*)$$

for every finite collection of scalars $\alpha_{i_1}, \cdots, \alpha_{i_N}$.

A second case occurs when the semi-norm p is generated by a bounded operator T defined on a linear subspace of X^* to X.

COROLLARY 2.2. *Let $\{c_i\}_{i \in I}$ be a set of scalars and $\{x_i^*\}_{i \in I}$ a corresponding set of functionals in X^*. Suppose that T is a bounded, conjugate-linear operator from the linear span A of $\{x_i^*\}_{i \in I}$ to X. There exists an $x \in X$ such that*
  (i) *$x_i^*(x) = c_i$ for all $i \in I$,*
  (ii) *$|x^*(x)|^2 \leqq x^*(Tx^*)$ for all x^* in the closure of A, and*
  (iii) *$\|x\|^2 \leqq \|T\|_A$*
if and only if

$$(4) \qquad\qquad \left| \sum_{k=1}^{N} \alpha_{i_k} c_{i_k} \right|^2 \leqq \sum_{k,\kappa=1}^{N} \alpha_{i_k} \bar{\alpha}_{i_\kappa} x_{i_k}^*(Tx_{i_\kappa}^*)$$

for every finite collection of scalars $\alpha_{i_1}, \cdots, \alpha_{i_N}$.

Proof. The function $p(x^*) = \sqrt{\|T\|_A} \, \|x^*\|$ defines a (semi-)norm on X^*. Since $y^* = \sum_{k=1}^N \alpha_{i_k} x_{i_k}^*$ is in A, the condition (4) implies

$$\left| \sum_{k=1}^N \alpha_{i_k} c_{i_k} \right|^2 \leq |y^*(Ty^*)| \leq \|T\|_A \|y^*\|^2 = [p(y^*)]^2$$

so that (1) holds. Now by the Helly-Hahn theorem an $x \in X$ exists such that (i) holds, and (iii) follows from the inequality

$$|x(x^*)| = |x^*(x)| \leq p(x^*) = \sqrt{\|T\|_A} \, \|x^*\| \, .$$

Finally, (4) implies that $|x^*(x)|^2 \leq x^*(Tx^*)$ for $x^* \in A$, and by continuity for all x^* in the closure of A. The converse is trivial since (4) follows from (i) and (ii).

REMARKS. It is evident that x is unique whenever $\{x_i^*\}_{i \in I}$ is a complete set in X^* (i.e., the closure of A is X^*).

Positive, continuous, conjugate-linear operators $T: X^* \to X$ occur naturally. For example, if X is a Hilbert space, the familiar identification of X^* with X has these properties. See also §5.

3. An example. As an application of Corollary 2.2, we shall obtain the following result, and then note in the next section that a similar proof implies several significant generalizations.

THEOREM 3.1 (*FitzGerald* [2, *Thm.* 1.1]). *Let $\{z_n\}_{n=1}^\infty$ be a sequence of distinct points in a domain D of C, with an accumulation point in D. Suppose $K(z, \zeta)$ is defined on $D \times D$ and is analytic in z and anti-analytic in ζ. Suppose $\{c_n\}_{n=1}^\infty$ is a sequence of complex numbers. If*

$$(5) \qquad \left| \sum_{n=1}^N \alpha_n c_n \right|^2 \leq \sum_{m,n=1}^N \alpha_m \bar{\alpha}_n K(z_m, z_n)$$

for every finite collection of complex numbers $\alpha_1, \cdots, \alpha_N$, then there exists a unique analytic function f in D such that $f(z_n) = c_n$ for $n = 1, 2, \cdots$. Furthermore,

$$(6) \qquad \left| \sum_{n=1}^N \alpha_n f(z_n') \right|^2 \leq \sum_{m,n=1}^N \alpha_m \bar{\alpha}_n K(z_m', z_n')$$

holds for every finite collection of complex numbers $\alpha_1, \cdots, \alpha_N$ and any sequence $\{z_n'\}$ of points.

Proof. Let D_1 be a relatively compact subdomain of D, containing an accumulation point of the sequence $\{z_n\}$. Consider the Hilbert space $X = L_a^2(D_1)$ of square integrable analytic functions on D_1. Then the point evaluation functionals $x_n^*: g \to g(z_n)$, for $z_n \in D_1$,

are a complete set in X^*. Define $T: X^* \to X$ by

$$(Tx^*)(z) = \int_{D_1} K(z, \zeta)\phi(\zeta)dm_\zeta$$

where $\phi \in L_a^2(D_1)$ represents x^* and m is area measure. Then T is a bounded conjugate-linear operator and $x_m^*(Tx_n^*) = K(z_m, z_n)$. By Corollary 2.2 (or Corollary 2.1) a unique $f \in L_a^2(D_1)$ exists such that $f(z_n) = c_n$ for all $z_n \in D_1$. Furthermore, by choosing $x^*: g \to \alpha_1 g(z_1') + \cdots + \alpha_N g(z_N')$ in part (ii) of the corollary, we obtain (6) whenever the points $z_1', \cdots, z_N'$ belong to D_1. The theorem now follows from exhausting D by such subdomains D_1.

4. Some generalizations. In our proof of FitzGerald's theorem the particular choice of functionals and the complex plane were inessential. A similar proof yields the following generalization for $H(D)$, the space of analytic functions on a domain D in C^n, with the topology of locally uniform convergence.

First, we need some notation: If g is anti-analytic in D and $x^* \in H^*(D)$, define $\bar{x}^*g = \overline{x^*(\bar{g})}$. In addition, if $K(z, \zeta)$ is defined in $D \times D$, analytic in z and anti-analytic in ζ, and $x^*, y^* \in H^*(D)$, we define $x^*\bar{y}^*K = x_{(z)}^*(\bar{y}_{(\zeta)}^*K(z, \zeta))$. In particular, we write $|x^*|^2 = x^*\bar{x}^*$. Furthermore, K is a positive kernel if $|x^*|^2K \geqq 0$ for every $x^* \in H^*(D)$.

THEOREM 4.1. *Let D be a domain in C^n and $K(z, \zeta)$ a positive kernel on $D \times D$ that is analytic in z and anti-analytic in ζ. Suppose that $\{x_i^*\}_{i \in I}$ is a set of functionals in $H^*(D)$ and that $\{c_i\}_{i \in I}$ is a corresponding set of complex numbers. If*

$$(7) \qquad \left| \sum_{k=1}^N \alpha_{i_k} c_{i_k} \right|^2 \leqq \sum_{k, \kappa=1}^N \alpha_{i_k} \bar{\alpha}_{i_\kappa} x_{i_k}^* \bar{x}_{i_\kappa}^* K$$

for every finite collection of complex numbers $\alpha_{i_1}, \cdots, \alpha_{i_N}$, then there exists a function $f \in H(D)$ such that $x_i^*(f) = c_i$ for all $i \in I$. Furthermore, we have

$$(8) \qquad\qquad |x^*(f)|^2 \leqq |x^*|^2 K$$

for all $x^* \in H^*(D)$.

Proof. Let $\{D_m\}$ be an exhaustion of D by relatively compact subdomains. Each functional x_i^* may be represented by a complex measure with compact support in D. Let I_m be the subset of I for which the support of x_i^* is in D_m. As before, let $L_a^2(D_m)$ be the Hilbert space of square integrable analytic functions on D_m. Since

the topology of $L_a^2(D_m)$ implies uniform convergence on each compact subset of D_m, the representing measure for x_i^* may be used to extend x_i^* to $L_a^2(D_m)$ for each $i \in I_m$.

Define $T_m: L_a^2(D_m)^* \to L_a^2(D_m)$ by $(T_m x^*)(z) = \bar{x}_{(\zeta)}^*(K(z, \zeta)|_{D_m \times D_m})$. Then T_m is continuous, conjugate-linear, and

$$\| T_m \|^2 = \int_{D_m \times D_m} | K(z, \zeta) |^2 dm_z dm_\zeta .$$

It is even positive since point evaluation functionals form complete sets in $H^*(D)$ and $L_a^2(D_m)^*$. By Corollary 2.1 there exists a function $f_m \in L_a^2(D_m)$ such that $x_i^*(f_m) = c_i$ for each $i \in I_m$ and

(9) $$|x^*(f_m)|^2 \leqq |x^*|^2 K|_{D_m \times D_m} \quad \text{for all} \quad x^* \in L_a^2(D_m)^* .$$

In particular, for the point evaluation functionals we have

$$|f_m(z)|^2 \leqq K(z, z) \quad \text{for} \quad z \in D_m .$$

Therefore $\{f_m\}$ forms a normal family, and the limit f of any convergent subsequence interpolates: $x_i^*(f) = c_i$ for all $i \in I$. Furthermore, since (9) holds for each finite linear combination of point evaluations if m is sufficiently large, then (8) holds for such functionals. Such functionals are dense in $H^*(D)$; hences (8) holds for all $x^* \in H^*(D)$.

REMARKS. 1. In case $\{x_i^*\}_{i \in I}$ is a complete set of functionals, then (7) implies that K is a positive kernel. This occurred in FitzGerald's theorem; however, we do not need to assume that the set of functionals is complete.

2. If the kernel $K(z, \zeta)$ of Theorem 4.1 is already square integrable over $D \times D$, then the exhaustion is not needed in the proof. As a consequence, we may conclude that there exists an interpolating function belonging to $L_a^2(D)$. Unfortunately, the Bergman kernel does not have this property. However, we shall see in Theorem 5.1 that it also produces interpolating functions in $L_a^2(D)$.

3. In case $D = C^n$, Theorem 4.1 gives sufficient conditions for existence of interpolating entire functions of several variables.

4. In Theorem 4.1 the index set I may be uncountable. It therefore gives sufficient conditions for analytic continuation of functions defined on subsets of D.

5. Some obvious choices for the set of functionals are the

following:

EXAMPLE 4.1. If $\{z_i\}_{i \in I}$ is a set of points in $D \subset C^n$, define $x_i^*(g) = g(z_i)$. Then (7) takes the form

$$\left| \sum_{k=1}^{N} \alpha_{i_k} c_{i_k} \right|^2 \leqq \sum_{k,\kappa=1}^{N} \alpha_{i_k} \bar{\alpha}_{i_\kappa} K(z_{i_k}, z_{i_\kappa}) \ .$$

If, for example, there is a point $\hat{z} \in D$ such that each coordinate of $\hat{z}$ is an accumulation point of corresponding coordinates of the set $\{z_i\}_{i \in I}$, then the set $\{x_i^*\}_{i \in I}$ is complete and the interpolating function is unique.

EXAMPLE 4.2. If I is a set of multi-indices $i = (m_1, \cdots, m_n)$ and $z_0 \in D \subset C^n$ is fixed, define $x_i^*(g) = (\partial^i g / \partial z^i)(z_0)$. Then (7) takes the form

$$\left| \sum_{k=1}^{N} \alpha_{i_k} c_{i_k} \right|^2 \leqq \sum_{k,\kappa=1}^{N} \alpha_{i_k} \bar{\alpha}_{i_\kappa} \frac{\partial^{i_k} \partial^{i_\kappa}}{\partial z^{i_k} \partial \bar{\zeta}^{i_\kappa}} K(z, \zeta) \bigg|_{z=\zeta=z_0} \ .$$

If I contains all multi-indices, then the set $\{x_i^*\}_{i \in I}$ is complete and the interpolating function is again unique.

There are many other possible choices for functionals. For example, one can use point evaluations of the function and its derivatives at various points, their linear combinations, or integrals over compact sets.

The proof of Theorem 4.1 applies equally well to the space $h(D)$ of harmonic functions on a domain $D \subset R^n$, with the topology of locally uniform convergence. We simply state it:

THEOREM 4.1′. *Let D be a domain in R^n and $K(s, t)$ a continuous function on $D \times D$, that is harmonic in each variable. Assume that K is a positive kernel in the sense that $(x^*)^2 K = x_{(s)}^* (x_{(t)}^* K(s, t)) \geqq 0$ for every $x^* \in h^*(D)$. Suppose that $\{x_i^*\}_{i \in I}$ is a set of functionals in $h^*(D)$ and that $\{c_i\}_{i \in I}$ is a corresponding set of real numbers. If*

$$(10) \qquad \left| \sum_{k=1}^{N} \alpha_{i_k} c_{i_k} \right|^2 \leqq \sum_{k,\kappa=1}^{N} \alpha_{i_k} \alpha_{i_\kappa} x_{i_k}^* x_{i_\kappa}^* K$$

for every finite collection of real numbers $\alpha_{i_1}, \cdots, \alpha_{i_N}$, then there exists a function $u \in h(D)$ such that $x_i^*(u) = c_i$ for all $i \in I$. Furthermore, we have $|x^*(u)|^2 \leqq (x^*)^2 K$ for all $x^* \in h^*(D)$.

REMARKS. Just as in Example 4.1 we may choose point evaluations for the set $\{x_i^*\}_{i \in I}$ of functionals and then (10) assumes an analogous form. This set of functionals will be complete and u will

be unique if, for example, $D \subset \boldsymbol{R}^2$ and there are infinitely many points on some analytic Jordan curve in D whose interior is also in D.

If, as in Example 4.2, we specify certain partial derivatives at a fixed point $t_0 \in D$, then (10) will assume a corresponding form. Furthermore, u will be unique if we interpolate all its partial derivatives at t_0.

Another proof of Theorems 4.1 and 4.1' could be based on the fact that Montel spaces, such as $H(D)$ and $h(D)$, are reflexive. In this case one can apply Corollary 2.1 directly after showing that the conjugate-linear operator $Tx^* = \bar{x}_t^* K$ is continuous.

5. Reproducing kernels. Let X be a Hilbert space with a reproducing kernel. That is, X is a space of functions on a set D. In addition, there is a kernel function $K(z, \zeta)$ on $D \times D$ such that

(a) $\overline{K(z, \cdot)} \in X$ for each $z \in D$, and

(b) $x(z) = (x, \overline{K(z, \cdot)})$ for each $x \in X$ and $z \in D$.

Since X is a Hilbert space, the natural identification of X^* with X defines a positive, bounded, conjugate-linear operator T. It has the property that $x^*(Ty^*) = x^* \bar{y}^* K$. Thus Corollary 2.2 implies the following.

THEOREM 5.1 (*Duren-Williams*). *Let* X *be a Hilbert space with a reproducing kernel* K. *Suppose that* $\{x_i^*\}_{i \in I}$ *is a set of functionals in* X^* *and that* $\{c_i\}_{i \in I}$ *is a corresponding set of scalars. Then there exists an* $x \in X$ *such that*

(i) $x_i^*(x) = c_i$ *for all* $i \in I$,

(ii) $|x^*(x)|^2 \leq |x^*|^2 K$ *for all* $x^* \in X^*$, *and*

(iii) $\|x\| \leq 1$,

if and only if

$$\left| \sum_{k=1}^{N} \alpha_{i_k} c_{i_k} \right|^2 \leq \sum_{k, \kappa = 1}^{N} \alpha_{i_k} \bar{\alpha}_{i_\kappa} x_{i_k}^* \bar{x}_{i_\kappa}^* K$$

for every finite collection of scalars $\alpha_{i_1}, \cdots, \alpha_{i_N}$.

REMARKS. This theorem was proved by P. L. Duren and D. L. Williams in [1, Thm. 2]. They stated it for a family $\{x_i^*\}_{i \in I}$ of point evaluation functionals, but their argument carries over in the present form. They obtained it as an application of a very useful, general interpolation theorem for function spaces, which in turn is based on the proof of Theorem 2.1.

Since each positive kernel generates a proper functional inner product space for which it is the reproducing kernel, a proof of Theorems 4.1 and 4.1' could be based on Theorem 5.1. In this

framework one needs to identify points in the proper functional completion with analytic or harmonic functions when the kernel has these properties.

We note a few examples of interesting kernels:

EXAMPLE 5.1. Let D be a domain in C^n. A reproducing kernel K for $L_a^2(D)$ is called a *Bergman kernel*. For example, if $D = D^n$ is the unit polydisk in C^n, then the Bergman kernel is

$$K(z, \zeta) = \pi^{-n} \prod_{k=1}^{n} (1 - z^{(k)}\overline{\zeta^{(k)}})^{-2}$$

where $z = (z^{(1)}, \cdots, z^{(n)})$ and $\zeta = (\zeta^{(1)}, \cdots, \zeta^{(n)})$, and can be used in Theorem 5.1. In addition, $2\operatorname{Re} K(z, \zeta) - 1$ is the reproducing kernel for $L_{ph}^2(D^n)$, the space of square integrable pluriharmonic functions, to which Theorem 5.1 applies.

EXAMPLE 5.2. Let $H^2(D^n)$ be the Hardy space of analytic functions g in the polydisk D^n with the property that

$$\| g \|^2 = \limsup_{r \to 1} \int_0^{2\pi} \cdots \int_0^{2\pi} |f(re^{i\theta_1}, \cdots, re^{i\theta_n})|^2 d\theta_1 \cdots d\theta_n$$

is finite. Its reproducing kernel is

$$K(z, \zeta) = (2\pi)^{-n} \prod_{k=1}^{n} (1 - z^{(k)}\overline{\zeta^{(k)}})^{-1}$$

and is called the *Szegö kernel*. In addition, $2\operatorname{Re} K(z, \zeta) - 1$ is a reproducing kernel for the Hardy space $h_{ph}^2(D^n)$ of pluriharmonic functions. Theorem 5.1 applies in both cases.

One easily gives many additional examples. We have mentioned these to suggest the wide applicability of Theorem 5.1.

Since Corollary 2.2 applies already to reflexive Banach spaces, one may produce interpolating functions in the spaces L_a^p, L_{ph}^p, H^p, h_{ph}^p of the polydisk for $1 < p \leq 2$ for interpolating functionals in the corresponding conjugate spaces. In these cases the reproducing kernels of the examples above may be used.

For further information we refer the reader to Duren and Williams [1].

6. **Interpolation and continuation.** The following theorem is a generalization of a result of FitzGerald [2, Thm. 1.2], but the proof is quite different.

THEOREM 6.1. *Let D_1 be a domain in C^m and D_2 a domain in*

C^n.   For $\nu = 1, 2$ let $K_\nu(z, \zeta)$ be functions on $D_\nu \times D_\nu$ that are analytic in z and anti-analytic in ζ. Suppose that $\{x_i^*\}_{i \in I}$ and $\{y_j^*\}_{j \in J}$ are complete sets of functionals in $H^*(D_1)$ and $H^*(D_2)$, respectively, and that $\{c_{ij}\}_{i \in I, j \in J}$ is a set of complex numbers. If

$$(11) \qquad \left| \sum_{k, \kappa = 1}^{N_1, N_2} \alpha_{i_k} \beta_{j_\kappa} c_{i_k j_\kappa} \right|^2 \leqq \left[\sum_{k, \kappa = 1}^{N_1} \alpha_{i_k} \bar{\alpha}_{i_\kappa} x_{i_k}^* \bar{x}_{i_\kappa}^* K_1 \right]\!\left[\sum_{k, \kappa = 1}^{N_2} \beta_{j_k} \bar{\beta}_{j_\kappa} y_{j_k}^* \bar{y}_{j_\kappa}^* K_2 \right]$$

for all finite collections of complex numbers $\alpha_{i_1}, \cdots, \alpha_{i_{N_1}}$ and $\beta_{j_1}, \cdots, \beta_{j_{N_2}}$, then there exists a unique function $V \in H(D_1 \times D_2)$ such that $x_i^* y_j^* V = c_{ij}$ for every $i \in I$ and $j \in J$. Furthermore, we have

$$(12) \qquad |x^* y^* V|^2 \leqq [\,|x^*|^2 K_1\,][\,|y^*|^2 K_2\,]$$

for all $x^* \in H^*(D_1)$ and $y^* \in H^*(D_2)$.

  Proof. Choose $N_2 = 1$ in (11). Then we have

$$\left| \sum_{k=1}^{N_1} \alpha_{i_k} c_{i_k j} \right|^2 \leqq \left[\sum_{k, \kappa = 1}^{N_1} \alpha_{i_k} \bar{\alpha}_{i_\kappa} x_{i_k}^* \bar{x}_{i_\kappa}^* K_1 \right]\![\,|y_j^*|^2 K_2\,]$$

for all finite collections $\alpha_{i_1}, \cdots, \alpha_{i_{N_1}}$. By Theorem 4.1 there exists an $f_j \in H(D_1)$ such that $x_i^*(f_j) = c_{ij}$ for all $i \in I$ and also

$$|x^*(f_j)|^2 \leqq [\,|x^*|^2 K_1\,][\,|y_j^*|^2 K_2\,]$$

for every $x^* \in H^*(D_1)$.

  Let $C_i = \sum_{\kappa=1}^{N_2} \beta_{j_\kappa} c_{i j_\kappa}$ for a fixed collection $\beta_{j_1}, \cdots, \beta_{j_{N_2}}$ of complex numbers. Then (11) implies

$$\left| \sum_{k=1}^{N_1} \alpha_{i_k} C_{i_k} \right|^2 \leqq b \sum_{k, \kappa = 1}^{N_1} \alpha_{i_k} \bar{\alpha}_{i_\kappa} x_{i_k}^* \bar{x}_{i_\kappa}^* K_1$$

where $b = \sum_{k, \kappa=1}^{N_2} \beta_{j_k} \bar{\beta}_{j_\kappa} y_{j_k}^* \bar{y}_{j_\kappa}^* K_2$. By Theorem 4.1 there exists an $F \in H(D_1)$ such that $x_i^*(F) = C_i$ for all $i \in I$ and, moreover,

$$(13) \qquad |x^*(F)|^2 \leqq b\,|x^*|^2 K_1 \quad \text{for every} \quad x^* \in H^*(D_1)\,.$$

Since $\sum_{k=1}^{N_2} \beta_{j_k} f_{j_k}$ also interpolates the set $\{C_i\}_{i \in I}$ and the family $\{x_i^*\}_{i \in I}$ is complete, we conclude that $F = \sum_{\kappa=1}^{N_2} \beta_{j_\kappa} f_{j_\kappa}$. Therefore, taking x^* in (13) to be point evaluation at z, we have

$$(14) \qquad \left| \sum_{\kappa=1}^{N_2} \beta_{j_\kappa} f_{j_\kappa}(z) \right|^2 \leqq K_1(z, z) \sum_{k, \kappa = 1}^{N_2} \beta_{j_k} \bar{\beta}_{j_\kappa} y_{j_k}^* \bar{y}_{j_\kappa}^* K_2\,.$$

  Holding z fixed, inequality (14) is valid for every finite collection $\beta_{j_1}, \cdots, \beta_{j_{N_2}}$. Again by Theorem 4.1 there exists a function $G_z \in H(D_2)$ such that $y_j^*(G_z) = f_j(z)$ for every $j \in J$ and also

$$(15) \qquad |y^*(G_z)|^2 \leqq K_1(z, z)\,|y^*|^2 K_2$$

for all $y^* \in H^*(D_2)$. Since the family $\{y_j^*\}_{j \in J}$ is complete, we may find finite linear combinations $y_\nu^* = \sum_{\mu=1}^{M_\nu} \gamma_{j_{\mu\nu}} y_{j_{\mu\nu}}^*$ that converge to the point evaluation functional at a fixed point ζ_0. In this case (15) implies a uniform bound on compact subsets of D_1 for the analytic functions $y_\nu^*(G_z) = \sum_{\mu=1}^{M_\nu} \gamma_{j_{\mu\nu}} f_{j_{\mu\nu}}(z)$ of z. Consequently, their limit $G_z(\zeta_0)$ is analytic in z.

Define $V(z, \zeta) = G_z(\zeta)$. Then V is analytic in each variable and so belongs to $H(D_1 \times D_2)$. Furthermore, $x_i^* y_j^* V = x_i^*(f_j) = c_{ij}$ for all $i \in I$ and $j \in J$. Finally, (12) follows from (11) since the families $\{x_i^*\}_{i \in I}$ and $\{y_j^*\}_{j \in J}$ are complete. V is unique for the same reason.

REMARKS. FitzGerald's theorem [2, Thm. 1.2] is the special case $m = n = 1$ where the sets of functionals are sequences of point evaluations, both having an interior accumulation point. As in the examples following Theorem 4.1, there are many other possible choices for functionals. Furthermore, the index sets I and J do not need to be denumerable.

The following corollaries are fairly direct consequences of Theorem 6.1. We omit their proofs since they are analogous to those of FitzGerald [2, Theorems 1.3 and 1.4].

COROLLARY 6.1. *Let D be a domain in C^n and $K(z, \zeta)$ a function on $D \times D$, that is analytic in z and anti-analytic in ζ. Suppose that $\{x_i^*\}_{i \in I}$ is a complete set of functionals in $H^*(D)$ and that $\{c_{ij}\}_{i,j \in I}$ is a symmetric ($c_{ij} = c_{ji}$) set of complex numbers. If*

$$\left| \sum_{k,\kappa=1}^{N} \alpha_{i_k} \alpha_{i_\kappa} c_{i_k i_\kappa} \right| \leqq \sum_{k,\kappa=1}^{N} \alpha_{i_k} \bar{\alpha}_{i_\kappa} x_{i_k}^* \bar{x}_{i_\kappa}^* K$$

for every finite collection of complex numbers $\alpha_{i_1}, \cdots, \alpha_{i_N}$, then there exists a unique function $V \in H(D \times D)$ such that $x_i^* x_j^* V = c_{ij}$ for all $i, j \in I$. Furthermore, V is symmetric ($V(z, \zeta) = V(\zeta, z)$) and $|(x^*)^2 V| \leqq |x^*|^2 K$ for every $x^* \in H^*(D)$.

COROLLARY 6.2. *Let D_1 and D_2 be domains in C^n with nonempty intersection and $K(z, \zeta)$ a function defined on $(D_1 \times D_1) \cup (D_2 \times D_2)$, that is analytic in z and anti-analytic in ζ. Suppose that $\{x_i^*\}_{i \in I}$ is a complete set of functionals in $H^*(D_1 \cap D_2)$. If*

$$\sum_{k,\kappa=1}^{N} \alpha_{i_k} \bar{\alpha}_{i_\kappa} x_{i_k}^* \bar{x}_{i_\kappa}^* K \geqq 0$$

for every finite collection of complex numbers $\alpha_{i_1}, \cdots, \alpha_{i_N}$, then $K(z, \zeta)$ can be continued to a positive kernel on $(D_1 \cup D_2) \times (D_1 \cup D_2)$ that is analytic in z and anti-analytic in ζ.

By using Theorem 4.1' in place of Theorem 4.1 one obtains the following parallel results for harmonic interpolation and continuation.

THEOREM 6.1'. *Let D_1 be a domain in $\mathbf{R}^m$ and D_2 a domain in $\mathbf{R}^n$. For $\nu = 1, 2$ let $K_\nu(s, t)$ be continuous functions on $D_\nu \times D_\nu$, that are harmonic in each variable. Suppose that $\{x_i^*\}_{i \in I}$ and $\{y_j^*\}_{j \in J}$ are complete sets of functionals in $h^*(D_1)$ and $h^*(D_2)$, respectively, and that $\{c_{ij}\}_{i \in I, j \in J}$ is a set of real numbers. If*

$$\left| \sum_{k,\kappa=1}^{N_1, N_2} \alpha_{i_k} \beta_{j_\kappa} c_{i_k j_\kappa} \right|^2 \leqq \left[\sum_{k,\kappa=1}^{N_1} \alpha_{i_k} \alpha_{i_\kappa} x_{i_k}^* x_{i_\kappa}^* K_1 \right]\!\!\left[\sum_{k,\kappa=1}^{N_2} \beta_{j_k} \beta_{j_\kappa} y_{j_k}^* y_{j_\kappa}^* K_2 \right]$$

for all finite collections of real numbers $\alpha_{i_1}, \cdots, \alpha_{i_{N_1}}$ and $\beta_{j_1}, \cdots, \beta_{j_{N_2}}$, then there exists a unique continuous function V on $D_1 \times D_2$ that is harmonic in each variable and satisfies $x_i^* y_j^* V = c_{ij}$ for every $i \in I$ and $j \in J$. Furthermore, we have $|x^* y^* V|^2 \leqq [(x^*)^2 K_1][(y^*)^2 K_2]$ for all $x^* \in h^*(D_1)$ and $y^* \in h^*(D_2)$.

COROLLARY 6.1'. *Let D be a domain in $\mathbf{R}^n$ and $K(s, t)$ a continuous function on $D \times D$ that is harmonic in each variable. Suppose that $\{x_i^*\}_{i \in I}$ is a complete set of functionals in $h^*(D)$ and that $\{c_{ij}\}_{i,j \in I}$ is a symmetric set of real numbers. If*

$$\left| \sum_{k,\kappa=1}^{N} \alpha_{i_k} \alpha_{i_\kappa} c_{i_k i_\kappa} \right| \leqq \sum_{k,\kappa=1}^{N} \alpha_{i_k} \alpha_{i_\kappa} x_{i_k}^* x_{i_\kappa}^* K$$

for every finite collection of real numbers $\alpha_{i_1}, \cdots, \alpha_{i_N}$, then there exists a unique continuous function V on $D \times D$ that is harmonic in each variable and satisfies $x_i^* x_j^* V = c_{ij}$ for all $i, j \in I$. Furthermore, V is symmetric and $|(x^*)^2 V| \leqq (x^*)^2 K$ for every $x^* \in h^*(D)$.

COROLLARY 6.2'. *Let D_1 and D_2 be domains in $\mathbf{R}^n$ with nonempty intersection and $K(s,t)$ a symmetric continuous function on $(D_1 \times D_1) \cup (D_2 \times D_2)$ that is harmonic in each variable. Suppose that $\{x_i^*\}_{i \in I}$ is a complete set of functionals in $h^*(D_1 \cap D_2)$. If*

$$\sum_{k,\kappa=1}^{N} \alpha_{i_k} \alpha_{i_\kappa} x_{i_k}^* x_{i_\kappa}^* K \geqq 0$$

for every finite collection of real numbers $\alpha_{i_1}, \cdots, \alpha_{i_N}$, then $K(s, t)$ can be continued to a positive kernel on $(D_1 \cup D_2) \times (D_1 \cup D_2)$ that is continuous and is harmonic in each variable.

7. An application. This section contains an extension of Pick-Nevanlinna interpolation and some remarks.

THEOREM 7.1. *Let* $\{z_i\}_{i \in I}$ *be a set of uniqueness for* $H(\boldsymbol{D}^n)$ *and* $\{w_i\}_{i \in I}$ *a set of complex numbers. For* z *and* ζ *in the polydisk* $\boldsymbol{D}^n$ *assume that a kernel* K *has the form*

$$K(z, \zeta) = 1 + \sum a_{m_1 m_2 \cdots m_n} (z^{(1)} \overline{\zeta^{(1)}})^{m_1} (z^{(2)} \overline{\zeta^{(2)}})^{m_2} \cdots (z^{(n)} \overline{\zeta^{(n)}})^{m_n}$$

where all $a_{m_1 m_2 \cdots m_n} \geqq 0$. *Then there exists an analytic function* $f \colon \boldsymbol{D}^n \to \bar{\boldsymbol{D}}$ *such that*

(i) $f(z_i) = w_i$ *for all* $i \in I$ *and*

(ii) $|x^*|^2([1 - f(z)\overline{f(\zeta)}]K(z, \zeta)) \geqq 0$ *for all* $x^* \in H^*(\boldsymbol{D}^n)$

if and only if

(16) $$\sum_{k, \kappa = 1}^{N} \alpha_{i_k} \bar{\alpha}_{i_\kappa} (1 - w_{i_k} \bar{w}_{i_\kappa}) K(z_{i_k}, z_{i_\kappa}) \geqq 0$$

for all finite collections of complex numbers $\alpha_{i_1}, \cdots, \alpha_{i_N}$.

Proof. Inequality (16) is an immediate consequence of (i) and (ii) for the functional x^* defined by $x^*(g) = \sum_{k=1}^{N} \alpha_{i_k} g(z_{i_k})$.

Conversely, if (16) holds, then

$$\sum_{k, \kappa = 1}^{N} \alpha_{i_k} \bar{\alpha}_{i_\kappa} K(z_{i_k}, z_{i_\kappa}) \geqq \sum_{k, \kappa = 1}^{N} \alpha_{i_k} \bar{\alpha}_{i_\kappa} w_{i_k} \bar{w}_{i_\kappa} K(z_{i_k}, z_{i_\kappa})$$

$$= \left| \sum_{k=1}^{N} \alpha_{i_k} w_{i_k} \right|^2 + \sum a_{m_1 \cdots m_n} \left| \sum_{k=1}^{N} \alpha_{i_k} w_{i_k} (z_{i_k}^{(1)})^{m_1} \cdots (z_{i_k}^{(n)})^{m_n} \right|^2$$

$$\geqq \left| \sum_{k=1}^{N} \alpha_{i_k} w_{i_k} \right|^2$$

for all $\alpha_{i_1}, \cdots, \alpha_{i_N} \in \boldsymbol{C}$. Since the family $\{x_i^*\}_{i \in I}$ of point evaluations at z_i is complete, K is a positive kernel and by Theorem 4.1 there exists a (unique) function $f \in H(\boldsymbol{D}^n)$ that satisfies (i). Now (ii) follows from (16) and the completeness of $\{x_i^*\}_{i \in I}$. Finally, to verify that $f(\boldsymbol{D}^n) \subset \bar{\boldsymbol{D}}$ one inserts point evaluation functionals into (ii) and uses the positivity of K.

REMARKS. Note that the Bergman and Szegö kernels of Examples 5.1 and 5.2 satisfy the hypothesis of Theorem 7.1.

One easily obtains analogous versions of Theorem 7.1 for harmonic or pluriharmonic functions.

For our final remark, let us restrict our attention to the unit disk $\boldsymbol{D}$ and the kernels

$$K(z, \zeta) = \frac{1}{(1 - z\bar{\zeta})^p}$$

where p is a positive integer. In this case, if $\{z_n\}_{n=1}^{\infty}$ is a sequence

of distinct points in D, with an accumulation point in D, and if

$$(17) \qquad \sum_{m,n=1}^{N} \alpha_m \bar{\alpha}_n \frac{1 - w_m \bar{w}_n}{(1 - z_m \bar{z}_n)^p} \geqq 0$$

for all finite collections $\alpha_1, \cdots, \alpha_N \in C$, then Theorem 7.1 implies that there is an analytic function f in D that is bounded by 1 and that satisfies $f(z_n) = w_n$ for every $n = 1, 2, 3, \cdots$. In (17) the special case $p = 1$ is, of course, the classical Pick-Nevanlinna conditions.

It is interesting to note that the conditions (17) are necessary for all $p = 1, 2, 3, \cdots$. For $p = 1$ one usually bases a proof on the Herglotz representation for analytic functions with positive real part. This proof extends easily to (17). Alternatively, one may deduce the cases $p = 2, 3, \cdots$ from $p = 1$ by writing

$$\frac{1 - w_m \bar{w}_n}{(1 - z_m \bar{z}_n)^p} = \frac{1 - w_m \bar{w}_n}{1 - z_m \bar{z}_n} \frac{1}{(1 - z_m \bar{z}_n)^{p-1}} \ .$$

Each factor generates a nonnegative hermitian form, and the Schur product theorem ensures that their product does also. Consequently, the conditions (17) are necessary and sufficient for every $p = 1, 2, 3, \cdots$.

REFERENCES

1. P. L. Duren and D. L. Williams, *Interpolation problems in function spaces*, J. Functional Analysis, **9** (1972), 75-86.
2. C. H. FitzGerald, *Quadratic inequalities and analytic continuation*, J. Analyse Math., **31** (1977), 19-47.
3. C. H. FitzGerald and R. A. Horn, *On quadratic and bilinear forms in function theory*, Proc. London Math. Soc., to appear.
4. G. Köthe, *Topological Vector Spaces* II, Grundlehren der math. Wiss. 237, Springer-Verlag, 1979.

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