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CRYSTALLISATION MOVES

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A crystallisation of a closed PL-manifold is an edge-coloured graph, which represents it via a contracted triangulation. Any two crystallisations of the same manifold are proved to be joined by a finite sequence of moves, two alternative sets of which are defined. A further move for dimension 3 is introduced.

1. Introduction and notation. Throughout this work balls, spheres, manifolds and maps are piecewise-linear (in the sense of [9] and [17]).

An n-dimensional ball-complex K will be said to be a *pseudocomplex*[1] if each h-ball, considered with all its faces, is abstractly isomorphic with an h-simplex. Further, K will be said to be a *contracted n-complex* if the number of its vertices is $n+1$. We shall also call *simplex* each element of a pseudocomplex.

By a *pseudodissection* (resp. *contracted triangulation*) of a polyhedron $\boldsymbol{P}$ we mean a pair (K, f), where K is a pseudocomplex (resp. a contracted complex) and $f: |K| \to \boldsymbol{P}$ is a homeomorphism. About pseudodissections, compare also [11].

A theorem of Pezzana states that every closed, connected, n-dimensional manifold admits a contracted triangulation (for proofs see [14] and [6] and the sketch contained in this work, § 4). The theorem can be extended to manifolds with connected boundary and to more general spaces (see [2]). A method for constructing pseudocomplexes from a set of disjoint n-simplexes is shown in [6] and [7].

Note that even in a pseudodissected manifold, stars and links of simplexes are not necessarily balls and spheres. But there exists a minimal set of severings on them, which makes them balls and spheres. The so modified stars (resp. links) are called *disjoined stars* (resp. *disjoined links*); compare the quoted papers for detailed definitions and proofs.

Closely related is the notion of $(n+1)$-coloured graph. Let $\Gamma = (\boldsymbol{X}, \boldsymbol{E})$ be a finite nonoriented multigraph without loops, $\mathscr{C}$ a set (called *colour-set*), $\gamma: \boldsymbol{E} \to \mathscr{C}$ a map (called a *coloration*). Such a pair (Γ, γ) is defined to be an *h-coloured graph with boundary* if:

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[1] In all references pseudocomplexes are symbolised with a tild on top of the letter, which we omit throughout this work.

(1) Card $\mathscr{C} = h$,

(2) $\gamma(e_1) \neq \gamma(e_2)$ for each pair of adjacent edges e_1, e_2.

Observe that properties (1) and (2) imply that each vertex of Γ has degree $\leqq h$. A vertex of degree $< h$ will be called a *boundary-vertex.* If all vertices have degree exactly h, then (Γ, γ) will be simply called an h-coloured graph. If $\mathscr{B} \subseteqq \mathscr{C}$, we define $\Gamma_{\mathscr{B}}$ to be the partial graph of Γ determined by the edge set $\gamma^{-1}(\mathscr{B})$. For each $c \in \mathscr{C}$, $\hat{c}$ will be the set $\mathscr{C} - \{c\}$.

In order to evidence the possibly existing boundary-vertices, we will draw as many "one-ended" edges coming out of them, as to make their degree equal to h. Figure 1 shows a 4-coloured graph with just one (boundary-) vertex.

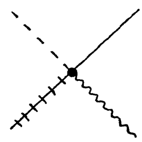


FIGURE 1

For every $(n + 1)$-coloured graph (Γ, γ) (either with or without boundary), we can construct a finite n-dimensional pseudocomplex $K(\Gamma)$, whose n-simplexes are in bijection with the vertices of Γ, and two have a common $(n - 1)$-face iff the corresponding vertices are adjacent. Actually Γ turns out to be the 1-skeleton of the ball-complex dual to $K(\Gamma)$. For the general construction and properties see [4] and [7]. We just recall that $K(\Gamma)$ is a contracted complex iff (Γ, γ) also satisfies:

(3) for each colour $c \in \mathscr{C}$, the partial graph $\Gamma_{\hat{c}}$ is connected.

We will call *contracted* a graph (Γ, γ) satisfying 3). Note that, if (Γ, γ) is contracted, to each colour c there corresponds a vertex v_c of $K(\Gamma)$. Otherwise, $K(\Gamma)$ has as many vertices corresponding to c as the connected components of $\Gamma_{\hat{c}}$.

On the other hand, if K is a contracted n-complex triangulating a closed manifold M, then there exists an $(n + 1)$-coloured graph (Γ, γ), such that $K(\Gamma) = K$.[2]

Pezzana's theorem then enables us to represent every closed, connected, n-dimensional manifold M by a (contracted) $(n + 1)$-coloured graph related with a contracted triangulation.

Such a graph is called a *crystallisation* of M. Further information on crystallisations is available in [8], [15], [5].

Let now (Γ, γ) and (Γ', γ') be two $(n + 1)$-coloured graphs (either with or without boundary), and $\mathscr{C}$, $\mathscr{C}'$ the respective colour-sets.

[2] The theorem holds under much wider hypotheses: see [7], Prop. 8.

DEFINITION. Given a bijection $\psi: \mathscr{C} \to \mathscr{C}'$, a graph isomorphism $\Psi: \Gamma \to \Gamma'$ will be called a *colour-isomorphism compatible with* ψ if

$$\gamma' \circ \Psi = \psi \circ \gamma .$$

(Γ, γ) and (Γ', γ') will then be said to be *isomorphic*.

Since, by the construction itself, colour-isomorphism of crystallisations is equivalent to abstract isomorphism of the related ball-complexes, colour-isomorphism also implies homeomorphism of their spaces. The converse is not true, as nonisomorphic contracted triangulations — hence also nonisomorphic edge-coloured graphs — can be given for the same polyhedron.

The aim of this work is to solve the equivalence problem for crystallisations of closed manifolds of arbitrary dimension. This is the analogous for crystallisations, to what has already been obtained for Heegaard diagrams and splittings of 3-manifolds (in [18], [16], and [1]), for framed links (in [12] and [3]) and, recently, for generalised Heegaard diagrams of 4-manifolds (in [13]).

A possible development of the results of this work could be the determination of topological invariants of manifolds, as graph — theoretical invariants of crystallisations, which are left unaltered by the moves (e.g., the number of vertices of a 2-manifold crystallisation is such an invariant, which classifies 2-manifolds up to orientability character).

We also are exploring the problem of detecting S^3 and $S^1 \times S^2$, at the light of the results presented here.

2. Connected sum of h-coloured graphs. Let (Γ_1, γ_1), (Γ_2, γ_2) be h-coloured graphs, ψ a bijection between their colour-sets $\mathscr{C}_1$ and $\mathscr{C}_2$. Let then Λ_1, Λ_2 be subgraphs of Γ_1 and Γ_2 respectively, such that:

(1) there exists a colour-isomorphism $\Psi: \Lambda_1 \to \Lambda_2$ compatible with ψ;

(2) $j_i(|K(\Lambda_i)|)$ is an $(h-1)$-ball $(i = 1, 2)$, j_i being the inclusion of $|K(\Lambda_i)|$ into $|K(\Gamma_i)|$.

DEFINITION. We define *polyhedral connected sum of* Γ_1 *and* Γ_2, *with respect to* Λ_1, Λ_2, ψ *and* Ψ, the h-coloured graph $\Gamma_1 \#_{\Lambda_1\Lambda_2} \Gamma_2$ so constructed:

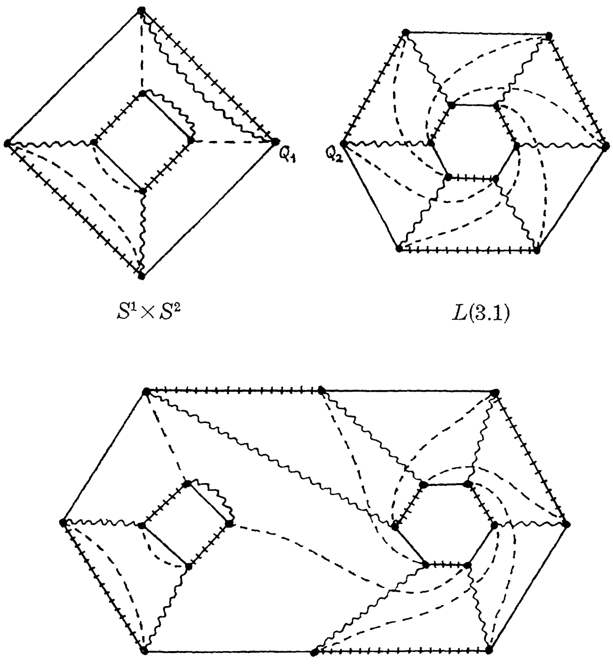
(a) consider the graphs Γ_1', Γ_2' complementary of Λ_1, Λ_2 in Γ_1, Γ_2 respectively;

(b) join two boundary-vertices P_1 of Γ_1' and P_2 of Γ_2' by an s-coloured edge ($s \in \mathscr{C}_1$) iff P_1 is joined with a vertex Q_1 of Λ_1 in Γ_1 by an s-coloured edge, and P_2 with a vertex Q_2 of Λ_2 in Γ_2 by a $\psi(s)$-coloured edge, so that $\Psi(Q_1) = Q_2$.

(c) colour the edges of the resulting graph by the coloration induced by γ_1 and $\psi^{-1} \circ \gamma_2$.

DEFINITION 3. If (with the above notation) the graphs Λ_1 and Λ_2 reduce to the two vertices Q_1 and Q_2 of Γ_1 and Γ_2 respectively, then $\Gamma_1 \#_{Q_1 Q_2} \Gamma_2$ will be simply called *connected sum* of Γ_1 and Γ_2.

Figure 2 illustrates the latter operation on crystallisations of $L(3, 1)$ and $S^1 \times S^2$, yielding a crystallisation of their connected sum.



$S^1 \times S^2$ $L(3.1)$

$S \times S^2 \# L(3, 1)$

FIGURE 2

3. The moves.

Move of type I (Adding or cancelling of a nondegenerate dipole).

Let (Γ, γ) be an $(n+1)$-coloured graph. Assume Γ admits a partial subgraph Θ, composed of two vertices, X and Y say, joined by h edges, where $1 \leqq h \leqq n$, coloured by $c_0, c_1, \cdots, c_{h-1} \in \mathscr{C}$. Let also $\mathscr{B} = \mathscr{C} - \{c_0, \cdots, c_{h-1}\}$ be the set of the remaining colours, and $C_{\mathscr{B}}(X)$, $C_{\mathscr{B}}(Y)$ the connected components of $\Gamma_{\mathscr{B}}$, containing X and Y respectively.

DEFINITION 4. Such a partial subgraph Θ is said to be an n-*dimensional dipole of type* h, if $C_{\mathscr{B}}(X) \neq C_{\mathscr{B}}(Y)$.

A dipole of type 1 or n will be called *degenerate*.

Let now (Γ, γ), (Γ', γ') be two $(n+1)$-coloured graphs and Θ a dipole in Γ. We can assume, w.l.o.g., that $\mathscr{C}$ is the colour set for both graphs.

DEFINITION 5. We will say that *Γ' is obtained from Γ by cancelling the dipole Θ*, if:

(a) $\Gamma'_{\mathscr{B}}$ is obtained from $\Gamma_{\mathscr{B}}$, by substituting, to $C_{\mathscr{B}}(X) \cup C_{\mathscr{B}}(Y)$, the connected sum $C_{\mathscr{B}}(X) \#_{XY} C_{\mathscr{B}}(Y)$, accomplished with respect to the identity on $\mathscr{B}$;

(b) two vertices A, B of Γ' are joined by a c_i-coloured edge $(h \leqq i \leqq n)$ iff the corresponding vertices of Γ so are.

We will also say that *Γ is obtained from Γ' by adding the dipole Θ*.

Observe that if Θ is a non-degenerate dipole, then (Γ, γ) is contracted iff (Γ', γ') is. This is not true if Θ is degenerate.

The *move of type* I on a contracted $(n+1)$-coloured graph is defined to be the adding or cancelling of a nondegenerate dipole.

Figure 3a shows the adding of a dipole of type 2 to a crystallisation of $\boldsymbol{RP}$.[3]

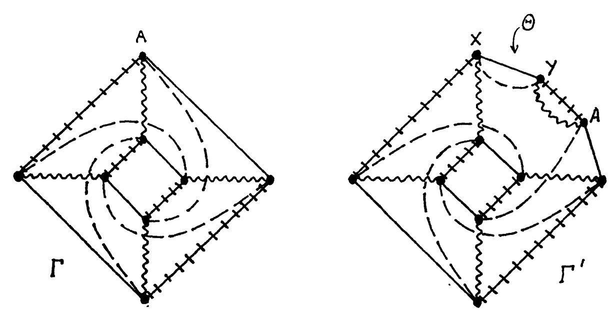


FIGURE 3a

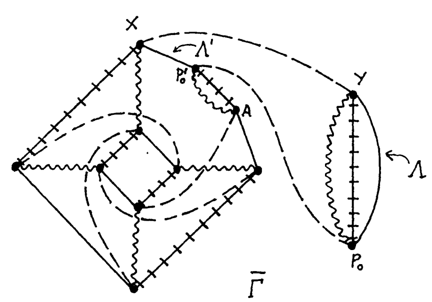


FIGURE 3b

Move of type II (simple cut-and-glue).

Let (Γ, γ), (Γ', γ') be two contracted[3] $(n+1)$-coloured graphs.

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[3] The move can also be defined for noncontracted graphs.

As before, we can assume that $\mathscr{C}$ is the common colour set.

DEFINITION 6. We will say that *Γ' is obtained from Γ by a simple cut-and-glue move*, or *move of type* II (and conversely) if there exist a noncontracted $(n+1)$-coloured graph $(\bar{\Gamma}, \bar{\gamma})$ and a colour $c \in \mathscr{C}$, such that:

(a) $\bar{\Gamma}$ is obtained from Γ (resp. from Γ') by adding a degenerate dipole Θ (resp. Θ') of type 1.

(b) the edges of both dipoles are c-coloured.

In other words, $\bar{\Gamma}_{\hat{c}}$ must have two connected components, Γ_1 and Γ_2 say, and there must exist two pairs of vertices (P_1, P_2), (Q_1, Q_2) joined by a c-coloured edge, with $P_i, Q_i \in \Gamma_i$ $(i = 1, 2)$, such that $\Gamma_{\hat{c}} = \Gamma_1 \#_{P_1 P_2} \Gamma_2$, $\Gamma'_{\hat{c}} = \Gamma_1 \#_{Q_1 Q_2} \Gamma_2$, both connected sums being accomplished with respect to the identity on $\mathscr{C} - \{c\}$. Moreover, two vertices $A, B \neq P_1, P_2$ (resp. $A, B \neq Q_1, Q_2$) of $\bar{\Gamma}$ are joined by a c-coloured edge iff the corresponding vertices of Γ (resp. Γ') so are.

The pass from Γ (resp. Γ') to $\bar{\Gamma}$ will be called also a *simple cut*; the inverse pass a *simple glueing*, Figure 4 shows two crystalli-

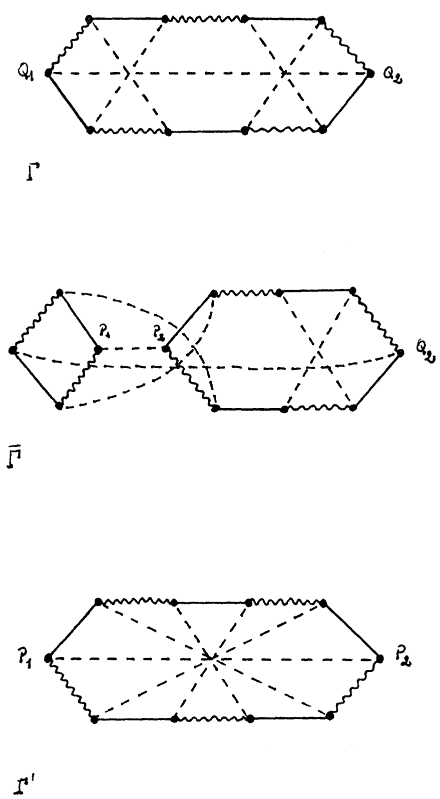

FIGURE 4

sations of the 2-torus (compare [8]) which are obtained from each other by a simple cut-and-glue.

DEFINITION 7. Two contracted $(n+1)$-coloured graphs (Γ, γ) and (Γ', γ') will be said to be (I-II)-*equivalent* if there exists a finite sequence $\{(\Xi_j, \rho_j)\}_{j \in \Delta_s}$[4] of contracted $(n+1)$-coloured graphs, such that

$$(\Xi_0, \rho_0) = (\Gamma, \gamma)\,; \qquad (\Xi_s, \rho_s) = (\Gamma', \gamma')\,,$$

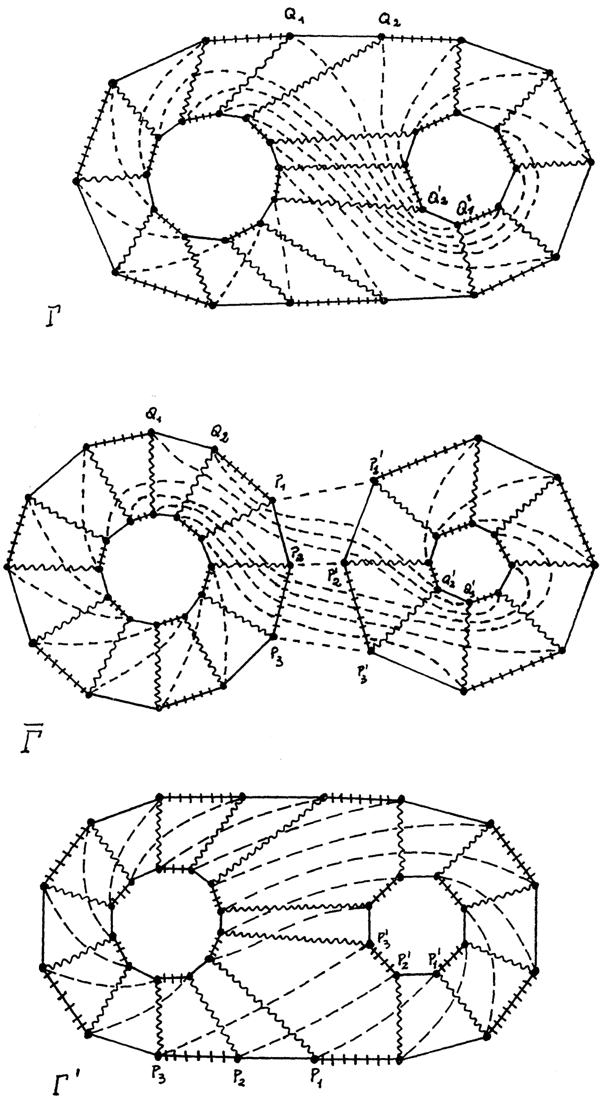

FIGURE 5

[4] We denote by Δ_s, s a positive integer, the set $\{i \in \mathbf{Z} \mid 0 \leqq i \leqq s\}$.

and Ξ_i is obtained from Ξ_{i-1} by a move of type I (adding or cancelling of a nondegenerate dipole) or of type II (simple cut-and-glue).

Move of type A (polyhedral cut-and-glue).

DEFINITION 8. With the same notations as in Definitions 2 and 6 we will say that *Γ' is obtained from Γ by a polyhedral cut-and-glue move* or *move of type* A (and conversely) if there exists a non-contracted $(n+1)$-coloured graph $(\bar{\Gamma}, \bar{\gamma})$ and a colour $c \in \mathscr{C}$, such that:

(a) $\bar{\Gamma}_{\hat{c}}$ has two connected components, Γ_1 and Γ_2 say, and $\Gamma_{\hat{c}} = \Gamma_1 \#_{\Lambda_1\Lambda_2} \Gamma_2$, $\Gamma'_{\hat{c}} = \Gamma_1 \#_{\bar{\Lambda}_1\bar{\Lambda}_2} \Gamma_2$, where both connected sums are accomplished with respect to two isomorphisms $\varphi: \Lambda_1 \to \Lambda_2$, $\bar{\varphi}: \bar{\Lambda}_1 \to \bar{\Lambda}_2$ compatible with the identity on $\mathscr{C} - \{c\}$;

(c) every vertex A (resp. B) of Λ_1 (resp. of $\bar{\Lambda}_1$) is joined with $\varphi(A)$ (resp. $\varphi(B)$) by a c-coloured edge;

(c) two vertices P, Q not in $\Lambda_1 \cup \Lambda_2$ (resp. $\bar{\Lambda}_1 \cup \bar{\Lambda}_2$) are joined by a c-coloured edge iff the corresponding vertices of Γ (resp. Γ') so are;

Definitions of *polyhedral cut*, *polyhedral glueing* and *A-equivalence* are analogous to the ones of simple cut, simple glueing and (I, II)-equivalence respectively. A polyhedral cut-and-glue binding two crystallisations of the Dodecahedral-spherical space is shown in Figure 5, where $\bar{\Lambda}_1$ (resp. $\bar{\Lambda}_2$) is the edge Q_1Q_2 (resp. $Q'_1Q'_2$) and Λ_1 (resp. Λ_2) is the subgraph spanned by the vertices P_1, P_2, P_3 (resp. P'_1, P'_2, P'_3).

4. Main theorem.

THEOREM. *Let (Γ, γ) and (Γ', γ') be crystallisations of two n-dimensional, closed, connected manifolds M and M'. The following three sentences are equivalent:*

(i) *M and M' are homeomorphic;*

(ii) *(Γ, γ) and (Γ', γ') are* (I, II)*-equivalent;*

(iii) *(Γ, γ) and (Γ', γ') are A-equivalent.*

Proof. (ii) $\Rightarrow$ (iii). The proof is trivial for $n \leq 2$. It is also evident that, for $n > 2$, a simple cut-and-glue is a particular case of a polyhedral cut-and-glue. Finally, cancelling of a nondegenerate dipole Θ corresponds to a polyhedral cut-and-glue accomplished by (1) isolating a vertex of Θ by means of a simple cut and (2) doing a polyhedral glueing. Figure 3b illustrates, as an example, how the move of Figure 3a can be obtained by such an operation, c being the "dashed" colour, with $\Gamma'_{\hat c} = \Gamma_1 \#_{P_1 P'_0} \Gamma_2$, $\Gamma'_{\hat c} = \Gamma_1 \#_{\Lambda\Lambda'} \Gamma_2$. □

(iii) ⟹ (i). Assume that (Γ, γ), (Γ', γ') are obtainable from each other by a polyhedral cut-and-glue. Set $K = K(\Gamma)$, $K' = K(\Gamma')$, and let v_c be the vertex of K (resp. v'_c the vertex of K'), corresponding to the colour c.

$\Gamma_{\hat c}$ (resp. $\Gamma'_{\hat c}$) represents the pseudocomplex $\mathrm{Lkd}(v_c, K)$ (resp. $\mathrm{Lkd}(v'_c, K')$), disjoined link of v_c in K (resp. of v'_c in K'), which is an $(n-1)$-sphere. M (resp. M') can be obtained from the n-ball $|\mathrm{Std}(v_c, K)|$ (resp. $|\mathrm{Std}(v'_c, K')|$), space of the disjoined star of v_c in K (resp. of v'_c in K'), by pairwise identification of the $(n-1)$-simplexes of its boundary $|\mathrm{Lkd}(v_c, K)|$ (resp. $|\mathrm{Lkd}(v'_c, K')|$) and of all their faces. These identifications are represented in Γ (resp. in Γ') by the c-coloured edges.

In order to show that M and M' are homeomorphic, it suffices to prove that both are homeomorphic to the manifold $\bar M = |K(\bar\Gamma)|$. This comes from explicit construction of $K(\bar\Gamma)$, if one starts from the pair of disjoint balls, which are bounded by $|K(\Gamma_1)|$ and $|K(\Gamma_2)|$ respectively, where $\bar\Gamma_{\hat c} = \Gamma_1 \cup \Gamma_2$. □

REMARK. Notice that the hypotheses on the graphs Γ and Γ' can be weakened, the implication (iii) ⟹ (i) still remaining true. In particular, if Γ (resp. Γ') is not contracted, or, more precisely, if $\Gamma_{\hat c}$ (resp. $\Gamma'_{\hat c}$) is nonconnected, the implication still holds, and the proof is the same, in its general lines. If, further, Γ (resp. Γ') does not represent a manifold, the move can still be applied, provided that $\Gamma_{\hat c}$ (resp. $\Gamma'_{\hat c}$) represents an $(n-1)$-sphere, if connected, or a disjoint union of $(n-1)$-spheres, one for each connected component. This condition implies that all vertices associated with the components of $\Gamma_{\hat c}$ (resp. $\Gamma'_{\hat c}$) have neighborhoods in $|K(\Gamma)|$ (resp. $|K(\Gamma')|$), which are homeomorphic with closed n-balls.

The proof of the last implication ((i) ⟹ (ii)) needs some notation and some lemmas.

1. If Θ is a dipole in (Γ, γ), we will say that the pseudocomplex $\mathscr{D} = K(\Theta)$ is itself a *dipole* in $K(\Gamma)$ (*of type h* if such is Θ). $\mathscr{D}$ obviously satisfies the two following properties:

(a) $\mathscr{D}$ consists of two n-simplexes X^n, Y^n of K, which have h common $(n-1)$-faces;

(b) if A^{n-h} is the intersection of such common faces, and X^{h-1} and Y^{h-1} are the $(h-1)$-faces of X^n and Y^n respectively, opposite to A^{n-h}, then $X^{h-1} \neq Y^{h-1}$.

We will further call a subcomplex $\mathscr{D}$ of a pseudocomplex K a *dipole* if $\mathscr{D}$ satisfies (a) and (b), even if K is not associated with any $(n+1)$-coloured graph. Two pseudocomplexes K and K', both triangulating closed manifolds, will be called *equal up to dipoles* if one can be obtained from the other by adding and/or cancelling a finite number of dipoles.

On the same line, we will say that K' *is obtained from* K *by a simple cut-and-glue move* if K' is obtained from K by adding a dipole of type 1 and cancelling one in the resulting pseudocomplex.

Note that in dimension 2 this is nothing but the elementary move used for normalisation, from which the idea itself of cut-and-glue is taken. Actually there is a classical (Dehn's) normal form (see [19], §12) which consists of a contracted triangulation.

LEMMA 1. *If K and L, both triangulating closed manifolds, are equal up to dipoles, then* $|K| \cong |L|$.

Proof. It follows from $|\mathscr{D}|$ being a ball. □

Notice that, if $\mathscr{D}$ is a dipole of type h in K, which consists of two n-simplexes X^n an Y^n, and $\mathscr{D}'$ is a further dipole of type $n+1-h$, consisting of the n-simplexes Y^n and Z^n, then $\mathscr{D} \cup \mathscr{D}'$ provides a contracted triangulation of an n-simplex A^n. Elimination of $\mathscr{D}$ or of $\mathscr{D}'$ from K is then equivalent to the substitution of X^n, Y^n, Z^n by A^n.

2. A vertex w of an n-pseudocomplex K will be called a *cone-vertex*, if it belongs to all n-simplexes of K, or, equivalently, if $\mathrm{St}(w, K) = K$.

Let now M be an n-dimensional, closed, connected manifold. We indicate by $\mathfrak{C}^q$, with $q \in \Delta_{n+1}$, the set of all pseudocomplexes L having q cone-vertices, and such that $|L| \cong M$. Obviously, every pseudocomplex triangulating M belongs to at least one of such sets, and $\mathfrak{C}^{n+1}$ is the set of the *contracted* complexes triangulating M (in fact, if $L \in \mathfrak{C}^{n+1}$, it cannot have any other vertices than its $n+1$ cone-vertices).

We will call *cone-algorithm* any process $\mathscr{A}$, which associates to $L \in \mathfrak{C}^i$ the pseudocomplex $L' = \mathscr{A}(L) \in \mathfrak{C}^{i+1}$ in the following way:

(a) Call $w_0, \cdots, w_{i-1}$ the cone-vertices of L. One can form (in many ways) a tree (ε, ζ), whose "vertex"-set ε consists of all disjoined stars of the $(n-i)$-simplexes of L not containing any of $w_0, \cdots, w_{i-1}$, and where an "edge" of ζ is an $(n-1)$-simplex which is a common face, in K, of the two stars at its "end-points". Choose such a tree, and orient the ζ-edges so, as to make (ε, ζ) an out-tree (with arbitrary source) (see [10], p. 201).

(b) Consider the disjoined stars of ε, and attach any two of them which are joined by a ζ-edge (an $(n-1)$-simplex), by re-identifying the two copies of this $(n-1)$-simplex. The pseudocomplex D so built is an n-ball, and all of its vertices lie on the boundary $\Sigma = \partial D$, excepted for the case $i = n$; in such a case, anyway, none of the inner vertices is a cone-vertex. Identification of suitable pairs of $(n-1)$-simplexes of Σ and of all their faces gets L out of D. The $(n-1)$-simplexes of Σ which have to be identified, in order to get L, will be said to be *twin*.

(c) Re-triangulate $|D|$, leaving the triangulation of the boundary unaltered, by making the join $w_i * \Sigma$ from an arbitrarily chosen, inner point w_i.

(d) Build the complex $L' = \mathscr{A}(L)$ by re-identification of the twin simplexes of Σ.

Now, $|\mathscr{A}(L)| \cong |L| \cong M$. Moreover $L' = \mathscr{A}(L)$ has a new cone vertex w_i, hence it belongs to $\mathfrak{C}^{i+1}$.

Iterated application of $n+1-i$ cone-algorithms, starting from a pseudocomplex in $\mathfrak{C}^i$ (in particular, $n+1$ cone-algorithms starting from a simplicial complex) provides a construction of a contracted triangulation of M. As every (PL)-manifold admits a simplicial triangulation, this process sketches a proof of Pezzana's theorem. For further details, and for the formalisation, see [14] and [6].

To every pseudocomplex $L \in \mathfrak{C}^i$, $i \in \Delta_n$, there corresponds a set $\mathfrak{A}(L) \subseteqq \mathfrak{C}^{i+1}$ consisting of all pseudocomplexes obtainable from L by a cone-algorithm.

3. Let now $\mathscr{A}$ be a cone-algorithm on L, (ε, ζ) and D respectively the out-tree and the n-ball of its construction. Build a different cone-algorithm $\mathscr{A}'$ as follows.

(a′) Consider a disjoined star $S \in \varepsilon$ and call (ε_S, ζ_S) the sub-out-tree of (ε, ζ) determined by S and by all of its "descendants"; further call $(\varepsilon'_S, \zeta'_S)$ the sub-out-tree determined by $\varepsilon'_S = \varepsilon - \varepsilon_S$.

(b′) Consider the ball D_1, bounded by Σ_1, built on the out-tree (ε_S, ζ_S), and the ball D_2, bounded by Σ_2, built on the out-tree $(\varepsilon'_S, \zeta'_S)$.

(c′) Identify a pair of twin $(n-1)$-simplexes $B_1^{n-1} \in \Sigma_1$ and $B_2^{n-1} \in \Sigma_2$, so getting a ball D' with boundary Σ'.

(d′) Proceed as in (c) and (d) to get $L'' = \mathscr{A}'(L)$ from D' and Σ' (noting that Σ' contains two new $(n-1)$-simplexes to be considered twin).

Observe that if $L' = \mathscr{A}(L)$ and $L'' = \mathscr{A}'(L)$, then L'' is obtained from L' by a simple cut-and-glue move.

LEMMA 2. *Let $K \in \mathfrak{C}^i$, $i \in \Delta_n$. If $K', K'' \in \mathfrak{A}(K)$, then K''* (resp. *K'*) *is obtained from K'* (resp. *from K''*) *by a finite sequence of cut-and-glue moves.*

Proof. By induction on the cardinality of $\zeta_2 - \zeta_1$, where (ε_1, ζ_1), (ε_2, ζ_2) are respectively the out-trees of the construction of K' and K''. □

4. Now consider $L \in \mathfrak{C}^i$, $i \in \Delta_n$, and let $K \in \mathfrak{A}(L)$ be the pseudocomplex obtained from the ball $w_i * \Sigma$ by identification of twin simplexes of Σ. Let then $p: w_i * \Sigma \to K$ be the canonical projection.

Let $\mathscr{D}$ be a dipole of type $h(1 \leqq h \leqq n-1)$ in Σ, and Σ' the pseudocomplex obtained from Σ by cancelling $\mathscr{D}$.

LEMMA 3. *If the two $(n-1)$-simplexes forming $\mathscr{D}$ are twin, then $p(w_i * \mathscr{D})$ is a dipole of type $h+1$ in K. Moreover, the pseudocomplex K' obtained from K by cancelling $p(w_i * \mathscr{D})$ is $p(w_i * \Sigma')$.*

Proof. Straightforward. □

Observe that K' belongs to $\mathfrak{C}^{i+1}$ too.

LEMMA 4a. *If $K \in \mathfrak{C}^{i+1}$, $i \in \Delta_n$, then there exist $L \in \mathfrak{C}^i$ and $K' \in \mathfrak{A}(L)$, such that K and K' are equal up to dipoles of type $h \geqq 2$. In particular, if $i = n$, K, L, K' coincide.*

Proof. Recall that if $K \in \mathfrak{C}^{i+1}$, then $K \in \mathfrak{C}^j$ for each $j \leqq i$. Then consider $L = K$ as belonging to $\mathfrak{C}^i$ by the cone-vertices $w_1, \cdots, w_i$. L has a further cone-vertex w_{i+1}, whose disjoined star, $\operatorname{Std} w_{i+1} = w_{i+1} * \Sigma$ say, is strongly connected. Consider the graph whose "vertices" are the disjoined stars of $(n-i)$-simplexes of L not containing $w_1, \cdots, w_i$, and whose "edges" are the $(n-1)$-simplexes joining them in $\operatorname{Std} w_{i+1}$. This graph is connected, and one can get a tree (ε, ζ) from it by neglecting a set of edges. The ball D' built on (ε, ζ) has a boundary Σ' equal to Σ up to as many dipoles as the neglected edges. Note that, for $i = n$, the graph reduces to the single vertex corresponding to $\operatorname{Std} w_{i+1}$ itself, thus no edge is erased, and no dipole arises.

Now, application of Lemma 3 proves the statement. □

LEMMA 4b. *If K, $L \in \mathfrak{C}^0$, then there exist $K' \in \mathfrak{A}(K)$, $L' \in \mathfrak{A}(L)$ which are equal up to dipoles of type $h \geqq 2$.*

Proof. K and L are in stellar equivalence, as one can take them to their barycentric subdivisions, which are simplicial complexes, by elementary stellar subdivision.

It is then sufficient to consider the case of L being obtained from K by an elementary stellar subdivision. But in this case a construction similar to the one of Lemma 4a proves the statement. □

LEMMA 4c. *If $K, L \in \mathfrak{C}^i$, $i \in \Delta_n$, are equal up to dipoles of type $h \geqq 2$, then there exist $K' \in \mathfrak{A}(K)$ and $L' \in \mathfrak{A}(L)$, which are equal up to dipoles of type $h \geqq 2$ if $i < n$, up to dipoles of type h, $2 \leqq h \leqq n-1$ (i.e., nondegenerate), if $i = n$.*

Proof. Let us assume that $K, L \in \mathfrak{C}^i$, $i \in \Delta_n$, and that K is obtained from L by cancelling a dipole $\mathscr{D}$ of type $h \geqq 2$ formed by two n-simplexes X^n and Y^n meeting at h $(n-1)$-faces $F_1^{n-1}, \cdots, F_h^{n-1}$.

If $\mathscr{D}$ is *degenerate*, i.e., if $h = n$, set $w = \bigcap_{j=1}^n F_j^{n-1}$; w is a vertex, which is not a cone-vertex for L. Cancelling $\mathscr{D}$ from L can be thought of as follows: dig the inside of $|\mathscr{D}|$ out of $|L|$, so setting a manifold $\bar{M}$ with a boundary, which is the sphere formed by the two $(n-1)$-simplexes X^{n-1} and Y^{n-1} opposite to w in X^n and Y^n respectively; then identify X^{n-1} and Y^{n-1} to produce $|K|$.

As $\bar{M}$ is strongly connected, one can build an out-tree (ε, ζ) for K, so that no ζ-edge represents the $(n-1)$-simplex Z^{n-1} arisen from the identification of X^{n-1} with Y^{n-1}. Call D the ball, bounded by Σ, built on (ε, ζ), and $K' \in \mathfrak{A}(K)$ the out-coming pseudocomplex.

We must now distinguish the three cases $i = n$, $0 < i < n$, $i = 0$.

If $i = n$, the only noncone-vertex belonging to $\mathscr{D}$ is w, and its disjoined star is $\mathscr{D}$ itself. Hence, one can build an out-tree (ε', ζ') on L by adding to (ε, ζ) a new edge ending in a new vertex which represents $\mathscr{D}$. Call D' the ball, bounded by Σ', relative to (ε', ζ'), and $L' \in \mathfrak{A}(L)$ the pseudocomplex obtained from it. Now, Σ' is isomorphic with Σ by an isomorphism, which is compatible with the projections $D \to K'$, $D' \to L'$. Therefore K' and L' are equal.

If $0 < i < n$, the only $(n-i)$-simplex of $\mathscr{D}$ not containing any cone-vertices is a join $w * F^{n-i-1}$, where F^{n-i-1} is that face of $\partial\mathscr{D}$, common to X^n and Y^n, which contains no cone-vertices. The disjoined star of $w * F^{n-i-1}$ in L is obtained from $\mathscr{D}$ by doubling the $(n-1)$-faces of X^n and Y^n not containing $w * F^{n-i-1}$. The ball D' built as for $i = n$ has boundary Σ', equal to Σ up to dipoles (as many as the doubled faces). Lemma 3 then implies that K' and L' are equal up to dipoles of type $h \geqq 2$.

Finally, if $i = 0$, an out-tree (ε', ζ') can be built by adding to (ε, ζ) a new vertex corresponding to X^n and Y^n attached together by a common $(n-1)$-face, and a new edge corresponding to a non-common $(n-1)$-face. The result comes again from Lemma 3.

Let now $\mathscr{D}$ be *nondegenerate*, i.e., of type $h < n$, and set $F^{n-h} =$

$\bigcap_{j=1}^{h} F_j^{n-1}$; let then $X_1^{n-1}, \cdots, X_{n-h+1}^{n-1}$, $Y_1^{n-1}, \cdots, Y_{n-h+1}^{n-1}$ be the noncommon $(n-1)$-faces of X^n and Y^n, where faces with equal lower index are opposite to the same vertex; set $Q = \bigcup_k X_k^{n-1}$, $R = \bigcup_k Y_k^{n-1}$.

Elimination of the inside of $|\mathscr{D}|$ from $|L|$ yields the manifold $\bar{M}$ with the sphere $Q \cup R$ as boundary. Identification of each X_k^{n-1} with Y_k^{n-1} produces $|K|$.

Now, let (ε, ζ) be an out-tree for K, such that the ball D built on it has all simplexes of R (hence also of Q) on its boundary Σ. These simplexes will not in general form an $(n-1)$-ball isomorphic with Q; but, if (ε, ζ) is carefully built, elimination of nondegenerate dipoles will yield a sphere $\bar{\Sigma}$ in which this happens. Let $v_1, \cdots, v_i$ be the cone-vertices of L, and let F_X^{n-i}, F_Y^{n-i} be the faces of X^n and Y^n respectively, not containing $v_1, \cdots, v_i$. If $F_X^{n-i} = F_Y^{n-i} = F^{n-h}$, then its disjoined star is $\mathscr{D}$ itself. The ball D' obtained by attaching $\mathscr{D}$ to D by one of the simplexes of R, has a sphere Σ' as boundary, such that $\bar{\Sigma}$ can be obtained from Σ' by cancelling nondegenerate dipoles. This leads to the result.

Analogous constructions prove the statement in the cases $F_X^{n-i} = F_Y^{n-i} \neq F^{n-h}$ and $F_X^{n-i} \neq F_Y^{n-i}$. □

LEMMA 4d. *Let $L, L' \in \mathfrak{C}^i$, $i \in \Delta_n - \{0\}$, be obtained from each other by a simple cut-and-glue move, and let $H \in \mathfrak{A}(L)$, $H' \in \mathfrak{A}(L')$. Then H and H' are obtained from each other by a finite sequence of simple cut-and-glue moves and of addings and cancellings of nondegenerate dipoles.*

Proof. First, assume $n > 2$. Let $\bar{L} \in \mathfrak{C}^{i-1}$ be the pseudocomplex obtained from L (resp. from L') by adding the dipole of type 1 $\mathscr{D}$ (resp. $\mathscr{D}'$), formed by two n-simplexes X^n, Y^n meeting at the $(n-1)$-face F^{n-1}(X'^n, Y'^n, F'^{n-1} resp.). Let $v_1, \cdots, v_i$ be the cone-vertices of L (and of L'); $v_1, \cdots, v_{i-1}$, say, will be cone-vertices also for $\bar{L}$. Call w, w' the vertices common to $\mathscr{D}$ and $\mathscr{D}'$, which are opposite to F^{n-1} in $\mathscr{D}$, to F'^{n-1} in $\mathscr{D}'$.

If $\mathscr{D} \cap \mathscr{D}'$ is not contained in the $(n-3)$-skeleton of $\bar{L}$, adding of a suitable nondegenerate dipole yields a pseudocomplex $L^\#$, in which this happens. Call $K^\#$ and $K'^\#$ the pseudocomplexes obtained by adding the same dipole in L and L' respectively.

The pseudocomplex $\bar{\bar{L}}$, closure of $L^\# - (\mathscr{D} \cup \mathscr{D}')$ is then strongly connected. In order to show this, let A_0^n, A_t^n be two n-simplexes of $\bar{\bar{L}}$, joined in $L^\#$ by a sequence of n-simplexes $A_0^n, \cdots, A_t^n$, where A_{j-1}^n, A_j^n meet at an $(n-1)$-face A_j^{n-1}, for each $j \in \Delta_t$. Let A_k^n be the first simplex of the sequence which belongs to $\mathscr{D} \cup \mathscr{D}'$. A_{k-1}^n and A_{k+1}^n meet at a face $A^{n-2} = A_k^{n-1} \cap A_{k+1}^{n-1}$ of A_k^n. Now, the star of A^{n-2} is cyclic, so we can substitute A_k^n by a sequence $B_0^n = A_{k-1}^n$,

$B_1^n, \cdots, B_{k+1}^n$, where all B's are adjacent and belong, B_r^n possibly excluded, to $\bar{\bar{L}}$. A finite number of such substitutions shows the strong connectedness.

Therefore one can build an out-tree $(\bar{\varepsilon}, \bar{\zeta})$ for $L^\#$ so that

(1) each $\bar{\varepsilon}$-vertex represents the disjoined star of an $(n-i)$-simplex of L not containing $v_1, \cdots, v_{i-1}, w, w'$;

(2) if $i > 1$, no $\bar{\zeta}$-edge represents faces of $\mathscr{D} \cup \mathscr{D}'$;

(3) if $i = 1$, the $\bar{\varepsilon}$-vertices representing the dipoles themselves are terminal in the out-tree.

Call $\bar{D}$ the ball obtained as in point (b) of the construction of a cone-algorithm relative to $(\bar{\varepsilon}, \bar{\zeta})$. Set $W = F^{n-1} \cap \partial\bar{D}$, $W' = F'^{n-1} \cap \partial\bar{D}$. Accomplish point (c) by means of a new vertex v_{i+1}; set $\bar{W} = v_{i+1} * W$, $\bar{W}' = v_{i+1} * W'$. Carry out the identifications of point (d) to get a pseudocomplex $\bar{K}$.

Let (ε, ζ), (ε', ζ') be the analogous out-trees for $K^\#$, $K'^\#$ respectively (isomorphic with $(\bar{\varepsilon}, \bar{\zeta})$ if $i > 1$, only homeomorphic if $i = 1$); call K, K' respectively the out-coming pseudocomplexes. Let S, S' be the disjoined stars of w, w', respectively, in $\bar{K}$.

Now, the ball obtained from S and S' by re-identifying the pairs of copies of the $(n-1)$-simplexes of $\bar{W}$, has boundary isomorphic with the disjoined link of v_i in K; re-triangulation of the ball as a join from v_i then gets the disjoined star of v_i in K. Similarly, one gets the disjoined star of v_i in K' by using $\bar{W}'$. In $\bar{K}$, each of these operations is equivalent to: a simple glueing (corresponding to the re-identification of the first $(n-1)$-simplex), and cancelling of $n-2$ dipoles if $i = 1$ ($n - i$ if $i > 1$), which, by Lemma 3, are nondegenerate.

Resuming, the passage from K to K' consists of: adding of nondegenerate dipoles, one simple cut-and-glue, and cancelling of nondegenerate dipoles. Now, apply Lemma 4c to L and $L^\#$ (to L' and $L'^\#$), then Lemma 2 to the out-coming pseudocomplexes, to H, and to K (to H', and to K' resp.) to get the final result.

In dimension 2, the analogous of $\bar{\bar{L}}$ may be non strongly connected, but then the cut-and-glue can be substituted by adding and cancelling of a finite sequence of (obviously degenerate) dipoles (i.e., by a finite sequence of cut-and-glues) in which any pair of consecutive dipoles does not disconnect. □

Proof of (i) $\Rightarrow$ (ii). The statement will be proved as last step of an inductive argument.

Let $K_1, L_1 \in \mathfrak{G}^1$. By Lemma 4a, there exist $K_0, L_0 \in \mathfrak{G}^0$, $K_1' \in \mathfrak{A}(K_0)$, $L_1' \in \mathfrak{A}(L_0)$, such that K_1 and K_1' (L_1 and L_1' respectively) are equal up to dipoles of type $h \geqq 2$. On the other hand, Lemma 4b assures that there exist $K_1'' \in \mathfrak{A}(K_0)$, $L_1'' \in \mathfrak{A}(L_0)$, which are equal up to dipoles of type $h \geqq 2$. Now, by Lemma 2, K_1' and K_1'' (L_1' and L_1'' respectively) are obtained from each other by a finite sequence of cut-and-glue moves; therefore, K_1 and L_1 are obtained from each other by a finite sequence of cut-and-glues and addings and/or cancellings of dipoles of type $h \geqq 2$.

Now, let $K_i, L_i \in \mathfrak{C}^i$, $2 \leqq i \leqq n$. Application of Lemma 4a yields pseudocomplexes $K_{i-1}, L_{i-1} \in \mathfrak{C}^{i-1}$, $K_i', L_i' \in \mathfrak{C}^i$ analogous to the previous K_0, L_0, K_1', L_1' and with the same properties. The inductive hypothesis that K_{i-1}, L_{i-1} are obtained from each other by a finite sequence of cut-and-glues and addings and/or cancellings of dipoles of type $h \geqq 2$, together with Lemmas 2, 4c and 4d, assure again that K_i and L_i are obtained from each other in the same way.

Finally, the same argument applies for $K, L \in \mathfrak{C}^{n+1}$ with the following stronger inductive thesis, due to the particular form which Lemmas 4a and 4c assume for $\mathfrak{C}^{n+1}$: K and L are obtained from each other by a finite sequence of cut-and-glue moves and addings and/or cancellings of *nondegenerate* dipoles. This means that the corresponding crystallisations are (I, II)-equivalent. □

5. Dimension 3: generalised dipoles. We now describe a further move for dimension 3, which, however redundant, often turns useful. In the natural correspondence between crystallisations of 3-manifolds and Heegaard diagrams, a "genus" can be assigned to such crystallisations. This move permits to lower the genus even in cases when a (standard) dipole of type 2 is not present. Actually, it is the closest analogous, in crystallisation theory, of the stabilisation move of Reidemeister and Singer for Heegaard diagrams.

Let (Γ, γ) be a 4-coloured graph, $\mathscr{C} = \{c_i\}_{i \in \Delta_3}$ being its colour set. Assume that for two colours (c_0 and c_1 say) there are a connected component C of $\Gamma_{\{c_0,c_1\}}$ and a connected component C' of $\Gamma_{\{c_2,c_3\}}$ with only one common vertex. Let $\{x_0, x_1, \cdots, x_m\}$, $\{x_0, y_1, \cdots, y_n\}$ be the sets of vertices of C and C' respectively (see Figure 6a for $m = 3$, $n = 5$).

DEFINITION 9. The subgraph Θ of Γ determined by C and C' will be called an *(m, n)-dipole*.

Assume x_1, x_m, y_1, y_n to be the vertices joined with x_0 by edges of colours c_0, c_1, c_2, c_3 respectively. Let (Γ', γ') be a further 4-coloured graph with $\mathscr{C}$ as colour set.

DEFINITION 10. We shall say that *Γ' is obtained from Γ by cancelling the (m, n)-dipole* if:

(1) Γ' is obtained from Γ by substituting to Θ the product

(see [10], p. 21) Ξ of the subgraphs $C - \{x_0\}$ and $C' - \{x_0\}$ (see Figure 6b);

(2) for all $i, i' \in \Delta_m - \{0\}$, $j, j' \in \Delta_n - \{0\}$, an edge joining the vertex (x_i, y_j) with the vertex $(x_i, y_{j'})$ (resp. $(x_{i'}, y_j)$) in Ξ is coloured like the edge joining y_j and $y_{j'}$ (resp. x_i and $x_{i'}$);

(3) for all $i \in \Delta_m - \{0\}$, $j \in \Delta_n - \{0\}$, if a vertex z of $\Gamma - \Theta$ is joined to $\begin{matrix} y_j \\ x_i \end{matrix}$ by a $\begin{matrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{matrix}$-coloured edge in Γ, then v is joined to $\begin{matrix} (x_1, y_j) \\ (x_m, y_j) \\ (x_i, y_1) \\ (x_i, y_n) \end{matrix}$ by a $\begin{matrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{matrix}$-coloured edge in Γ'.

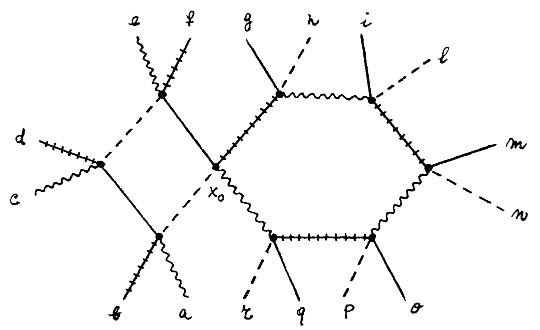

FIGURE 6a

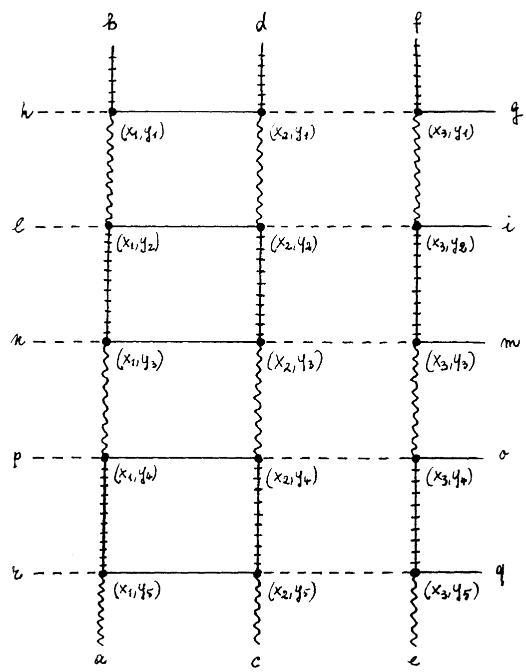

FIGURE 6b

Conversely, we will say that *Γ is obtained from Γ' by adding the (m, n)-dipole Θ.*

PROPOSITION. *Let (Γ, γ), (Γ', γ') be crystallisations of two closed, connected, 3-dimensional manifolds M, M' respectively. If Γ' is obtained from Γ by cancelling an (m, n)-dipole, then M and M' are homeomorphic.*

Sketch of proof. Cancelling of an (m, n)-dipole, for $m, n > 1$, can be accomplished by a sequence of $(n-1)/2$ (or $(m-1)/2$) cut-and-glue moves and cancelling of a dipole of type 2. For $m = 1$ or/and $n = 1$, it is just cancelling of a dipole of type 2. □

Note that the n-simplexes corresponding to the vertices of an (m, n)-dipole form a pseudocomplex, the inside of whose space is an open ball; cancelling the (m, n)-dipole results in re-triangulating this ball.

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