PACIFIC JOURNAL OF MATHEMATICS
Vol. 100, No. 1, 1982

MAXIMAL GROUPS IN SANDWICH SEMIGROUPS OF BINARY RELATIONS

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A sandwich semigroup is given as follows. Let R be an arbitrary but fixed binary relation on a finite set X. For relations A and B on X we say $(a, b) \in A * B$ (the product of A and B) if there are c and d in X such that $(a, c) \in A$, $(c, d) \in R$ and $(d, b) \in B$. This semigroup is denoted $B_X(R)$. In this paper we study maximal groups in $B_X(R)$ for various classes of R.

Sandwich semigroups of binary relations were introduced in [2]. These semigroups arise naturally in automata theory, and their role in automata theory is studied in [3]. Montague and Plemmons [5] have shown that given a finite group G there is some set X such that G is a maximal group in B_X, the usual semigroup of binary relations. We show there are classes of R for which this result holds and others for which it does not hold.

If R is a relation and E is a nonzero idempotent in $B_X(R)$, then we write $G_E(R)$ for the maximal group determined by E and call E an R-idempotent. In § 1 we give a class of relations for which $G_E(R)$ is trivial for any relation R in this class and any R-idempotent E. In § 2 we produce a class of relations for which the Montague-Plemmons result holds. That is, any finite group G arises as a maximal group for some X and some relation R in this class. Finally, in § 3 we show there is a class of relations for which some but not all finite groups arise.

Throughout we use Boolean matrix representation for relations. That is, if R is a relation over X where $|X| = n$, then R is represented by an $n \times n$ matrix where the (i, j) entry is a 1 if (x_i, x_j) is in R and 0 otherwise. These matrices are multiplied using Boolean arithmetic.

This paper is part of a Ph. D. thesis prepared under the direction of C. J. Maxson whom I wish to thank for his guidance and many helpful suggestions.

1. $B_X(R)$ containing only trivial groups.

Let Γ be the collection of (nonzero) matrices with the property that all nonzero columns are the same. For R in Γ it is easy to see that if the (i, j) entry of R is zero then either row i or column j of R is zero. The following theorem characterizes R-idempotents for any R in Γ and shows that $G_E(R)$ is trivial for any R in Γ and any R-idempotent E.

THEOREM 1. *Let R be in Γ. Then*

(i) *A is an R-idempotent if and only if all nonzero rows of A are the same and for some i and j such that the (i, j) entry of R is nonzero we have the (j, i) entry of A is nonzero.*

(ii) *If E is an R-idempotent, then $G_E(R)$ is trivial.*

Proof. Throughout the proof let a_{ij} (r_{ij}) denote the (i, j) entry of the matrix $A(R)$.

(i) Assume A is an R-idempotent. AR has zero columns where R does and since all nonzero columns of R are alike, all nonzero columns of AR are alike. Let

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

denote the nonzero columns of AR. Writing out the product ARA we see that for each i such that $b_i = 1$ we have a nonzero row of A and each nonzero row is identical.

Assume for each k and m such that $r_{km} = 1$ we have $a_{mk} = 0$. Clearly, if column j of R is zero, then column j of AR is zero. We show if column j of R is nonzero, then row j of A is zero. These two statements imply $(AR)A = 0$, a contradiction. Let column j of R be nonzero and denote by b_{ji} the (j, i) entry of AR. Then for any i

$$b_{ji} = \sum_{k=1}^{n} a_{jk}r_{ki} = \begin{cases} \sum_{k=1}^{n} a_{jk}r_{kj} & \text{if column } i \text{ of } R \text{ is nonzero} \\ & \text{(hence } r_{ki} = r_{kj}) \\ 0 & \text{otherwise} \end{cases}$$
$$= 0 \quad \text{in either case by the assumption.}$$

Thus row j of AR is zero which implies row j of $(AR)A = A$ is zero.

Conversely, assume $r_{ij} = 1$ and $a_{ji} = 1$. If row k of A is nonzero, then $a_{ki} = 1$. From $a_{ki} = r_{ij} = a_{ji} = 1$ we have the (k, i) entry of ARA is 1 and so row k of ARA is nonzero. Since $a_{ki} = 1$, row k of AR is row i of R and so the (k, j) entry of AR is nonzero. Furthermore, since $a_{ji} = 1$ we have row k of ARA is row j of A. But all rows of A are the same so row k of ARA is row k of A. If row k of A is zero, then row k of ARA is zero. Hence we have $ARA = A$ and A is an R-idempotent.

(ii) Let E be an R-idempotent and A be in $G_E(R)$. Throughout the remainder of the proof we use the following:

e_{ij} denotes the (i, j) entry of E,
b_{ij} denotes the (i, j) entry of AR,
c_{ij} denotes the (i, j) entry of ARE.
We show $a_{ij} = e_{ij}$ for any i and j.

Let $e_{ij} = 0$. Then, by the remark preceding the theorem, either row i or column j of E is zero. If row i is zero, then row i of $ERA = A$ is zero and so $a_{ij} = 0$. If column j is zero, then column j of $ARE = A$ is zero and so $a_{ij} = 0$.

Let $e_{ij} = 1$. We show $a_{ij} = 1$. Assume not, that is assume $a_{ij} = 0$. We first show row i and column j of A are zero. We have

$$c_{ij} = \sum_{k=1}^{n} b_{ik} e_{kj} .$$

Since all nonzero columns of E are alike, then for any nonzero columns n and j of E it follows that $c_{ij} = c_{im}$. But $ARE = A$ implies $c_{ij} = a_{ij} = 0$ and so row i of A is zero. Similarly column j of A is zero.

We now show $A = 0$, a contradiction. If row k of E is zero, then $ERA = A$ implies row k of A is zero. If row k of E is nonzero, then $e_{kj} = 1$ since $e_{ij} = 1$. By the above we know column j of A is zero, so $a_{kj} = 0$. Thus we have $e_{kj} = 1$ and $a_{kj} = 0$. Using the above arguments, this implies row k of A is zero.

2. **$B_X(R)$ containing all finite groups.** Let Γ be any class of matrices such that for every positive integer n the matrix

$$\begin{pmatrix} I_n & A \\ B & C \end{pmatrix}$$

is in Γ where I_n is the $n \times n$ identity matrix, A is an arbitrary $n \times k$ matrix, B is an arbitrary $k \times n$ matrix and C is an arbitrary $k \times k$ matrix.

THEOREM 2. *If G is a finite group, then G is a maximal group in $B_X(R)$ for some nonidentity matrix R in Γ and some X.*

Proof. From Montague and Plemmons [5] we know there is an X' such that G is isomorphic to $G_{E'}(I)$ where E' is an idempotent in $B_{X'}(I)$ (I is the identity relation). Let X' have n elements and

$$R = \begin{pmatrix} I_n & A \\ B & C \end{pmatrix}$$

where R is $k \times k$ with k greater than n and A, B and C are arbitrary. The matrix E where

$$E = \begin{pmatrix} E' & 0 \\ 0 & 0 \end{pmatrix}$$

is an R-idempotent. Let A be in $G_E(R)$ where

$$A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}.$$

Then, $A * E = A = E * A$ gives $Q = R = S = 0$ and $PE' = E'P = P$. Let B be the R-inverse of A in $G_E(R)$. Then

$$B = \begin{pmatrix} P' & 0 \\ 0 & 0 \end{pmatrix}$$

and $B * A = E = A * B$ give $PP' = E = P'P$ and so P is in $G_{E'}$. Thus the map θ from $G_{E'}(I)$ to $G_E(R)$ given by

$$\theta(P) = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$$

is an isomorphism.

We remark here that the R and X of the theorem are not unique. In fact G is in $B_X(R)$ for all X containing at least n elements. Also, if R is as in the theorem and $R' = PRQ$ where P and Q are invertible, then the map θ from $B_X(R)$ onto $B_X(R')$ given by $\theta(A) = QAP$ is an isomorphism.

The following theorem shows the symmetric groups arise in $B_X(R)$ where R is a permutation.

THEOREM 3. *Let R be a permutation in $B_X(I)$ for some arbitrary but fixed X where X has n elements. Then R', the inverse of R in $B_X(I)$, is an R-idempotent and $G_{R'}(R)$ is isomorphic to S_n, the symmetric group on n elements.*

Proof. It is clear that R' is an R-idempotent, and for all A in $B_X(R)$ we have $A * R' = R' * A = A$. It remains to be shown that only permutations have an R-inverse with respect to R'. If A is a permutation, then AR and RA are permutations and $(R'A'R')(RA) = (AR)(R'A'R') = R'$ where A' is the I-inverse of A. Thus, $R'A'R'$ is the R-inverse of A'.

Conversely, assume for some A we have a B such that $A * B = B * A = R'$. If A is not a permutation, then either $xA = \varnothing$ for some x in X or for some x and y in X with $x \neq y$ we have $xA = yA$. In the former case we have $\varnothing = x(A * B) = xR'$. In the latter case since R is a permutation, we have $x(A * B) = y(A * B)$ and so $x(R') = y(R')$ for $x \neq y$. Neither case is tenable and so A must be a permutation.

We show in the next section that there is a class of matrices such that some groups are not in $B_X(R)$ for any R in this class.

The question now arises, "Do we always have either all groups or only trivial groups?" This is answered negatively in the next section.

3. $B_X(R)$ containing only some groups. In this section we look at a class of matrices for which some, but not all, groups appear in $B_X(R)$ for R in this class. We show that for any R in this class the maximal groups in $B_X(R)$ are a special type.

Consider the class Γ of matrices having the block form

$$\begin{pmatrix} I_k & A \\ 0 & 0 \end{pmatrix}$$

where I_k is the $k \times k$ identity matrix and A is a $k \times n$ matrix whose (1, 1) entry is a 1 and all other entries are 0. We will establish our results for matrices in this class and show the results also hold for matrices of the forms

$$\begin{pmatrix} I_k & A \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_k & 0 \\ A & 0 \end{pmatrix}$$

where A has exactly one nonzero entry. Throughout this section all sandwich matrices R will be in Γ.

THEOREM 4. *The following are necessary and sufficient for E to be an R-idempotent.*

(i) *Assume row j has a 1 in the $(j, 1)$ position. If row j also has a 1 in positions $P_1, \cdots, P_m$, then row j is the sum of rows 1, $k+1$ and rows $P_1, \cdots, P_m$. Otherwise it is just the sum of rows 1 and $k+1$.*

(ii) *Assume row j has a 0 in the $(j, 1)$ position. If row j also has a 1 in positions $P_1, \cdots, P_m$, then row j is the sum of rows $P_1, \cdots, P_m$. If there are no such rows p_i, then row j is zero.*

Proof. Let $ERE = E$. Since rows $k+1$ through n of R are zero, then columns $k+1$ through n of E do not affect the product ER. Thus, we consider entries in columns 1 through k of E.

(i) If row j has a 1 in the $(j, 1)$ position, then $\{x_1, x_{k+1}\}$ is in x_jER. Thus $\{x_1, x_{k+1}\}E$ is in $x_jERE = x_jE$ and rows 1 nad $k+1$ are in row j. That is, row j has 1's at least where rows 1 and $k+1$ have 1's. If row j has a 1 in the (j, p_i) position for p_i in $\{2, \cdots, k\}$, then x_{p_i} is in x_jER and xE_{p_i} is in $x_jERE = x_jE$ and row p_i is contained in row j. Clearly if the (j, p_i) entry is 0, then x_{p_i} is not in

x_jER and hence row p_i is not in $x_jERE = x_jE$. Thus, $x_jE = x_jERE = \{x_1, x_{p_1}, \cdots, x_{p_m}, x_{k+1}\}E$ where the (j, p_i) entries are nonzero, and the result follows.

(ii) From the proof of (i) we see $x_jE = x_jERE = \{x_{p_1}, \cdots, x_{p_m}\}E$ where the (j, p_i) entry is a 1, and the result follows.

Conversely, consider row j of E. We show $x_jE = x_jERE$. If row j has a 1 in the $(j, 1)$ position and in the $(j, p_i), \cdots, (j, p_m)$ positions for p_t in $\{2, \cdots, k\}$, then $x_jERE = \{x_1, x_{p_1}, \cdots, x_{p_m}, x_{k+1}\}RE = \{x_1, x_{p_1}, \cdots, x_{p_m}, x_{k+1}\}E$. By hypothesis, row j is the sum of rows $1, p_1, \cdots, p_m, k+1$ and $x_jE = \{x_1, x_{p_1}, \cdots, x_{p_m}, x_{k+1}\}E$. If row j has a 0 in the $(j, 1)$ position, then the proof is similar except we exclude x_1 and x_{k+1}.

EXAMPLE 1. If $n = 7$ and $k = 4$, then the matrix

$$E = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is an R-idempotent, but the matrix

$$F = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is not an R-idempotent.

We now look at elements in $G_E(R)$.

THEOREM 5. *Let A be in $G_E(R)$.*

(i) *Row m of A is zero if and only if row m of E is zero.*

(ii) *Rows j and m of A are equal if and only if rows j and m of E are equal.*

(iii) *Row m of A is the sum of a subset of the rows 1 through $k+1$ of E.*

(iv) *Row j of A is the sum of rows $p_1, \cdots, p_t$ of A if and only if row j of E is the sum of rows $p_1, \cdots, p_t$ of E.*

Proof. (i) and (ii) follow directly from $ARA' = E$ and $ERA = A$ where A' denotes the R-inverse of A.

(iii) From $ARE = A$ we have row m of A is

$$\begin{pmatrix} a_{m1}e_{11} + a_{m2}e_{21} + \cdots + a_{mk}e_{k1} + a_{m1}e_{k+1,1} \\ a_{m1}e_{12} + a_{m2}e_{22} + \cdots + a_{mk}e_{k2} + a_{m1}e_{k+1,2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}e_{1n} + a_{m2}e_{2n} + \cdots + a_{mk}e_{kn} + a_{m1}e_{k+1,n} \end{pmatrix}$$

where $a_{ij}(e_{ij})$ is the (i, j) entry of $A(E)$. If $a_{mp_1}, \cdots, a_{mp_j} = 0$ and $a_{mp_{j+1}}, \cdots, a_{mp_q} = 1$ where p_i is in $\{1, \cdots, k\}$, then row m of A is

$$(e_{p_{j+1},1} + e_{p_{j+2},1} + \cdots + e_{p_q,1} \cdots e_{p_{j+1},n} + e_{p_{j+2},n} + \cdots + e_{p_q,n})$$

which is the sum of rows $p_{j+1}, \cdots, p_q$ of E. If $a_{m1} = 1$, then we also have $e_{k+1,t}$ in each entry where t runs from 1 to n.

(iv) If row j of A is the sum of rows $p_1, \cdots, p_t$ of A and if A' denotes the R-inverse of A we have

$$x_jE = x_jARA' = \{x_{p_1}, \cdots, x_{p_t}\}ARA' = \{x_{p_1}, \cdots, x_{p_t}\}E\,.$$

The converse is similar.

Thus, for example, if X has 7 elements and $k = 4$ and row m of A is (1 0 1 1 0 0 1), then this row is the sum of rows 1, 3, 4 and 5 of E.

We remark here that this theorem is also valid if R has the form

$$\begin{pmatrix} I_k & A \\ 0 & 0 \end{pmatrix}$$

where A has exactly one nonzero entry, say the (i, j) entry where $j \geqq k + 1$ is nonzero. For in the above proof we use row i where we previously used row 1 and column j where we used column $k + 1$. Similarly, by using the word "column" where we used "row" the result also holds for any R of the form

$$\begin{pmatrix} I_k & 0 \\ A & 0 \end{pmatrix}$$

where A has exactly one nonzero entry.

The goal now is to show how to construct an arbitrary A in $G_E(R)$ and thereby show only certain groups arise in $B_X(R)$. From Theorems 4 and 5 (iv) we see that we need only show the construction of the first $k + 1$ rows of A. The remaining rows are determined by their pattern in E. That is, if row m of E, for $m > k + 1$, is the of rows $p_1, \cdots, p_t$ or E where p_i is between q and $k + 1$ inclusive, then row m of A is the sum of rows $p_1, \cdots, p_t$ of A. We make the following definitions which are illustrated in Example 2.

DEFINITION 1. Let S be a sum of a subset of the first $k+1$ rows of A, but S is not one of the first $k+1$ rows of A (and may not even be any row of A). Then S is called a row *associated* with A. If any row of A or row associated with A is the sum of rows $p_1, \cdots, p_t$, then each p_i is called a *summand*. S is the *maximal sum* of rows $p_1, \cdots, p_t$ if every one of the first $k+1$ rows contained in A is a p_i. We also refer to S as a maximal row associated with A.

DEFINITION 2. Each row m of A is the sum of a subset of the first $k+1$ rows of A and some of the associated rows of A. Let row m be listed as a summand only if it is not the sum of rows distinct (not necessarily different) from itself. Then we say the sum is *maximal* if all rows contained in row m and all maximal rows associated with A contained in row m are listed as summands. If row m is the maximal sum of N rows we write $S_m(A) = N$ and say row m has *order* N.

When we say row m of A is a sum of N rows of A, we mean each summand is either one of the first $k+1$ rows of A or a row associated with A.

We now make the following classification of the nonzero rows of A and the rows associated with A.

DEFINITION 3. If every summand of row m is identical to row m, then row m is called an *independent* row. If at least one summand of row m is proper and if row m is not the sum of its proper summands, then it is called *fixed*. If at least one summand of row m is proper and if row m is the sum of its proper summands, then it is called *dependent*.

By this definition rows associated with A are dependent. Thus, when we refer to a dependent row, it may or may not be in A.

EXAMPLE 2. Let A be given below where $k = 8$.

$$A = \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$S_i(A) = 1$ for $i = 3, 4, 5, 7$ and 8 and $S_1(A) = 4$ (sum of rows 1, 2, 3 and 4), $S_2(A) = 2$ (sum of rows 3 and 4) and $S_{10}(A) = 2$ (sum of rows 4 and 5). We also have row 6 is the sum of rows 6, 7 and 8 and S where S is the sum of rows 7 and 8 and so $S_6(A) = 4$. Row 9 is the sum of rows 1 through 5 and S_1, S_2 and S_3 where S_1 is the sum of rows 3 and 5, S_2 is the sum of rows 4 and 5 and S_3 is the sum of rows 2, 3, 4 and 5. Therefore, $S_9(A) = 8$. Note that (1 1 1 1 1 0 0 0 0 0) considered as the sum of rows 1 and 5 of A is associated with A, but would not be a maximal row associated with A unless we considered it as the sum of rows 1, 2, 3, 4, 5 and 9 of A. Rows 3, 4, 5, 7 and 8 are independent, rows 1, 6 and 9 are fixed, and rows 2 and 10 are dependent.

The following sequence of propositions will enable us to construct an arbitrary element in $G_E(R)$ for an R-idempotent E. Throughout we let A be in $G_E(R)$.

PROPOSITION 1.

(i) *Row m of E is independent if and only if row m of A is independent.*

(ii) *Row m of E is fixed if and only if row m of A is fixed.*

(iii) *Row m of E is dependent if and only if row m of A is dependent.*

Proof. We prove the "if" part of (i), (ii) and (iii) and the "only if" parts must follow.

(i) Let row m of E be the maximal sum of rows $p_1, \cdots, p_t$ of E. Each of these rows will be identical to row m. Thus, by Theorem 6 (ii) and (iv) row m of A is the maximal sum of rows $p_1, \cdots, p_t$ all just like row m of A and row m of A is independent.

(ii) Let row m of E be the maximal sum of rows $p_1, \cdots, p_t$ where either m is a p_i or some row p_i is identical to row m. Apply Theorem 6 (ii) and (iv) to show row m of A is the maximal sum of rows $p_1, \cdots, p_t$ of A where either m is a p_i or some row p_i is identical to row m. Thus, row m of A is fixed.

(iii) As above, apply the definition of dependent row along with Theorem 6 (ii) and (iv).

PROPOSITION 2. *$S_m(E) = N$ if and only if $S_m(A) = N$.*

Proof. Assume $A \neq E$ or there is nothing to prove. Assume $S_m(E) = N$ and row m of E is the maximal sum of rows $p_1, \cdots, p_N$ of E. Assume rows $p_1, \cdots, p_j$ are in E (as usual p_i is between 1 and $k + 1$ inclusive) and rows $p_{j+1}, \cdots, p_N$ are maximal associated with E. Thus, row m of E is the sum of rows $p_1, \cdots, p_j$ of E (not maximal unless $j = N$), and so row m of A is the sum of rows $p_1, \cdots, p_j$ of A.

Assume row p_q is one of the dependent rows associated with E and is the sum of rows $p_{z_1}, \cdots, p_{z_t}$ of E where p_{z_i} is between 1 and j inclusive. Then the sum of rows $p_{z_1}, \cdots, p_{z_t}$ of A is associated with A. For if it were one of the first $k+1$ rows of A, say row q, then by Theorem 6 (ii) row q of E would be the sum of rows $p_{z_1}, \cdots, p_{z_t}$ of E. But this sum is not a row of E. Similarly, for each row p_t associated with E, we get a corresponding row p_t associated with A. Furthermore, each is maximal in A since it was in E. Thus $S_m(A)$ is greater than or equal to N. If $S_m(A)$ is strictly greater than N, then either there is another row in A in the sum of row m or another row associated with A in the sum. In the former case, we contradict Theorem 5 (ii), in the latter case this associated row of A will give rise to another associated row of E contradicting the fact that the sum was maximal.

Conversely assume $S_m(A) = N$ and $S_m(E) = M \neq N$. But by the above $S_m(E) = M$ implies $S_m(A) = M$ and we have a contradiction.

PROPOSITION 3. *Given the fixed and independent rows of A we can determine the dependent rows of A.*

Proof. The dependent rows of A will be in the same positions as the dependent rows of E. Let row m of E be dependent and the maximal sum of rows $p_1, \cdots, p_t$ of E where rows $p_1, \cdots, p_j$ are dependent. By the definition of maximal sum, every summand of any row p_i for i between 1 and j inclusive will be one of the rows $p_1, \cdots, p_t$ and by the definition of dependent row, each summand is proper. Thus, dependent rows are redundant in a maximal sum, and row m of E is the sum of rows $p_{j+1}, \cdots, p_t$ of E where each p_i is independent or fixed. By Theorem 5 (ii) and Proposition 3 row m of A is the sum of rows $p_{j+1}, \cdots, p_t$ of A which will be fixed or independent as they are in E.

From Theorem 5 (ii) and Propositions 1 and 2 we have the following proposition.

PROPOSITION 4. *Row m of A has the same unmber and types of summands as row m of E.*

Proposition 4 is useful in constructing the independent and fixed rows of A. Recall, each independent row of E is a row of E. That is, it cannot be associated with E. By Theorem 5 (ii) and Proposition 1 each of these rows must be an independent row of A. Similarly, each fixed row of E must be some fixed row of A. The following definitions help us apply Proposition 4.

DEFINITION 4. If an independent row is a summand of a fixed row, it is called *Type* 1. Otherwise it is *Type* 2.

Propositions 1 and 4 now give the following.

PROPOSITION 5. *Row m of E is independent of Type* 1 (*Type* 2) *if and only if row m of A is independent of Type* 1 (*Type* 2).

DEFINITION 5. A fixed row of A is called a *maximal fixed row* (MFR) if it is not the summand of any fixed row different from itself. An MFR together with its summands is called a *maximal fixed block* (MFB). MFRs (or MFBs) with the same number and types of summands are said to be in the same class. We define a *sub*-MFR (*sub*-MFB) to be any MFR (MFB) within an MFR (MFB). A fixed row is a *minimal fixed row* (mFR) if it does not contain any fixed summands. An mFR together with its summands is called a *minimal fixed block* (mFB).

We remark that a fixed row may be both an MFR and an mFR. Every MFB is either an mFB or contains an mFB.

EXAMPLE 3. Let

$$A=\begin{pmatrix}1&1&1&1&1&1&1&1&1&1&1\\1&1&1&1&0&0&0&0&0&0&0\\0&0&1&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&1&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&0&0&0\\0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&1&1&0\\0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0\end{pmatrix}\begin{array}{l}\\ \\ \left.\right]C\ \ \left.\right]B \\ \\ \\ \left.\right]E\ \ \left.\right]D\ \ \left.\right]H \\ \\ \left.\right]F \\ \\ \\ \\ \end{array}.$$

(Brackets: C = rows 3–4; B = rows 2–4; E = rows 6–7; D = rows 5–7; F = rows 8–9; H = rows 1–9.)

H is an MFB witn B, D and F as sub-MFBs. B and D are in the same class. C and E are sub-MFBs of B and D respectively and are mFBs. F is also an mFB.

Proposition 4 now gives us the following:

PROPOSITION 6. *Row m of E is an* MFR *with an associated* MFB *in class Γ if and only if row m of A is an* MFR *with associated* MFB *in class Γ.*

We now give the construction of the first $k+1$ rows of A.

Step 1. If any rows of E are zero, then the corresponding rows in A are zero.

Step 2. Distict independent rows of Type 2 in E are permuted observing Theorem 5 (ii).

Step 3. MFBs of the same class in E are permuted to form MFBs of this class in A. We must observe Propositions 1 and 2. That is, subblocks may need to be permuted within an MFB.

Step 4. If within an MFB there are independent rows of Type 2 (thus, they are actually independent rows of Type 1 in E), then they may be permuted.

Step 5. Repeat Steps 3 and 4 with sub-MFBs. That is, sub-MFBs of the same MFB and of the same class may be permuted and within them, independent rows of Type 2 may be permuted.

Step 6. Repeat Step 4 until mFBs have been permuted and their independent rows of Type 2 have been permuted.

Step 7. Calculate the dependent rows by the fixed and independent rows and the pattern of E (as in the proof of Proposition 3).

THEOREM 6. *A is in $G_E(R)$ if and only if A is constructed as above.*

Proof. If A is in $G_E(R)$, then Propositions 1 through 6 show that is A constructed as above. Conversely, let A be constructed as above. We must show $A*E = A = E*A$ and the existence of an inverse. We first show $A*E = E*A = A$.

Case 1. Row m of A is independent or fixed. Then it is some row of E, say row p. Thus, $x_mA = x_pE$ and $x_mA*E = x_pE*E = x_pE = x_mA$. Assume row m of E has ones in the $p_1, \cdots, p_t$ positions for p_i between 1 and k inclusive. Row m is the sum of rows $p_1, \cdots, p_t$ if the $(m, 1)$ position is a zero and so $x_mE = x_mER$. It is the sum of rows $p_1, \cdots, p_t, k+1$ if the $(m, 1)$ position is a 1. In the former case, row m of A is the sum of rows $p_1, \cdots, p_t$ of A and $x_m E * A = x_m EA = \{x_{p_1}, \cdots, x_{p_t}\}A = x_m A$. In the latter case row m of A is the sum of rows $p_1, \cdots, p_t$, $k+1$ of A and $x_m ERA = \{x_{p_1}, \cdots, x_{p_t}, x_{k+1}\}RA = \{x_{p_1}, \cdots, x_{p_t}, x_{k+1}\}A = x_m A$.

Case 2. Row m of A is dependent. Then row m of E is dependent. Assume row m of E is the sum of rows $p_1, \cdots, p_t$ of E where row p_i is fixed or independent. Thus, row m of A is the sum of rows $p_1, \cdots, p_t$ of A where row p_i is fixed or inedpendent in A. Thus, from Case 1, for each p_i we have $x_{p_i} A * E = x_{p_i} A = x_{p_i} E * A$. Now, $x_m A * E = \{x_{p_1}, \cdots, x_{p_t}\}A * E = x_{p_1} A * E + x_{p_2} A * E + \cdots + x_{p_t} A * E = x_{p_1} A + x_{p_2} A + \cdots + x_{p_t} A = \{x_{p_1}, \cdots, x_{p_t}\}A = x_m A$. Similarly $x_m E * A = x_m A$.

We now construct a B by the above rules and show B is an R-inverse of A.

Step 1. If row m of E is zero, then row m of B is zero.

Step 2. Independent rows of Type 2. Assume rows $p_1, \cdots, p_t$ of E are distinct independent rows of Type 2. Let θ be the permutation on $p_1, \cdots, p_t$ where row p_i of E is row $\theta(p_i)$ of A. Let these independent rows be permuted in B by θ^{-1}. That is, row $\theta(p_i)$ of E is row p_i of B.

Step 3. MFBs of the same class. Permute these in B following the same scheme above for independent rows of Type 2.

Step 4. Independent rows of Type 2 within an MFB. Let MFBs $B_1, \cdots, B_t$ be of the same class and let each B_i have distinct independent rows $b_{i1}, b_{i2}, \cdots, b_{it}$ of Type 2. Assume θ permutes the blocks as they are permuted in A (similar to θ in Step 2). Then in A, block B_i occupies the position $\theta(B_i)$ occupies in E and in B, block $\theta(B_i)$ occupies the position block B_i does in E. If rows $b_{i1}, \cdots, b_{it}$ of block B_i have been permuted in A, then apply the same permutation to the corresponding rows in block $\theta(B_i)$ of B.

Step 5. Sub-MFRs. These are formed in B following the same scheme as for independent rows in Step 4.

Step 6. Continue as in Steps 4 and 5 for independent rows of Type 2 within sub-MFBs and for sub-MFBs within the sub-MFBs until the process terminates with mFBs.

Step 7. Dependent rows. These are determined by independent and fixed rows.

Thus we have a B such that $B*E = B = E*B$. Let the independent rows of Type 2 in A and B be as in Step 2 above. Then for each i, $x_{\theta(p_i)}(A*B) = x_{p_i}(E*B) = x_{p_i}(B) = x_{\theta(p_i)}(E)$. Similarly for each i, $x_{p_i}(B*A) = x_{\theta(p_i)}(E*A) = x_{\theta(p_i)}(A) = x_{p_i}(E)$. Thus, for any independent row, say x_m, of Type 2 we have $x_m(A*B) = x_mE = x_m(B*A)$. Similar proofs give the same result for MFRs. Now consider independent rows of Type 2 within an MFB as in Step 4. By the construction, if row m of E is row p of A, then row p of E is row m of B where row m is in B_i and row p is in $\theta(B_i)$. This implies $x_m(E) = x_p(A)$ and $x_p(E) = x_m(B)$ and for each row m in B_i we have $x_m(E) = x_p(A) = x_p(E*A) = x_m(B*A)$. Similarly, if row m of E is row q of B, then row q of E is row m of A and $x_m(E) = x_q(B) = x_q(E*B) = x_m(A*B)$. Thus, for these rows $x_m(A*B) = x_mE = x_m(B*A)$. Sub-MFRs satisfy $x_m(A*B) = x_mE = x_m(B*A)$ by the same type of proof. We now show the result for dependent rows. Let row m of E be dependent. Then it is the sum of rows $p_1, \cdots, p_t$ of E which are fixed or independent, and rows m of A and B are the sums of rows $p_1, \cdots, p_t$ of A and B respectively. Since $x_m(A*B) = x_mE = x_m(B*A)$ for row x_m fixed or independent, we have $x_mE = \{x_{p_1}, \cdots, x_{p_t}\}E = \{x_{p_1}\}E + \cdots + \{x_{p_t}\}E = \{x_{p_1}\}A*B + \cdots + \{x_{p_t}\}A*B = \{x_{p_1}, \cdots, x_{p_t}\}A*B = x_m(A*B)$. Similarly, $x_mE = x_m(B*A)$.

COROLLARY 1. *$C_E(R)$ is trivial if and only if*

(i) *No two distinct independent rows of Type 2 are in E.*

(ii) *No independent rows of Type 1 can be permuted.*

(iii) *No two fixed rows of E are in the same class.*

COROLLARY 2. *$G_E(R)$ is nontrivial if and only if it contains a nontrivial subgroup isomorphic to a permutation group.*

Proof. Assume $G_E(R)$ is nontrivial. Then at least one of the three statements of Corollary 1 must be false. Assume (i) is false and let $p_1, \cdots, p_t$ be the distrinct independent rows of Type 2. Let A be the set of all A in $G_E(R)$ formed by permuting rows $p_1, \cdots, p_t$ of E and leaving all other rows of E stationary. A is a subgroup of $G_E(R)$ isomorphic to the permutation group on $\{p_1, \cdots, p_t\}$. A similar proof establishes the result if we assume (ii) or (iii) is false.

The converse is clear.

If for each N_i in $\{N_1, \cdots, N_p\}$ there are n_i identical independent rows of Type 2 and also if for each C_k in the set $\{C_1, \cdots, C_j\}$ there are c_k MFBs of class C_k where c_k is greater than 1, then $G_E(R)$ contains a subgroup isomorphic to $G = P_p \times P_{C_1} \times P_{C_2} \times \cdots \times P_{C_t}$ where P_T is the permutation group on the set of T elements. As in the proof of Theorem 6 let $\mathscr{A}$ in $G_E(R)$ be the set of all A such that the independent rows of Type 1 are fixed. Then $A \simeq G$. Thus we have the following

COROLLARY 3. *If E contains no independent rows of Type 1 that can be permuted or if no* MFBs *are of the same class, then $G_E(R)$ is isomorphic to a direct product of permutation groups.*

EXAMPLE 4. Let $k = 6$ and

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Rows 1, 2, 3, 7 and 8 are independent of Type 2; but since rows 1, 7 and 8 are alike and 2 and 3 are alike, we only get one permutation from these. Row 4 is fixed and rows 5 and 6 are independent of Type 1. Thus, $G_E(R) = \{E, A\}$ where

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

EXAMPLE 5. Let $k = 8$ and

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Row 1 is independent of Type 2. Row 2 is an MFR with rows 2, 3 and 4 as summands and so $S_2(E) = 3$. Row 5 is an MFR with rows 5, 6, 7, 8 and 9 as summands and so $S_5(E) = 5$. Rows 6 through 9 form a sub-MFB of row 5. From the above we see no permutations can be formed and $G_E(R)$ is trivial.

EXAMPLE 6. Let $k = 16$ and R be 18×18. Let

$$E = \begin{pmatrix}
\begin{matrix}
1&1&1&1&1&1&1&1\\
0&1&1&1&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&1&1&1\\
0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1
\end{matrix} & \bigcirc \\
\bigcirc & \begin{matrix}
1&1&1&1&1&1&1&1&0&0\\
0&1&1&1&1&0&0&0&0&0\\
0&0&1&1&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&1&1&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{matrix}
\end{pmatrix}.$$

(In the printed matrix, brackets mark the blocks: B_1 with sub-blocks S_1, S_2, s_1 and rows I_1, I_2, I_3, I_4; B_2 with sub-blocks S_3, S_4, s_2 and rows I_5, I_6, I_7, I_8.)

B_1 and B_2 are MFBs of the same class and can be permuted. S_1 and S_2 are sub-MFBs of B_1, s_1 is a sub-MFB of S_2. Similarly, S_3 and S_4 are sub-MFBs of B_2 and s_2 is a sub-MFB of S_3. Note s_1 and s_2 are mFBs and I_1 through I_8 are independent of Type 1. S_1 and S_2 (and S_3 and S_4) are not of the same class. The pairs (I_1, I_2), (I_3, I_4), (I_5, I_6) and (I_7, I_8) are independent of Type 2 within blocks S_1, s_1, s_2 and S_4 respectively and can be permuted within these blocks. Observe, if we permute B_1 and B_2, then we must permute S_1 and S_2 and S_3 and S_4 within the blocks. Thus we can describe $G_E(R)$ as follows. If we do not permute B_1 and B_2, then we have 16 elements of this form—one for each of the possible permutations of the pairs of independent rows. If we do permute B_1 and B_2, then we again have 16 elements. Thus, $G_E(R)$ has 32 elements. The first 16 elements described form the subgroup $K = S_2 \times S_2 \times S_2 \times S_2$

where S_2 is the symmetric group on the set of two elements. For example the element

$$\left(\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}\right)$$

in K corresponds to the element A in $G_E(R)$ with rows I_3 and I_4 and I_5 and I_6 interchanged. Rows I_1, I_2, I_7 and I_8 are not permuted. We can consider elements of $G_E(R)$ as 5-tuples (A, B, C, D, E) where each entry is a permutation of 1, 2. A represents the permutation of B_1 and B_2, B, C, D and E represent the permutations of the pairs (I_1, I_2), (I_3, I_4), (I_5, I_6) and (I_7, I_8) respectively. Consider the elements where A is the identity to be of Type 1, and those where A represents the permutation of B_1 and B_2 to be of Type 2. Let $X = (A, B, C, D, E)$ and $Y = (A', B', C', D', E')$ be elements of $G_E(R)$. The multiplication in $G_E(R)$ is given by

$$XY = \begin{cases}(AA', BB', CC', DD', EE') & \text{if } X \text{ and } Y \text{ are both Type 1}\\ (AA', BE', CD', DC', EB') & \text{if either } X \text{ or } Y \text{ is Type 2.}\end{cases}$$

We remark that the above theorems and propositions are also valid if R has the form

$$\begin{pmatrix}I_k & A\\ 0 & 0\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}I_k & 0\\ A & 0\end{pmatrix}$$

where A has exactly one nonzero entry. The proofs would be as indicated in the remarks following Theorem 5.

It is not known if there is a way to determine the maximal groups in $B_X(R)$ for any given R. It would be interesting to find properties of the relation R that determine the maximal groups.

References

1. K. Chase, *New semigroups of binary relations*, Semigroup Forum, **18** (1979), 79–82.
2. ———, *Sandwich semigroups of binary relations*, Discrete Math., **28** (1979), 231–236.
3. ———, *Digraphs, automata and sandwich semigroups of binary relations*, Ph. D. Thesis, Texas A & M University, College Station, Texas (1978).
4. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. 1, Amer. Math. Soc., Providence, RI, 1961.
5. S. I. Montague and R. J. Plemmons, *Maximal subgroups of the semigroup of relations*, J. Algbra, **13** (1969), 575–587.

Received August 25, 1980.

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