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REAL HOMOLOGY OF LIE GROUP HOMOMORPHISMS

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Let $h: G_1 \to G_2$ be a homomorphism of compact, connected Lie groups and let $h_*: H_*(G_1) \to H_*(G_2)$ be the homomorphism of homology with real coefficients induced by h. The investigation of the properties of h that can be deduced from a knowledge of h_* goes back at least to work of Dynkin in the early 1950's. This paper presents several contributions to the investigation. The main result is a characterization of homomorphisms with abelian images as those whose induced homomorphisms annihilate all three-dimensional indecomposables. We then examine what the homology can tell us about the dimension of the abelian image. Next, an inequality relating the homology of the kernel of h to the kernel of h_* leads to sufficient conditions for h to have an abelian, semisimple, or finite kernel. The final sections present various relationships between h_* and the kernel and image of h and, in particular, show that if $h(G_1)$ is totally nonhomologous to zero in G_2, then h_* gives quite precise information about the behavior of h.

For the results of Dynkin, see [2] and [3].

In order to avoid frequently repeating the same hypotheses, we state that throughout this paper:

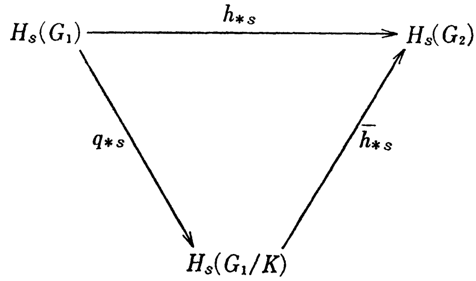
HYPOTHESES. *$h: G_1 \to G_2$ is a homomorphism where G_1 and G_2 are compact, connected Lie groups.*

1. Homomorphisms with abelian images.

A homomorphism $h: G_1 \to G_2$ induces homomorphisms of real homology $h_{*s}: H_s(G_1) \to H_s(G_2)$ for all s. We will need the following observation:

LEMMA 1.1. *If $h: G_1 \to G_2$ is surjective, then $h_{*s}: H_s(G_1) \to H_s(G_2)$ is surjective for all s.*

Proof. Since h is surjective, there is an isomorphism $\bar{h}: G_1/K \to G_2$, where K is the kernel of h. The quotient homomorphism $q: G_1 \to G_1/K$ induces a surjection of real homology, so the commutativity of



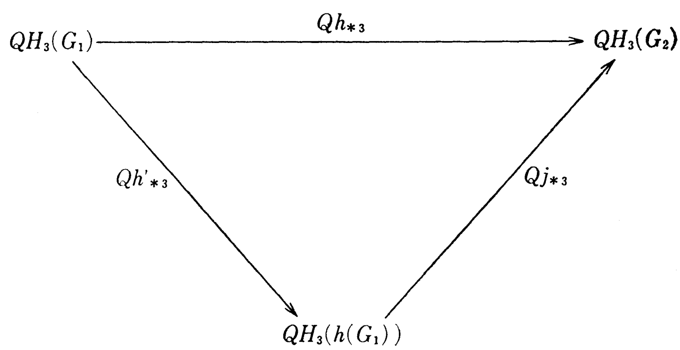
proves that h_{*s} is a surjection.

Let $QH_*(G)$ denote the graded vector space of indecomposables in the real homology of a Lie group G. A homomorphism $h: G_1 \to G_2$ induces $Qh_{*s}: QH_s(G_1) \to QH_s(G_2)$ by restricting h_* to the indecomposables in $H_s(G_1)$.

If a Lie group G is abelian, then $QH_s(G) = 0$ for all $s > 1$. Consequently, if $h: G_1 \to G_2$ such that $h(G_1)$ is abelian, then certainly $Qh_{*s} = 0$ for all $s > 1$ because the homomorphism factors through $QH_*(h(G_1))$. But the converse holds as well; in fact our main result requires that we consider only $s = 3$.

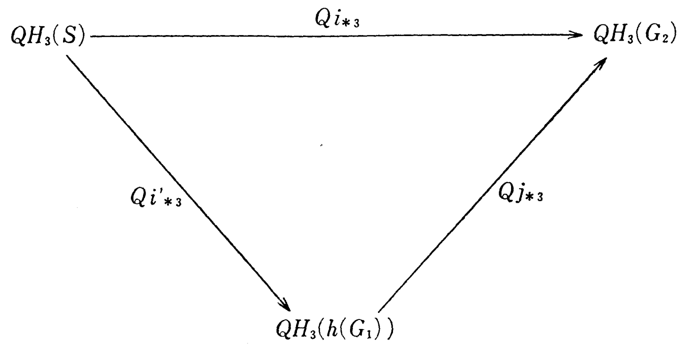
THEOREM 1.2. *If* $h: G_1 \to G_2$ *such that* $Qh_{*3}: QH_3(G_1) \to QH_3(G_2)$ *is the zero homomorphism, then the group* $h(G_1)$ *is abelian.*

Proof. We prove the contrapositive: assuming that $h(G_1)$ is not abelian we show that $Qh_{*3} \neq 0$. Let $h': G_1 \to h(G_1)$ be the same as h except for range and let $j: h(G_1) \to G_2$ be inclusion. Consider the diagram



The homomorphism Qh'_{*3} is surjective by Lemma 1.1 and the fact that the real homology of a Lie group is generated by indecomposables. Therefore, the diagram implies that $Qh_{*3} = 0$ if and only if $Q_{j*3} = 0$. So we will prove that if $h(G_1)$ is nonabelian then $Q_{j*3} \neq 0$. Because $h(G_1)$ is assumed nonabelian, it is possible to exploit the relationship between compact and complex semisimple

Lie groups to conclude that there is a closed subgroup S of $h(G_1)$ that is semisimple and three-dimensional. Let $i\colon S \to G_2$ and $i'\colon S \to h(G_1)$ be inclusions, so we have the diagram



By [6], because S is a semisimple and three-dimensional subgroup of G_2, then the homomorphism $i_*\colon H_*(S) \to H_*(G_2)$ is injective. Furthermore, the semisimplicity of S implies $QH_3(S) \neq 0$, so $Qj_{*3} \neq 0$ by the diagram and that completes the proof.

Two immediate consequences of the main result are:

COROLLARY 1.3. *Suppose $h\colon G_1 \to G_2$ where G_1 is semisimple. If $Qh_{*3} = 0$ then h is the constant homomorphism.*

COROLLARY 1.4. *If $Qh_{*3} = 0$ then $Qh_{*s} = 0$ for all $s > 1$.*

Note that it is not sufficient in Corollary 1.3 merely to assume that G_1 is nonabelian. For instance, if we let $h\colon S^1 \times S^3 \to S^3 \times S^3$ be the cartesian product of inclusion and the constant homomorphism, we obtain a nonconstant homomorphism for which $Qh_{*3}=0$.

The homological characterization of homomorphisms with abelian images in Theorem 1.2 was of course suggested by the homological characterization: a compact Lie group G is abelian if and only if $QH_3(G) = 0$. The characterization: a compact Lie group G is semisimple if and only if $H_1(G) = 0$ does not seem to lead to any characterization of homomorphisms with semisimple images. That is, the obvious condition $h_{*1} = 0$ is not sufficient for a semisimple image, as we can see by including abelian closed subgroups in simply-connected Lie groups. (In contrast, the homological characterizations are precisely what we use to give sufficient conditions for h to have an abelian or semisimple kernel in Corollaries 3.2 and 3.3 below.)

PROBLEM. *Find necessary and sufficient conditions on h_* so* *that $h(G_1)$ is semisimple.*

2. The dimension of an abelian image. Now that we have identified the homomorphisms with abelian images by means of their induced homology homomorphisms, we are led to investigate what homology can tell us about the only significant characteristic of such an image: its dimension.

It is convenient to recall the following concept from [5]. Let A be a closed, connnected subgroup of a compact Lie group G, then A is *totally nonhomologous to zero* in G, written $A \not\sim 0$ in G, if and only if the homomorphism j_* induced by the inclusion $j\colon A \to G$ is injective.

We have already made use of the concept in the proof of Theorem 1.2 because that proof depended on the fact that if A is semisimple and three-dimensional, then $A \not\sim 0$ in G.

We continue to use the notation K for the kernel of $h\colon G_1 \to G_2$ and denote by K_0 its maximal connected subgroup. We abbreviate dimension, of either a vector space or a Lie group, by dim, and image by im.

THEOREM 2.1. *If the image of $h\colon G_1 \to G_2$ is abelian, then*

$$\dim \operatorname{im} h_{*1} \leqq \dim h(G_1) \leqq \dim H_1(G_1)$$

with equality as follows:

(i) $\dim h(G_1) = \dim \operatorname{im} h_{*1}$ *if and only if* $h(G_1) \not\sim 0$ *in* G_2

(ii) $\dim h(G_1) = \dim H_1(G_1)$ *if and only if* K_0 *is semisimple.*

Proof. Since $h(G_1)$ is a torus, $\dim h(G_1) = \dim H_1(h(G_1))$. Defining $h'\colon G_1 \to h(G_1)$ and $j\colon h(G_1) \to G_2$ as before, we find that $\dim \operatorname{im} h_{*1} \leqq \dim \operatorname{im} h'_{*1}$ with equality if and only if j_{*1} is injective. Again using the fact that $h(G_1)$ is a torus, we conclude that j_{*1} is injective if and only if $h(G_1) \not\sim 0$ in G_2. Since

$$\dim H_1(G_1) = \dim H_1(K_0) + \dim H_1(h(G_1))$$

then $\dim h(G_1) \leqq \dim H_1(G_1)$ with equality if and only if $H_1(K_0) = 0$.

COROLLARY 2.2. *If the image of $h\colon G_1 \to G_2$ is abelian and h_{*1} is injective, then $\dim h(G_1) = \dim Z(G_1)$, where $Z(G_1)$ is the center of G_1.*

Proof. Since h_{*1} is injective, then $\dim \operatorname{im} h_{*1} = \dim H_1(G_1)$ so $\dim h(G_1) = \dim H_1(G_1)$ by Theorem 2.1. The fact that $\dim H_1(G_1) = \dim Z(G_1)$ completes the proof.

3. The kernel of a homomorphism. Since we have seen that Qh_* contains information about the image of $h: G_1 \to G_2$, at least when the image is abelian, we may hope that Qh_* will also reflect some of the properties of the kernel of h. We will show that to a certain extent this is indeed the case. Our results depend on the following useful inequality.

THEOREM 3.1. *Let* $h: G_1 \to G_2$, *then*

$$\dim QH_s(K_0) \leqq \dim \ker Qh_{*s}$$

for all $s > 0$, *where* ker *denotes kernel.*

Proof. Let $i: K_0 \to G_1$ be inclusion and let $\mathrm{di}: \mathscr{K} \to \mathfrak{G}_1$ be its differential, where $\mathscr{K}$ and $\mathfrak{G}_1$ are the Lie algebras of K_0 and G_1 respectively. There is a projection $\pi: \mathfrak{G}_1 \to \mathfrak{K}$ that is a Lie algebra homomorphism. Since $\pi(\mathrm{di})$ is the identity on $\mathfrak{K}$, the composition

$$H_s(\mathscr{K}) \xrightarrow{(\mathrm{di})_{*s}} H_s(\mathfrak{G}_1) \xrightarrow{\pi_{*s}} H_s(\mathscr{K})$$

is the identity for each s. We conclude that $(\mathrm{di})_{*s}$ is an injection. Then the Universal Coefficient Theorem and de Rham's Theorem imply that $i_{*s}: H_s(K_0) \to H_s(G_1)$ is injective; as is the restriction Qi_{*s}. Since hi is the constant homomorphism, the composition

$$QH_s(K_0) \xrightarrow{Qi_{*s}} QH_s(G_1) \xrightarrow{Qh_{*s}} QH_s(G_2)$$

is the zero homomorphism for each $s>0$. Therefore $Qi_{*s}(QH_s(K_0)) \subseteqq \ker Qh_{*s}$ and because we have proved that Qi_{*s} is injective

$$\dim QH_s(K_0) = \dim \mathrm{im}\, Qi_{*s} \leqq \dim \ker Qh_{*s} .$$

For an example where $\dim QH_s(K_0) \neq \dim \ker Qh_{*s}$, let $h: SO(8) \to SO(9)$ be inclusion. The Wang sequence of the fibration

$$SO(8) \xrightarrow{h} SO(9) \longrightarrow S^8$$

shows that $\dim \ker Qh_{*7} \geqq 1$.

COROLLARY 3.2. *If* h_{*1} *is injective, then* K_0 *is semisimple.*

Proof. Since $Qh_{*1} = h_{*1}$, then $\dim \ker Qh_{*1} = 0$ so $H_1(K_0) = 0$ by Theorem 3.1 and therefore K_0 is semisimple.

The injectivity of h_{*1} is sufficient for K_0 to be semisimple, but it is obvious that it is not a necessary condition.

COROLLARY 3.3. *If* Qh_{*3} *is injective, then* K_0 *is abelian.*

Once again, injectivity is a sufficient but not necessary condition. Define $h\colon \mathrm{Sp}(6) \times \mathrm{Sp}(6) \to \mathrm{Sp}(12)$ as follows. Given A and B in Sp(6), let $h(A, B)$ be the matrix

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Then h has trivial kernel, but Qh_{*3} cannot be injective because $\dim QH_3\,(\mathrm{Sp}(6) \times \mathrm{Sp}(6)) = 2$ and $\dim QH_3\,(\mathrm{Sp}(12)) = 1$.

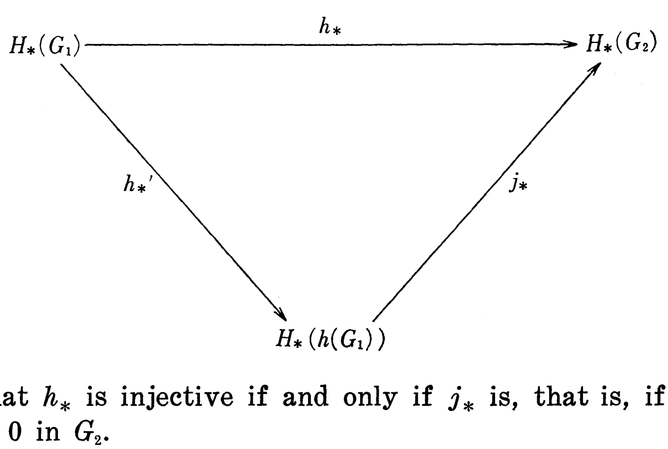
Combining the previous corollaries, we see that if h_{*1} and h_{*3} are both injective, then K_0 is trivial so

COROLLARY 3.4. *If h_{*s} is injective for $s \leqq 3$, then the kernel of h is finite.*

In common with the previous corollaries, the converse to Corollary 3.4 is not true in general. However

THEOREM 3.5. *Suppose $h\colon G_1 \to G_2$ has finite kernel. Then h_* is injective if and only if $h(G_1) \not\sim 0$ in G_2.*

Proof. Let $h'\colon G_1 \to h(G_1)$ be the same as h except for range. Since h' has finite kernel, its differential $dh'\colon \mathfrak{G}_1 \to \mathfrak{H}$ is an isomorphism ($\mathfrak{G}_1$ and $\mathfrak{H}$ are the Lie algebras of G_1 and $h(G_1)$, respectively). Therefore, $(dh')_*\colon H_*(\mathfrak{G}_1) \to H_*(\mathfrak{H})$ is an isomorphism and it follows that $h'_*\colon H_*(G_1) \to H_*(h(G_1))$ is an isomorphism also. For $j\colon h(G_1) \to G_2$ the inclusion, the commutativity of



implies that h_* is injective if and only if j_* is, that is, if and only if $h(G_1) \not\sim 0$ in G_2.

4. Rank and dimension of the kernel. We have seen in the previous section that Qh_* does contain information about the kernel K of $h\colon G_1 \to G_2$, at least to the extent of producing sufficient conditions for K to be semisimple or abelian. In this section, we improve a result from [1] to obtain upper bounds on the rank and dimension of K. Furthermore, we find a necessary and sufficient condition for the bounds to provide exact calculations.

The rank of a compact Lie group will be denoted by $\mathrm{rk}\,(G)$. (If G is disconnected, its rank is defined to be the rank of the maximal connected subgroup of G.) The dimension of a graded vector space, still denoted by dim, is the sum of the dimensions of the individual vector spaces.

Since the bounds on the rank and dimension of K are most naturally stated in the language of cohomology rather than homology, we consider the homomorphisms $h^{*s}\colon H^s(G_2) \to H^s(G_1)$, for each s, induced by $h\colon G_1 \to G_2$ on cohomology with real coefficients. Restricting to primitives, we have homomorphisms $Ph^{*s}\colon PH^s(G_2) \to PH^s(G_1)$.

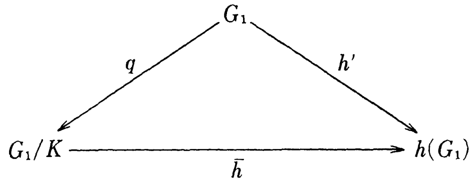
THEOREM 4.1. *Let K be the kernel of $h\colon G_1 \to G_2$, then*

(i) $\mathrm{rk}\,(K) \leqq \dim\ker Ph^* + (\mathrm{rk}\,(G_1) - \mathrm{rk}\,(G_2))$

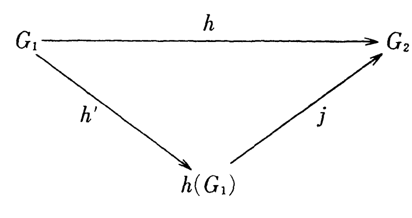
(ii) $\dim\,(K) \leqq \sum_s s(\dim\ker Ph^{*s}) + (\dim\,(G_1) - \dim\,(G_2))$.

The inequalities (i) *and* (ii) *are equalities if and only if* $h(G_1) \not\sim 0$ *in* G_2.

Proof. Since K is a normal subgroup of G_1, the quotient homomorphism $q\colon G_1 \to G_1/K$ induces an injective homomorphism of real cohomology, so Pq^{*s} is injective for all s. From the commutative diagram below, where $\bar{h}$ is an isomorphism and h' is the same as h except for range, we see that Ph'^{*s} is injective because Pq^{*s} is.



Therefore, the diagram



where j is inclusion, implies that $\ker Ph^{*s} = \ker Pj^{*s}$ and thus since

$$\dim PH^s(G_2) = \dim\ker Pj^{*s} + \dim\mathrm{im}\, Pj^{*s}$$

that

$$(*) \qquad \dim\ker Ph^{*s} - \dim PH^s(G_2) + \dim\operatorname{im} Pj^{*s} = 0 \ .$$

Now, for the proof of part (i), we know from [4] that for any compact Lie group, $\operatorname{rk}(G) = \dim PH^*(G)$. Since

$$PH^*(G_1) = PH^*(K) \oplus PH^*(h(G_1))$$

we see that

$$\operatorname{rk}(G_1) = \operatorname{rk}(K) + \operatorname{rk}(h(G_1)) \ .$$

Using (*) and rearranging terms, we have

$$\begin{aligned} \operatorname{rk}(K) = [&\dim\ker Ph^* + (\operatorname{rk}(G_1) - \operatorname{rk}(G_2))] \\ &- [\operatorname{rk}(h(G_1)) - \dim\operatorname{im} Pj^*] \ . \end{aligned}$$

Inequality (i) follows because $\operatorname{im} Pj^* \subseteqq PH^*(h(G_1))$. We have equality in (i) if and only if Pj^* is surjective which, by the Universal Coefficient Theorem, is equivalent to the condition $h(G_1) \not\sim 0$ in G_2. For part (ii) we use the fact that for any compact, connected Lie group

$$\dim(G) = \sum_s s(\dim PH^s(G)) \ .$$

Since

$$\dim(G_1) = \dim(K) + \dim(h(G_1))$$

we again use (*) and rearrange terms to conclude that

$$\begin{aligned} \dim(K) = [&\sum_s s(\dim\ker Ph^{*s}) + (\dim(G_1) - \dim(G_2))] \\ &- [\dim(h(G_1)) - \sum_s s(\dim\operatorname{im} Pj^{*s})] \ . \end{aligned}$$

Just as in the proof of part (i), inequality (ii) follows from this last equation, with equality in (ii) if and only if $h(G_1) \not\sim 0$ in G_2.

The less precise form of Theorem 4.1(i) in [1] was used there to study homomorphisms h for which h^* is an isomorphism.

COROLLARY 4.2. *If the image of* $h: G_1 \to G_2$ *is abelian, then*

(i) $\operatorname{rk}(K) \leqq \operatorname{rk}(G_1) - \dim\operatorname{im} h^{*1}$

(ii) $\dim(K) \leqq \dim(G_1) - \dim\operatorname{im} h^{*1}$

with equality if and only if $h(G_1) \not\sim 0$ *in* G_2.

Proof. We will prove inequality (i); the proof of (ii) is almost identical. Since $h(G_1)$ abelian implies $h_{*s} = 0$ for $s > 1$ the Universal Coefficient Theorem implies that $h^{*s} = 0$ for $s > 1$ also and thus

$$\dim\ker Ph^* = \dim\ker Ph^{*1} + \sum_{s>1} \dim PH^s(G_2) \ .$$

Because

$$\mathrm{rk}\,(G_2) = \dim H^1(G_2) + \sum_{s>1} \dim PH^s(G_2)$$

we may substitute into inequality (i) of Theorem 4.1 to conclude that

$$\mathrm{rk}\,(K) \leqq \mathrm{rk}\,(G_1) - (\dim H^1(G_2) - \dim \ker h^{*1}) .$$

We can apply Corollary 4.2 quite effectively when G_2 is abelian. For instance

COROLLARY 4.3. *Suppose $h: G_1 \to G_2$ is a homomorphism and G_2 is abelian, then*

$$\dim(K) = (\dim(G_1) - \dim(G_2)) + \dim \ker h^{*1} .$$

5. Surjective homomorphisms. The following homological characterization of surjective homomorphisms is easy to verify, but we include it for completeness.

PROPOSITION 5.1. *A homomorphism $h: G_1 \to G_2$ is surjective if and only if $h_{*n}: H_n(G_1) \to H_n(G_2)$ is a nonzero homomorphism, where $n = \dim(G_2)$.*

Proof. Necessity is immediate from Lemma 1.1 and the fact that since G_2 is a closed orientable n-manifold, $H_n(G_2) \neq 0$. Sufficiency is a consequence of the classical result that if a map of closed orientable manifolds has nonzero degree, then it is surjective. Alternatively, we may prove sufficiency using Theorem 4.1(ii) as follows. If $h_{*n} \neq 0$ then $h^{*n}: H^n(G_2) \to H^n(G_1)$ is nonzero. Let $z_1, z_2, \cdots, z_r$ generate the exterior algebra $H^*(G_2)$, then $\mu = z_1 z_2 \cdots z_r$ generates the vector space $H^n(G_2)$. We are assuming $h^*(\mu) \neq 0$ which implies Ph^* is injective. By inequality (ii) of Theorem 4.1, $\dim(K) \leqq \dim(G_1) - \dim(G_2)$. But then we have

$$\dim(G_1) - \dim(h(G_1)) \leqq \dim(G_1) - \dim(G_2) .$$

Since $h(G_1)$ is a closed submanifold of G_2, it must be that $h(G_1) = G_2$, that is, h is surjective.

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