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GLEASON'S THEOREM FOR TYPE I VON NEUMANN ALGEBRAS

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For a von Neumann algebra $\mathfrak{A}$ a G-measure m on $\mathfrak{A}$ is defined as a map from the projections of $\mathfrak{A}$ to the positive reals which satisfies the equation

$$m(\sum P_i)=\sum m(P_i)$$

for every family (P_i) of pairwise orthogonal projections. We prove the following generalization of Gleason's theorem: If m is a G-measure on a type I von Neumann algebra $\mathfrak{A}$ not containing a type I_2 direct summand, then there exists an extension of m to a positive normal linear for on $\mathfrak{A}$.

0. Let H be a complex Hilbert space, $\mathfrak{A}$ a von Neumann algebra acting on H, and $\mathscr{P}(\mathfrak{A})$ the lattice of projections in $\mathfrak{A}$. A map m defined on $\mathscr{P}(\mathfrak{A})$ with values in the positive reals is called a G-measure on $\mathfrak{A}$ if for every orthogonal family (P_i) in $\mathscr{P}(\mathfrak{A})$ the equation $m(\sum P_i) = \sum m(P_i)$ is satisfied. If the equation holds for finite orthogonal families then m is called a finitely additive G-measure.

Gleason proved [3] that for $\mathfrak{A} = L(H)$ (the bounded operators on H) and $\dim H \neq 2$ every G-measure is extendable to a (necessary unique and normal) linear form on $\mathfrak{A}$. A. A. Lodkin generalized this result to all von Neumann algebras acting on a separable Hilbert space and not having a type I_2 direct summand, but the proofs are only sketched and seem to be incomplete. This paper provides a complete proof of Gleason's theorem for those type I von Neumann algebras not having a type I_2 direct summand, so in particular no assumption about the separability of H is made.

THEOREM 1. *Let $\mathfrak{A}$ be a type* I *von Neumann algebra without a type I_2 direct summand. Let m be a G-measure on $\mathfrak{A}$. Then there is an extension of m to a linear form on $\mathfrak{A}$, which is necessarily normal and unique.*

The proof of the theorem is given in a series of lemmas. First we introduce some notation. Throughout the paper let X be a hyperstonean space. For every real or complex Hilbert space K of finite dimension n the space $C(X, K)$ of continuous functions from X to K is an n-dimensional module over $C(X)$. We can easily carry over many concepts known in the Hilbert space K to the module $C(X, K)$ by defining them pointwise, so let an inner product on $C(X, K)$ with values in $C(X)$ be defined by $(f, g)(x) = (f(x), g(x))$. Two elements f and g are said to be orthogonal to each other ($f \perp g$) iff $(f, g) = 0$. A module basis $\{f_1, \cdots, f_m\}$ is called an orthonormal basis if $(f_i, f_j) \equiv \delta_{ij}$. To avoid confusion, we define the function $| \, . \, |$ from $C(X, K)$ to $C(X)$ by $|f|(x) = \|f(x)\|$, leaving $\| \, . \, \|$ for the usual supremum norm on $C(X, K)$.

On $C(X, K)$ we define an equivalence relation $\sim$ by $f \sim g$ iff there is a $\varphi \in C(X)$ such that $\varphi(x) \neq 0$ for all $x \in X$ and $f = \varphi g$. Note that the following are equivalent: (i) $f \perp g$; (ii) there exists $g' \sim g$ and $f \perp g'$; (iii) for all $g' \sim g$ the equation $f \perp g'$ is satisfied.

For the case $K = \boldsymbol{R}^3$ define a cross product $\times$ on $C(X, \boldsymbol{R}^3)$ by $f \times g(x) = f(x) \times g(x)$ (ordinary cross product in $\boldsymbol{R}^3$). By S^2 we denote the usual unit sphere in $\boldsymbol{R}^3$.

Now we are ready to generalize Gleason's notion of a frame function on S^2 to the case of a function on $C(X, S^2)$ and to prove that every frame function on $C(X, S^2)$ is continuous. Note that for X a singleton this is just the essential lemma of Gleason's proof. We have tried to follow as closely as possible the outline of Varadarajan's proof [5]. The proofs of Lemmas 7 and 10 are almost exactly the same but are included for the convenience of the reader since they are short. On the other hand, the remainder of the proof requires substantially different arguments in several places.

DEFINITION 2. A function $\phi: C(X, S^2) \rightarrow \boldsymbol{R}^+$ is called a frame function if there is a positive real constant W, called the weight of ϕ, such that for every orthonormal basis $\{f_1, f_2, f_3\}$ of $C(X, \boldsymbol{R}^3)$ the equation $\phi(f_1) + \phi(f_2) + \phi(f_3) = W$ is satisfied.

So in particular if $f, g \in C(X, S^2)$ and $f \sim g$, then $\phi(f) = \phi(g)$.

NOTATION 3. Let $f \in C(X, S^2)$. Then we define

$$E(f) = \{g \in C(X, S^2) \,|\, g \perp f\} ;$$

$$L(f) = \{g \in C(X, S^2) \,|\, \forall x \in X \forall g' \in C(X, S^2)\colon\ g' \sim g \Longrightarrow f(x) \neq g'(x)\} ;$$

$$N(f) = L(f) \cap \bigcap \{L(h) \,|\, h \in E(f)\} .$$

For f and g in $C(X, S^2)$ we have $E(f) \subseteq L(f)$ and $N(f) \subseteq L(f)$. Furthermore, if one notes that X is hyperstonean, the following are easily seen to be equivalent: (i) $f \sim g$; (ii) $E(f) = E(g)$; (iii) $L(f) = L(g)$; (iv) $N(f) = N(g)$. Now let h be in $L(f)$. Then $|h \times f|$ is strictly positive on X, so $1/|h \times f|$ is in $C(X)$. Define $h \circ f = 1/|h \times f|(h \times f)$; then $h \circ f$ is in both $C(X, S^2)$ and $E(f) \cap E(h)$. In particular $h \in L(h \circ f)$, so $h \circ (h \circ f)$ is defined. We write

$$EW(h, f) = E(h \circ (h \circ f)) .$$

REMARK. The notations $E(f)$, $N(f)$ and $EW(h, f)$ are intended to bring to mind the notions equator with respect to f, northern hemisphere with respect to f and east-west great circle through h of Gleason's proof; the exact analogy breaks down if X is a singleton.

I. In this section we prove the continuity of a frame function.

LEMMA 4. *Let $f \in C(X, S^2)$, and ϕ be a frame function on $C(X, S^2)$ which is constant on $E(f)$. Then for any g in $N(f)$ and any h in $EW(g, f)$,*

$$\phi(g) \leqq \phi(h) + \phi(f) .$$

Proof. Denote by k the constant value of ϕ on $E(f)$, and let W be the weight of ϕ.

(a) For $l \in L(f)$, we claim that $\phi(l) \leqq k + \phi(f)$.
To see this, note that $\{l, l \circ f, (l \circ f) \circ l\}$ and $\{f, l \circ f, (l \circ f) \circ f\}$ are both orthonormal bases. Moreover $l \circ f$ and $(l \circ f) \circ f$ are in $E(f)$, so

$$W = \phi(f) + \phi(l \circ f) + \phi((l \circ f) \circ f) = \phi(f) + 2k$$

and

$$W = \phi(l) + \phi(l \circ f) + \phi((l \circ f) \circ l) \geqq \phi(l) + k .$$

It follows that $\phi(l) \leqq \phi(f) + k$.

(b) For $g \in N(f)$ and $h \in EW(g, f) = E(g \circ (g \circ f))$ define $h' = h \circ (g \circ (g \circ f))$. Then $\{g, g \circ f, g \circ (g \circ f)\}$ and $\{h', h, g \circ (g \circ f)\}$ are orthonormal bases and $g \circ f \in E(f)$, so

$$\phi(h') + \phi(h) = \phi(g) + \phi(g \circ f) = \phi(g) + k . \tag{1}$$

Moreover $h' \in L(f)$. Assume the contrary. Then there is an $x \in X$ such that $h'(x) = \pm f(x)$ so $f(x) \perp g \circ (g \circ f)(x)$; that is, $\{f(x), g \circ f(x), g \circ (g \circ f)(x)\}$ and $\{g(x), g \circ f(x), g \circ (g \circ f)(x)\}$ are orthonormal bases in $\boldsymbol{R}^3$. So $g(x) = \pm f(x)$ which contradicts the fact that g is in $N(f)$.

Since $h' \in L(f)$, part (a) yields the inequality

$\phi(h') \leqq k + \phi(f)$, which, combined with (1), gives
$\phi(h) \geqq \phi(g) - \phi(f)$. This completes the proof.

For the moment let X be a singleton and identify $C(X, S^2)$ with S^2, so the meaning for points in S^2 of the notation introduced in 3 is clear. Using cartesian coordinates in $\boldsymbol{R}^3$, define P to be the plane $\{x_3 = 1\}$ in $\boldsymbol{R}^3$, $N = \{x \in S^2 | 0 < x_3 < 1\}$ and $p = (0, 0, 1) \in S^2$. Denote by Π the central projection with center 0 of N into P. Then $\Pi(p) = p$ and Π is a homeomorphism of N onto $P \backslash \{p\}$ which maps the circles in N with centers p bijectively onto the circles in E with center p. If $K \neq E(p)$ is a great circle in S^2 not containing p then $\Pi(K \cap N)$ is a straight line in P not containing p. The mapping so defined is also a bijection. Moreover for y in N a great circle K is equal to $EW(y, p)$ iff $\Pi(K \cap N)$ is perpendicular to the straight line connecting p and $\Pi(y)$.

For $y \in N$ define $G_y = \Pi(EW(y, p) \cap N)$, and for $s \in P\backslash\{p\}$ define $G_s = G_{\Pi^{-1}(s)}$. Then for $s \in P\backslash\{p\}$, G_s is the unique line in P containing s such that the map $x \to \|x - p\|$ attains its infimum in s.

LEMMA 5. *Let X be a singleton, p be defined as above, $\alpha \in]0, \pi/2[$, and $z = (\cos\alpha, 0, \sin\alpha) \in S^2$. Define K_α to be the open disc in P with center $1/2(\Pi(z) + p)$ and radius $\|1/2(\Pi(z) - p)\|$, and define S_α to be the boundary of K_α. Then, when $s \in K_\alpha$, the intersection of G_s and S_α consists of two points $I_1(s, \alpha)$ and $I_2(s, \alpha)$ where I_1 and I_2 are continuous functions of s and α. Moreover,*

$$z \in EW(\Pi^{-1}(I_i(s, \alpha)), p) \textit{ and } \Pi^{-1}(I_i(s, \alpha)) \in EW(\Pi^{-1}(s), p) .$$

Proof. $\Pi(z)$ is of the form $\Pi(z) = (\rho(\alpha), 0, 1)$, where ρ is continuous. If s is in K_α, then $(s_1 - \rho(\alpha)/2)^2 + s_2^2 < (\rho(\alpha)/2)^2$, so $s_1\rho(\alpha)/(s_1^2 + s_2^2) - 1 > 0$. Now

$$S_\alpha = \{x \in P | x_1^2 + x_2^2 - x_1\rho(\alpha) = 0\}$$

and

$$G_s = \{x \in \boldsymbol{R}^3 | \exists \mu \in \boldsymbol{R} \text{ such that } x = (s_1, s_2, 1) + \mu(-s_2, s_1, 0)\} .$$

Thus for the points s in $G_s \cap S_\alpha$ we have

$$\mu^2 + \mu\left(\frac{s_2\rho(\alpha)}{s_1^2 + s_2^2}\right) - \left(\frac{s_1\rho(\alpha)}{s_1^2 + s_2^2} - 1\right) = 0 ;$$

that is, there are exactly two such points. Let $I_1(s, \alpha)$ (resp. $I_2(s, \alpha)$) be the point corresponding to the positive (resp. negative) square root of μ^2; then I_1 and I_2 are continuous functions of s and α. Since $I_i(s, \alpha)$ is in G_s, it follows that $\Pi^{-1}(I_i(s, \alpha))$ is in $EW(\Pi^{-1}(s), p)$. Let K be the disc in P with center p and radius $\|\Pi(z) - p\|$, and let G be the line containing $\Pi(z)$ and either $I_1(s, \alpha)$ or $I_2(s, \alpha)$, say $I_1(s, \alpha)$. Then $I_1(s, \alpha)$ is the midpoint of $G \cap K$, so $G = G_{I_1(s,\alpha)}$ and $z \in$

$EW(\Pi^{-1}(I_1(s, \alpha), p)$.

LEMMA 6. *Let* $f \in C(X, S^2)$ *and* ϕ *be a frame function defined on* $C(X, S^2)$ *and constant on* $E(f)$. *Let* $g \in N(f)$, *and define* $M_g = \{h \in N(f) \mid \exists l \in N(f) \cap EW(h, f)$ *such that* $g \in EW(l, f)\}$. *Then the interior of* M_g *is nonempty and for all* $h \in M_g$ *we have*:

$$\phi(h) \leqq \phi(g) + 2\phi(f) .$$

Proof. The inequality is an immediate consequence of Lemma 4 and the definition of the set M_g, so we only have to show that the interior of M_g is nonempty.

(a) First we show that we can assume that $f \equiv (0, 0, 1)$. With respect to the canonical module basis of $C(X, \boldsymbol{R}^3)$ we can write $f = (f_1, f_2, f_3)$, where the f_i's are continuous functions on X and $f_1^2 + f_2^2 + f_3^2 = 1$. Define $A = \{x \in X \mid f_3^2 \neq 1\}$, and define $\alpha', \beta' : A \to \boldsymbol{R}$ by $\alpha'(x) = (1 - f_3^2(x))^{-1/2} f_1(x)$ and $\beta'(x) = (1 - f_3^2(x))^{-1/2} f_2(x)$. Then α' and β' are continuous and bounded and X is hyperstonean, so there are continuous extensions α and β on cl A, the closure of A. Using the canonical basis of $\boldsymbol{R}^3$ we define $U' : X \to L(\boldsymbol{R}^3)$ by

$$U'(X) = \begin{cases} \begin{pmatrix} \alpha(x) & \beta(x) & 0 \\ -\beta(x) & \alpha(x) & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{if } x \in \operatorname{cl} A ; \\ \text{Identity map, elsewhere.} \end{cases}$$

Since cl A is open and closed the components U'_{ij} of U' are continuous functions on X. The mapping U' induces an isomorphism U of the vector space $C(X, \boldsymbol{R}^3)$ by $Uh(x) = U'(x)h(x)$. For all h in $C(X, \boldsymbol{R}^3)$ we have $|Uh| = |h|$ and $\|Uh\| = \|h\|$; that is, U is an isometry. Moreover, for all h, l in $C(X, \boldsymbol{R}^3)$ we have that $h \perp l$ iff $Uh \perp Ul$, so U maps $C(X, S^2)$ homeomorphically onto itself, taking M_g onto M_{Ug}. Furthermore Uf is of the form $Uf = ((Uf)_1, 0, f_3)$. Define $W : X \to L(\boldsymbol{R}^3)$ by

$$W(x) = \begin{pmatrix} f_3(x) & 0 & -(Uf)_1(x) \\ 0 & 1 & 0 \\ (Uf)_1(x) & 0 & f_3(x) \end{pmatrix} .$$

Then W has the same properties as U and $W(Uf) \equiv (0, 0, 1)$.

(b) Now we show that we can make the additional assumption that g is of the form $g(x) = (\cos \alpha(x), 0, \sin \alpha(x))$, where α is a continuous function on X with values in $]0, \pi/2[$. Since $f \equiv (0, 0, 1)$, g is in $N(f)$ iff $g_3(x) \in]-1, 0[\cup]0, 1[$ for all x in X. Reasoning as in

(a) we can assume that g is of the form $g = (g_1, 0, g_3)$. Now the function $x \to \operatorname{sign}(g_3(x))$ is a continuous function from X to $\{-1, 1\}$ and $g' = \operatorname{sign}(g_3)\cdot g$ is equivalent to g, so $M_{g'} = M_g$. Moreover we have $0 < g_3'(x) < 1$ for all x in X. Finally define $\alpha\colon X \to]0, \pi/2[$ by $\alpha(x) = \arcsin g_3'(x)$.

(c) Let $\alpha_0 = \sup \alpha(x) \in]0, \pi/2[$, and define $z_0 \in S^2$ by $z_0 = (\cos\alpha_0, 0, \sin\alpha_0)$. Let K be the open disc in the plane P with center $1/2(\Pi(z_0) + p)$ and radius $\|1/2(\Pi(z_0) - p)\|$. Let $B = C(X, \Pi^{-1}(K))$; then B is an open subset of $C(X, S^2)$. Choose $h \in B$; with the notation of Lemma 5, we have $\Pi(h(x)) \in K_{\alpha(x)}$ for every x in X. Define $l\colon X \to S^2$ by $l(x) = \Pi^{-1}(I_1(\Pi(h(x), \alpha(x))))$. It follows from Lemma 5 that $l \in C(X, S^2)$, $g \in EW(l, f)$, and $l \in EW(h, f)$. Moreover $0 < l_3(x) < 1$ and $0 < h_3(x) < 1$ for all x in X, so h is in M_g. We conclude that $U \subseteqq M_g$, which completes the proof of the lemma.

LEMMA 7. *With f and ϕ defined as in Lemma 6, let η be a positive real number with $\phi(f) < \eta$. Then there is a g in $N(f)$ and an open set M containing g such that*

$$0 \leqq \sup \phi(M) - \inf \phi(M) \leqq 3\eta .$$

Proof. (The proof follows exactly the proof of Lemma 7.16, p. 152 of [5].) Define $b = \inf \phi(N(f)) \geqq 0$. Let l be in $N(f)$ such that $b \leqq \phi(l) \leqq b + \eta$. By Lemma 6 we have $\phi(k) \leqq \phi(l) + 2\eta$ for every k in M_l, so $b \leqq \phi(k) \leqq b + 3\eta$. Define M to be the interior of M_l and let g be any point of M.

LEMMA 8. *Let f be in $C(X, S^2)$, $\varepsilon > 0$, and*

$$M(f, \varepsilon) = \{g \in C(X, S^2) \mid \forall f' \sim f \forall x \in X \| g(x) - f'(x) \| > \varepsilon\} .$$

Then there is a constant $k(\varepsilon)$ depending only on ε such that for all h_1, h_2 in $M(f, \varepsilon)$ we have

$$\| h_1 \circ f - h_2 \circ f \| \leqq k(\varepsilon) \| h_1 - h_2 \| .$$

Proof. For x in X and $i = 1, 2$ define $K(x, i)$ to be the great circle in S^2 containing $f(x)$ and $h_i(x)$. Define $M(x, \varepsilon) = \{t \in S^2 \mid \|f(x) - t\| > \varepsilon$ and $\|-f(x) - t\| > \varepsilon\}$. Let $A(x, i)$ be that component of $K(x, i) \cap M(x, \varepsilon)$ which contains $h_i(x)$, and let $v_i(x)$ be the point of intersection of $A(x, i)$ and $E(f(x))$. Then

$$\begin{aligned} \| h_1(x) \circ f(x) - h_2(x) \circ f(x) \| &= \| v_1(x) \times f(x) - v_2(x) \times f(x) \| \\ &= \| v_1(x) - v_2(x) \| . \end{aligned}$$

Moreover there is a constant $k(\varepsilon)$ such that

$$k(\varepsilon)\cdot d(A(x, 1), A(x, 2)) \geqq \| v_1(x) - v_2(x) \| ,$$

so

$$k(\varepsilon) \| h_1(x) - h_2(x) \| \geqq \| v_1(x) - v_2(x) \| = \| h_1(x)\circ f(x) - h_2(x)\circ f(x) \| .$$

Therefore

$$k(\varepsilon) \| h_1 - h_2 \| = \sup k(\varepsilon) \| h_1(x) - h_2(x) \| \geqq \| h_1(x)\circ f(x) - h_2(x)\circ f(x) \|.$$

LEMMA 9. *Let f be in $C(X, S^2)$ and ϕ be a frame function defined on $C(X, S^2)$. Let M be a neighborhood of f such that*

$$0 \leqq \sup \phi(M) - \inf \phi(M) \leqq \alpha .$$

Then for each g in $C(X, S^2)$ there is a neighborhood U_g of g such that

$$0 \leqq \sup \phi(U_g) - \inf \phi(U_g) \leqq 4\alpha .$$

Proof. (a) First we show that for each h in $E(f)$ there is a neighborhood V_h of h such that

$$0 \leqq \sup \phi(V_h) - \inf \phi(V_h) = 2\alpha .$$

As in Lemma 6 one can assume that $f \equiv (0, 0, 1)$ and $h \equiv (0, 1, 0)$. For $\delta \in \boldsymbol{R}$ such that $0 < \delta < 1/10$ and $K_\delta(f) \subseteqq M$, where $K_\delta(f)$ is the open ball with radius δ and center f, define $u \in C(X, S^2)$ by $u \equiv (0, [1 + (3/2)\delta + (2/9)\delta^2]^{-1/2}[1 + (1/3)\delta], \; -(1/3)\delta)$. Define $A = K_{\delta/9}(h)$; then $A \subseteqq M(u, \delta/9)$. So by Lemma 8 we have $\|h_1\circ u - h_2\circ u\| \leqq k(\delta/9) \| h_1 - h_2 \|$ for all h_1 and h_2 in A. Since $h_1\circ u$ and $h_2\circ u$ are elements of $M(h_1, \delta/9)$ and h_1 and h_2 are elements of $M(h_2\circ u, \delta/9)$, we have

$$\begin{aligned} &\| h_1\circ(h_1\circ u) - h_2\circ(h_2\circ u) \| \\ &\quad \leqq \| h_1\circ(h_1\circ u) - h_1\circ(h_2\circ u) \| + \| h_1\circ(h_2\circ u) - h_2\circ(h_2\circ u) \| \\ &\quad \leqq k(\delta/9) \| h_1\circ u - h_2\circ u \| + k(\delta/9) \| h_1 - h_2 \| \\ &\quad \leqq 2(1 + k(\delta/9))^2 \| h_1 - h_2 \| . \end{aligned}$$

This implies that the map $F\colon A \to C(X, S^2)$ defined by $F(l) = l\circ(l\circ u)$ is continuous. Since $F(h) = f$ there is a neighborhood V_h of h such that $V_h \subseteqq A$ and $F(V_h) \subseteqq K_{\delta/9}(f)$. Choose $l \in V_h$. Then l and u are in $E(l\circ u)$, so

$$\| F(l) - u\circ(l\circ u) \| = \| l\circ(l\circ u) - u\circ(l\circ u) \| = \| l - u \| .$$

Thus

$$\|f - u\circ(l\circ u) \| \leqq \delta/3 + \| l - u \| \leqq \delta/3 + \delta/9 + \| h - u \| < \delta ,$$

so $u\circ(l\circ u)$ is in M.

In addition, $\{l, l\circ u, l\circ(l\circ u)\}$ and $\{u, l\circ u, u\circ(l\circ u)\}$ are orthonormal bases, so we have

$$\phi(l) + \phi(l\circ(l\circ u)) = \phi(u) + \phi(u\circ(l\circ u)) .$$

Let l' be another element of V_h. Then

$$\phi(l') + \phi(l'\circ(l'\circ u)) = \phi(u) + \phi(u\circ(l'\circ u)) ,$$

and finally, by subtraction,

$$\begin{aligned}|\phi(l) - \phi(l')| \leqq |\phi(u\circ(l\circ u)) - \phi(u\circ(l'\circ u))| \\ + |\phi(l\circ(l\circ u)) - \phi(l'\circ(l'\circ u))| \leqq 2\alpha .\end{aligned}$$

This completes the proof of (a).

(b) Now let l be in $C(X, S^2)$. As in Lemma 6, we can assume that $f \equiv (0, 0, 1)$ and that l is of the form $l = (l_1, 0, l_3)$. Let $h \equiv (0, 1, 0)$; then $h \in E(f)$ and $l \in E(h)$. The proof of the lemma is completed by applying (a) twice.

LEMMA 10. *Let ϕ be a frame function defined on $C(X, S^2)$. Then ϕ is continuous.*

Proof. (The proof follows exactly the proof of Lemma 7.18, p. 154 of [5].) Let $\varepsilon > 0$. Since every positive constant function is a frame function we can assume that $\inf \phi(C(X, S^2)) = 0$. Let $\eta > 0$, $f \in C(X, S^2)$ be given such that $\phi(f) < \eta/2$. As in Lemma 6 we can assume that $f \equiv (0, 0, 1)$. Define $F: C(X, S^2) \to C(X, S^2)$ by $F((g_1, g_2, g_3)) = (-g_2, g_1, g_3)$. F is a homeomorphism which preserves orthogonality, so $\phi\circ F$ is again a frame function. Define $\psi = \phi + \phi\circ F$. Then ψ is a frame function on $C(X, S^2)$, and is constant on $E(f)$. Moreover $\psi(f) < \eta$. By Lemma 7 there is a g in $N(f)$ and a neighborhood U_g of g such that

$$0 \leqq \sup \psi(U_g) - \inf \psi(U_g) \leqq 3\eta .$$

By Lemma 9 there is a neighborhood U_f of f such that

$$0 \leqq \sup \psi(U_f) - \inf \psi(U_f) \leqq 12\eta ,$$

so

$$0 \leqq \sup \psi(U_f) \leqq 13\eta .$$

Thus

$$0 \leqq \sup \phi(U_f) \leqq 13\eta ,$$

from which it follows that

$$0 \leqq \sup \phi(U_f) - \inf \phi(U_f) \leqq 13\eta .$$

By Lemma 9 there exists for each h in $C(X, S^2)$ a neighborhood U_h of h such that

$$0 \leqq \sup \phi(U_h) - \inf \phi(U_h) \leqq 52\eta .$$

Hence ϕ is continuous.

II. In this section let H be a finite-dimensional complex Hilbert space of dimension $\dim H \geqq 3$ and let S be its unit sphere. Define $\mathscr{P}$ to be the set of all continuous functions on X with values in the projections on H and $\mathscr{P}_1$ to be the set of those members of $\mathscr{P}$ which have their values in the set of all one-dimensional projections. This section is devoted to the relationship between $C(X, S)$ and $\mathscr{P}_1$. For f in $C(X, S)$ define the map $P_f\colon X \to \mathscr{P}(L(H))$ by $P_f(x) = P_{f(x)}$ (the projection on the one-dimensional subspace of H spanned by $f(x)$).

LEMMA 11. *The map $f \to P_f$ is a map from $C(X, S)$ onto $\mathscr{P}_1$.*

Proof. (a) Let $f \in C(X, S)$ and $x, y \in X$. We have

$$\begin{aligned} \| P_f(x) - P_f(y) \| &= \sup \{ \| P_{f(x)}a - P_{f(y)}a \| \mid a \in S \} \\ &= \sup \{ \| (a, f(x))f(x) - (a, f(y))f(y) \| \mid a \in S \} = 2 \| f(x) - f(y) \| , \end{aligned}$$

so P_f is a continuous map, so $P_f \in \mathscr{P}_1$.

(b) Let $P \in \mathscr{P}_1$. For each x in X select an a_x in $(P(x)H) \cap S$ and define f_x to be the constant function which maps X to a_x. Then f_x is in $C(X, S)$ and $\| P(x)f_x(x) \| = 1$, so there is an open and closed neighborhood U_x of x such that $\| P(y)f_x(y) \| > 1/2$ for all y in U_x. Define $g_x\colon U_x \to S$ by $g_x(y) = \| P(y)f_x(y) \|^{-1} P(y)f_x(y)$. Then g_x is continuous and for all $y \in U_x$ we have $P(y) = P_{g_x(y)}$. Since $(U_x)_{x \in X}$ is an open covering of X, there is a finite subcovering $(U_{x_1}, \cdots, U_{x_n})$. Define $g \in C(X, S)$ by

$$g(x) = \begin{cases} g_{x_1}(x) , & \text{if } x \in U_{x_1} ; \\ g_{x_k}(x) , & \text{if } x \in U_{x_k} \backslash \cup \{ U_{x_i} \mid i < k \} . \end{cases}$$

Then we have $P = P_g$.

LEMMA 12. *Let n be a natural number and let Q be in $\mathscr{P}$ such that* $\dim Q(x) > n$ *for all x in X. Let $f_1, \cdots, f_n$ in $C(X, S)$ be pairwise orthogonal and such that $P_{f_i} \leqq Q$. Then there is a g in $C(X, S)$ such that $g \perp f_i$ $(1 \leqq i \leqq n)$ and $P_g \leqq Q$.*

Proof. For each x in X select a_x in $(Q(x)H) \cap S$ such that

$a_x \perp f_i(x)$ $(1 \leqq i \leqq n)$. Define g'_x to be the constant mapping from X to a_x. Then g'_x is in $C(X, S)$ and satisfies

$$\| Q(x)g'_x(x) \| = 1$$

and

$$\sum_{i=1}^{n} (f_i(x), g'_x(x))^2 = 0 .$$

Hence there exists an open and closed neighborhood U_x of x such that for $y \in U_x$ we have

$$\| Q(y)g'_x(y) \| > 1/2$$

and

$$\sum_{i=1}^{n} (f_i(y), Q(y)g'_x(y))^2 < 1/4 .$$

Define h_x: $U_x \to H$ by

$$h_x(y) = Q(y)g'_x(y) - \sum_{i=1}^{n} (Q(y)g'_x(y), f_i(y))f_i(y)$$

and g_x: $U_x \to H$ by

$$g_x(y) = \| h_x(y) \|^{-1} h_x(y) .$$

As in the proof of Lemma 11 one can define a function g with the desired properties.

LEMMA 13. *For f_1 and f_2 in $C(X, S)$ there exists a ρ in $C(X, \boldsymbol{C})$ such that $(f_1, \rho f_2)$ is in $C(X, \boldsymbol{R})$.*

Proof. Define $A = \{x \in X \mid (f_1(x), f_2(x)) \neq 0\}$. Then cl A is open and closed. Define ρ': $A \to \boldsymbol{C}$ by

$$\rho'(x) = |(f_1(x), f_2(x))|^{-1}(f_1(x), f_2(x)) .$$

Then ρ' is continuous and bounded on A; since X is a hyperstonean space there is a (unique) continuous extension ρ'' of ρ' to cl A. Finally define $\rho \in C(X, \boldsymbol{C})$ by $\rho = \rho''$ on A and $\rho \equiv 1$ on the complement of cl A.

LEMMA 14. *Let P_1, P_2 be in $\mathscr{P}_1$. Then there are functions f_1, f_2, f_3 in $C(X, S)$ such that the following hold:*

(i) *f_1, f_2, f_3 are pairwise orthogonal;*

(ii) *$P_1 = P_{f_1}$;*

(iii) *there is an f in $C(X, S)$ which is contained in the real submodule of $C(X, H)$ spanned by f_1 and f_2 and such that $P_2 = P_f$.*

Proof. By Lemma 11 there are functions g_1 and g_2 in $C(X, S)$ such that $P_1 = P_{g_1}$ and $P_2 = P_{g_2}$. Define $f_1 = g_1$. By Lemma 13 there is a ρ in $C(X, \boldsymbol{C})$ such that $(f_1, \rho g_2)$ is in $C(X, \boldsymbol{R})$; define $f = \rho g_2$. Then $P_2 = P_f$. Let $A = \{x \in X \mid (f_1, f) \neq 1\}$ and define $f_2' \colon A \to S$ by

$$f_2'(x) = \|(f - (f, f_1)f_1)(x)\|^{-1} (f - (f, f_1)f_1)(x) .$$

Then f_2' is continuous and bounded. H is finite-dimensional, so the fact that X is hyperstonean implies the existence of a continuous extension f_2'' of f_2' to the open and closed subset cl A of X. By Lemma 12 there exists an h in $C(\complement \operatorname{cl} A, S)$ such that h is orthogonal to the restriction of f_1 to the complement of cl A. Now define f_2 to be f_2'' on cl A and h on the complement of cl A. Then $f_1 \perp f_2$ and $f = (f, f_1)f_1 + (f, f_2)f_2$. Moreover, (f, f_2) is in $C(X, \boldsymbol{R})$, so f is contained in the real module spanned by f_1 and f_2. Again by Lemma 12 there is an f_3 having the desired properties.

III. In this section let $\mathfrak{A}$ be a von Neumann algebra of type I_n where $n \in N$ and $n \geqq 3$. Then by [1], pp. 239–240, $\mathfrak{A}$ is isomorphic to $\mathfrak{B} \otimes L(H)$, where $\mathfrak{B}$ is an abelian von Neumann algebra, H is an n-dimensional complex Hilbert space and $\otimes$ denotes the tensor product of von Neumann algebras. In particular, $\mathfrak{B}$ is isomorphic to a space $C(X, \boldsymbol{C})$, where X is hyperstonean. So by [1], p. 24, Proposition 4, (ii) (and the fact that $\dim H = n < \infty$), $\mathfrak{A}$ is isomorphic to $C(X, L(H))$. We identify $C(X, L(H))$ with $\mathfrak{A}$ and $C(X, \boldsymbol{C})$ with the center $\mathfrak{B}$ of $\mathfrak{A}$.

If m is a finitely additive G-measure on $\mathfrak{A}$ then the restriction of m to $\mathfrak{B}$ induces a finitely additive measure on the open and closed subsets of X and also induces a unique continuous positive linear form on $\mathfrak{B}$, denoted by $R(m)$.

Denote by M the set of all finitely additive G-measures m on $\mathfrak{A}$ with $m(\mathrm{Id}) = 1$, and let M_1^+ denote the set of all positive linear forms on $\mathfrak{B}$ with norm 1. Then R is a map from M onto M_1^+. Define $N = R^{-1}(\operatorname{ex} M_1^+)$ (ex A the extreme points of A). For $m \in M$ denote by $S(m)$ the support (in X) of $R(m)$. If m is in N we also denote the single element of $S(m)$ by $S(m)$.

LEMMA 15. *M is a compact convex subset of $\boldsymbol{R}^{\mathscr{P}(\mathfrak{A})}$ and N is a closed subset of M containing* ex M.

Proof. (a) M is a closed convex subset of $[0, 1]^{\mathscr{P}(\mathfrak{A})}$, so it is compact and convex.

(b) Equip M_1^+ with the w^*-topology of $\mathfrak{B}^* = C(X, \boldsymbol{C})^*$. We show that R is continuous, from which it follows that N is a closed subset of M.

Let $(m_i)_{i\in I}$ be a net in M converging to m, so $(m_i(P))$ converges to $m(P)$ for all $P \in \mathscr{P}(\mathfrak{A})$. Now for $P \in \mathscr{P}(\mathfrak{B})$ we have $m(P) = R(m)(P)$, so for all step functions T in $\mathfrak{B} = C(X, \boldsymbol{C})$ we have $R(m_i)(T) \to R(m)(T)$. Let $f \in \mathfrak{B}$ and $\varepsilon > 0$ be given. There is a step function $T \in \mathfrak{B}$ such that $\|f - T\| \leqq \varepsilon/3$ and an $i_0 \in I$ such that $|R(m_i)(T) - R(m)(T)| < \varepsilon/3$ for all $i \geqq i_0$. So for all $i \geqq i_0$ we have

$$|R(m)(f) - R(m_i)(f)| \leqq |R(m)(f) - R(m)(T)| + |R(m)(T) - R(m_i)(T)| + |R(m_i)(T) - R(m_i)(f)| < \varepsilon .$$

Thus $R(m_i)(f) \to R(m)(f)$, so R is continuous.

(c) Let $m \in \operatorname{ex} M$, and assume that $R(m)$ is not in $\operatorname{ex} M_1^+$. Then there exist $P_1, P_2 \in \mathscr{P}(\mathfrak{B})$ such that $P_1 + P_2 = \mathrm{Id}$ and $\lambda = m(P_1) \in]0, 1[$. Define m_1 and m_2 by $m_1(Q) = \lambda^{-1}m(P_1Q)$ and $m_2(Q) = (1-\lambda)^{-1}m(P_2Q)$. Then $m = \lambda m_1 + (1 - \lambda)m_2$ is a nontrivial convex decomposition of m, which contradicts the assumption. We conclude that $\operatorname{ex} M \subseteqq N$.

LEMMA 16. *Let m be in M, let U be an open and closed subset of X containing $S(m)$, and let $Q = 1_U$ (the indicator function of U); so $Q \in \mathfrak{B}$. Let P_1, P_2 be in $\mathscr{P}(\mathfrak{A})$ such that $P_1Q = P_2Q$, Then $m(P_1) = m(P_2)$.*

Proof. $S(m)$ is the support of $R(m)$, so $m(Q) = 1$ and $m(1 - Q) = 0$. Thus for all $P \in \mathscr{P}(\mathfrak{A})$ we have

$$m(P) = m(PQ) + m(P(1 - Q)) = m(PQ) .$$

LEMMA 17. *Let m be in N and let $P_1, P_2 \in \mathscr{P}_1$ be such that $P_1(S(m)) = P_2(S(m))$. Then $m(P_1) = m(P_2)$.*

Proof. Choose f_1, f_2, f_3 and f as in Lemma 14. Let g_1, g_2, g_3 be the canonical module basis of $C(X, \boldsymbol{R}^3)$. Define $I: C(X, \boldsymbol{R}^3) \to C(X, H)$ by $I(h) = (h, g_1)f_1 + (h, g_2)f_2 + (h, g_3)f_3$. Then I is an isometry of $C(X, \boldsymbol{R}^3)$ onto the real module spanned by $\{f_1, f_2, f_3\}$, and I maps $C(X, S^2)$ into $C(X, S)$ (where S is the unit sphere of H). Define a frame function ϕ on $C(X, S^2)$ by $\phi(f) = m(P_{I(f)})$. Now $P_1(S(m)) = P_2(S(m))$, so either $f_1(S(m)) = f(S(m))$ or $f_1(S(m)) = -f(S(m))$; we can assume that $f_1(S(m)) = f(S(m))$ (otherwise take $-f$ instead of f). Define $h = I^{-1}(f)$. Then we have $g_1(S(m)) = h(S(m))$. For each $n \in \boldsymbol{N}$ there exists an open and closed neighborhood U_n of $S(m)$ such that

$$\sup \{\| g_1(x) - h(x) \| \,|\, x \in U_n\} < 1/n .$$

Define $h_n \in C(X, S^2)$ to be h on U_n and g_1 on the complement. Then

$\|h_n - g_1\| < 1/n$, so h_n converges to g_1. For $x \in U_n$ we have $h(x) = h_n(x)$ and so also $f(x) = I(h)(x) = I(h_n)(x)$. By Lemma 16, $m(P_2) = m(P_{I(h_n)}) = \phi(h_n)$ for all $n \in N$. Moreover, $m(P_1) = m(P_{f_1}) = \phi(g_1)$. So by Lemma 10 we conclude that $m(P_1) = \phi(g_1) = \lim \phi(h_n) = \phi(h) = m(P_2)$.

For $P \in \mathscr{P}(\mathfrak{A})$ and $x \in X$ define $P_x \in \mathscr{P}(\mathfrak{A})$ by $P_x(y) = P(x)$ for all y in X.

LEMMA 18. *Let m be in N and $x = S(m)$. Then $m(P) = m(P_x)$ for all $P \in \mathscr{P}(\mathfrak{A})$.*

Proof. The map $d: X \to N$, defined by $d(x) = \text{trace } P(x) = \dim P(x)$ is continuous, so there is an open and closed neighborhood U of x such that d is constant on U. By Lemma 16 we may assume $U = X$. If $d \equiv 0$, then $P = 0$ and $0 = m(P) = m(P_x)$. Assume that $d \equiv n \in N$. By Lemma 12 there are $P_1, \cdots, P_n \in \mathscr{P}_1$ which are pairwise orthogonal and such that $P = \sum_{i=1}^n P_i$. Thus $m(P) = \sum_{i=1}^n m(P_i)$, and we may apply Lemma 17 to complete the proof.

LEMMA 19. *Let m be in N. Then there is a unique state f_m on $\mathfrak{A}$ extending m. The map $m \to f_m$ is continuous from N to the set S of all states of $\mathfrak{A}$ when S is equipped with the w^*-topology of the dual of $\mathfrak{A}$.*

Proof. (a) Denote by m' the restriction of m to $\boldsymbol{C} \cdot 1_X \otimes L(H) \cong L(H)$. Then m' is a G-measure on $L(H)$. Define h_m to be the unique extension of m' to a state on $L(H)$. Define $I: \mathfrak{A} \to L(H)$ by $I(P) = P(S(m))$. Then I is a representation of $\mathfrak{A}$, and $f_m = h_m \circ I$ is a state on $\mathfrak{A}$. We show that f_m extends m. Let $P \in \mathscr{P}(\mathfrak{A})$. Then we have $f_m(P) = h_m \circ I(P) = h_m(P(S(m))) = m'(P(S(m))) = m(P_{S(m)}) = m(P)$, the last equality being justified by Lemma 18. Since the dimension of H is finite, the linear combinations of elements must be dense in $\mathfrak{A}$ in the norm topology, so f_m is unique.

(b) Let $(m_i)_{i \in I}$ be a net in N converging to m. Then (m_i') converges to m'. As in the proof of Lemma 15 one can show that (h_{m_i}) converges to h_m and that (f_{m_i}) converges to f_m.

LEMMA 20. *Let m be in M. Then there is a unique state f_m of $\mathfrak{A}$ extending m.*

Proof. By Choquet's theorem there is a maximal measure μ on M which represents m. By Lemma 15 the support of μ is in N. This and Lemma 19 suffice to settle the question of existence of an extension. The uniqueness follows as in Lemma 19.

IV. Proof of the theorem. By [1], p. 240, second paragraph and Proposition 2, $\mathfrak{A}$ is of the form $\mathfrak{A} = \Pi\mathfrak{A}_n$, where the $\mathfrak{A}_n$ are of type I_n and n is out of a fixed set of cardinals. By the assumption $n \neq 2$ for all n. Since m is completely additive we can assume that $\mathfrak{A}$ is of type I_n and $n \neq 2$. If $n = 1$, then $\mathfrak{A}$ is commutative, so we may assume that $n \geqq 3$.

We first prove the following claim:

(2) If $P \in \mathscr{P}(\mathfrak{A})$ is a finite projection, then the restriction m_P of m to $P\mathfrak{A}P$ has a unique extension to a state on $P\mathfrak{A}P$.

Let $P \in \mathscr{P}(\mathfrak{A})$ be finite, that is, $P\mathfrak{A}P$ is a finite type I von Neumann algebra. As above we may restrict our attention to the case in which $P\mathfrak{A}P$ is of type I_n, where now n is a natural number. The case $n = 1$ is the commutative case and for $n \geqq 3$ the claim follows by Lemma 20. Only the case $n = 2$ remains. Let $P\mathfrak{A}P$ be of type I_2. Then there are projections P_1, P_2 in $P\mathfrak{A}P$ such that P_1 and P_2 are abelian relative to $P\mathfrak{A}P$, P_1 is orthogonal to P_2, P_1 is equivalent to P_2 relative to $P\mathfrak{A}P$, and $P_1 + P_2 = \mathrm{Id}_{P\mathfrak{A}P} = P$. Since $P_iP = PP_i = P_i$ we have that P_1 and P_2 are abelian relative to $\mathfrak{A}$ and since $P\mathfrak{A}P \subseteqq \mathfrak{A}$ they are equivalent relative to $\mathfrak{A}$ as well. Denote by $Z(Q)$ the central support of $Q \in \mathscr{P}(\mathfrak{A})$. Because of the additivity of m we may assume that $Z(P) = 1$. Let $Q = 1 - P$. Then by [1], p. 218, Théorème 1, there exists a projection G in $\mathfrak{B}$ such that

(a) $QG \prec P_1G$, and (b) $P_1(1 - G) \prec Q(1 - G)$.

By (a) and [1], p. 123, Définition 3 and the succeeding paragraph there must be a projection H in $\mathfrak{B}$ such that $H \leqq G$ and $QG \sim P_1H$. Then $Z(QG) = Z(P_1H) = Z(P_1)H = H$, so $Q(G - H) = 0$ or $P(G - H) = G - H$. Now $P\mathfrak{A}P$ is of type I_2 and $\mathfrak{A}$ is of type I_n where $n > 2$, so $G - H$ has to be 0. So we have $QG \sim P_1G$.

By (b) there is a $Q_1 \leqq Q(1 - G)$ such that $P_1(1 - G) \sim Q_1$. Let $P_3 = QG + Q_1$. Then $P_3 \sim P_1$, so P_3 is abelian and P_1, P_2 and P_3 are pairwise orthogonal. Define $P' = P_1 + P_2 + P_3$. Thus $P'\mathfrak{A}P'$ is of type I_3 and contains $P\mathfrak{A}P$, so again we may apply Lemma 20. This completes the proof of (2).

Now define $F' = \cup \{P\mathfrak{A}P \mid P \in \mathscr{P}(\mathfrak{A}),\ P \text{ finite}\}$, and let F be the vector subspace of $\mathfrak{A}$ which is spanned by F' and 1; that is, $F = F' \cup \{\lambda(1 - A) \mid \lambda \in \boldsymbol{C} \text{ and } A \in F'\}$. Let A be in F and P_1, P_2 be finite projections in $\mathscr{P}(\mathfrak{A})$; denote by Q their supremum in $\mathscr{P}(\mathfrak{A})$, which is again a finite projection. Then we have:

(i) If $A = P_1AP_1 = P_2AP_2$, then $A = QAQ$ and $f_{P_1}(A) = f_{P_2}(A) = f_Q(A)$.

(ii) If $A = P_1AP_1 = \lambda(1 - B)$ where $B = P_2BP_2$, then either $A = 0$ or $1 \in Q\mathfrak{A}Q$, so $f_{P_1}(A) = f_Q(A) = \lambda(1 - f_{P_2}(B))$.

(iii) If $A = \lambda_1(1 - P_1B_1P_1) = \lambda_2(1 - P_2B_2P_2)$ then $\lambda_1(1 - f_{P_1}(P_1B_1P_1)) = \lambda_2(1 = f_{P_2}(P_2B_2P_2))$. These properties of the family (f_P) allow us to define a linear form f' on F by

$$f'(A) = \begin{cases} f_P(A) , & \text{if } A \in P\mathfrak{A}P ; \\ \lambda(1 - f_P(B)) , & \text{if } A = \lambda(1 - B) \text{ and } B \in P\mathfrak{A}P . \end{cases}$$

We may assume that $m(1) = 1$. Then we have the following:

(i) F is self-adjoint and $1 \in F$;

(ii) $f'(1) = 1$;

(iii) $f'(A) \geqq 0$ for all $A \in F \cap \mathfrak{A}^+$;

(iv) $f'(A^*) = \overline{f'(A)}$ for all $A \in F$.

We only show that (iii) is true for $A = \lambda(1 - B)$ and $\lambda < 0$. In this case $B \geqq 1$, so $\operatorname{supp} B \geqq 1$. Now $B = PBP$ for some finite P; that is, $1 = P1P$. So $F = F' = \mathfrak{A}$ and $f'(A) \geqq 0$. By [2], p. 50, Lemme 2.10.1, the conditions (i)-(iv) ensure that there is a state f on $\mathfrak{A}$ extending f'. It remains to prove that f is an extension of m; the normality of f is then an immediate consequence of the complete additivity of m.

Let $P \in \mathscr{P}(\mathfrak{A})$. Then there exists a family $(P_\iota)_{\iota \in I}$ of pairwise orthogonal and finite projections in $\mathfrak{A}$ and a subset $J \subseteqq I$ such that

$$\sum_{\iota \in I} P_\iota = 1 \text{ and } \sum_{\iota \in J} P_\iota = P .$$

Then we have

$$f(P) \geqq \sum_J f(P_\iota) = \sum_J m(P_\iota) = m(P) . \tag{3}$$

$$f(1 - P) \geqq \sum_{I \setminus J} f(P_\iota) = m(1 - P) ,$$

so

$$f(P) = 1 - f(1 - P) \leqq 1 - m(1 - P) = m(P) . \tag{4}$$

By (3) and (4) we have $f(P) = m(P)$, so f is an extension of m and the proof is completed.

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