

INVARIANCE OF PROPERTIES UNDER AUTOMORPHISMS OF THE LATTICE OF RECURSIVELY ENUMERABLE SETS

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1. **Introduction.** An important question about the structure of \mathcal{E} , the lattice of r.e. sets, is the study of which properties of r.e. sets are *invariant* under $\text{Aut } \mathcal{E}$, the group of automorphisms of \mathcal{E} . Many of the more important properties of r.e. sets are easily seen to be invariant. Simplicity, maximality, and r -maximality are invariant since the usual definitions are in terms of the lattice operations of \mathcal{E} . Hyperhypersimplicity is invariant since Lachlan has characterized the hyperhypersimple sets as those which have Boolean algebras for their lattices of r.e. supersets. D. A. Martin has shown that hypersimplicity is not invariant. We show in this paper that dense simplicity is not invariant.

DEFINITION 1.1. (a) If $B \subseteq N$, p_B is the function that enumerates the elements of B in increasing order; i.e., $p_B(n) = n$ th element of B .

(b) A coinfinite r.e. set A is *hypersimple* (Post) if $p_{\bar{A}}$ is not dominated by any total recursive function (i.e., for each recursive function f there are infinitely many n for which $f(n) < p_{\bar{A}}(n)$).

(c) A coinfinite r.e. set A is *dense simple* (Martin), if $p_{\bar{A}}$ dominates every total recursive function. (It is clear that if A is dense simple, A is hypersimple.)

THEOREM 1.2. *There is a dense simple set A , an r.e. set B which is not hypersimple, and an automorphism Φ of \mathcal{E} such that $\Phi(A) = B$.*

COROLLARY 1.3. *Dense simplicity is not invariant under $\text{Aut } \mathcal{E}$.*

COROLLARY 1.4 (D. A. Martin). *Hypersimplicity is not invariant under $\text{Aut } \mathcal{E}$.*

A proof of Martin's theorem as well as a summary of most of what is known about the automorphisms of \mathcal{E} may be found in [5]. Martin's proof was a finite injury priority argument. In [5], however, Soare gives a powerful new technique of the infinite injury type for constructing automorphisms and uses this to prove that the maximal sets form an orbit of $\text{Aut } \mathcal{E}$. It is this technique that

we use and extend to prove Theorem 1.2.

In §2 we give the basic strategy for meeting each of the requirements of the theorem. In §3 we describe the “pinball machines” which give the basic combinatorial picture of the proof and we give the basic construction. In §4 we give the rules describing the movement of “balls” on the “pinball machines” and thereby determining the sets A, B and the automorphism Φ . In §5 we carry out the verification that the construction works. In §6 we discuss some open problems suggested by this theorem.

Our notation is for the most part standard; a reference is Rogers [3]. We identify a set with its characteristic function and let $A[x]$ denote the restriction of A to arguments $\leq x$. N always denotes the set of nonnegative integers. Let \mathcal{E}^* denote the lattice \mathcal{E} modulo \mathcal{F} where \mathcal{F} is the ideal of finite sets. If $A, B \in \mathcal{E}$, $A^*, B^* \in \mathcal{E}^*$ denote the equivalence classes of A, B . We let $A^* = B^*$ be denoted by $A =^* B$ (i.e., the symmetric difference of A and B is finite); $A \subseteq^* B$ denotes $A \cap \bar{B} =^* \phi$; and $B \subset_\infty A$ denotes $A - B$ is infinite. Often we will (implicitly) confuse A and A^* by choosing a particular representative of A^* . A *recursive array* is a recursive sequence of r.e. sets. A *simultaneous enumeration* of a given recursive array $\{X_n\}_{n \in N}$ is a 1:1 recursive function g with range $\langle m, n \rangle : m \in X_n \rangle$. Thus at each state s , $g(s) = \langle m, n \rangle$ enumerates one element m in one r.e. set X_n . If g is assumed, $X_{n,s}$ denotes the set of elements enumerated in X_n by the end of stage s , $X_n \setminus X_m = \{x : \exists s(x \in X_{n,s} - X_{m,s})\}$, and $X_n \setminus X_m = (X_n \setminus X_m) \cap X_m$. The e -state of an element x at stage s is $\sum_{i \leq e; x \in X_{i,s}} 2^{-i}$ and is denoted by $\sigma(s, e, x)$. We sometimes identify $\sigma(s, e, x)$ with $\{i \leq e : x \in X_{i,s}\}$.

2. The requirements. The strategy for making A dense simple is straightforward. If $\{\varphi_i\}_{i \in N}$ is an effective listing of all (partial) recursive functions, we simply insure that $\varphi_i(j) < p_{\bar{A}}(n)$ for all $i, j \leq n$. Note that the only negative requirement on A is to make \bar{A} infinite; we have considerable freedom in enumerating elements in A beyond those necessary to insure $p_{\bar{A}}$ dominant.

To guarantee that B is not hypersimple, we will use an alternative characterization of hypersimplicity. If $x_1 < x_2 < \dots < x_k$ and $y = 2^{x_1} + \dots + 2^{x_k}$, let D_y denote the finite set $\{x_1, \dots, x_k\}$. A sequence of r.e. sets $\{D_{h(n)}\}_{n \in N}$ determined by a recursive function h is a *strong array*. Kuznecov, Medvedev, and Uspenskii (see [3]) have shown that our definition of hypersimplicity is equivalent to the nonexistence of a strong array $\{D_{h(n)}\}_{n \in N}$ of pairwise disjoint sets such that $D_{h(n)} \cap \bar{B} \neq \phi$ for all n . Fix a strong array $\{D_{h(n)}\}_{n \in N}$ of disjoint sets such that $|D_{h(n)}| = n + 1$ and $\bigcup_n D_{h(n)} = N$. We

will arrange that $|D_{h(n)} \cap B| \leq n$. This requirement is clearly a negative requirement on B .

We will not actually construct $\Phi \in \text{Aut } \mathcal{E}$ so that $\Phi(A) = B$. Rather, we will construct $\Phi \in \text{Aut } \mathcal{E}^*$ such that $\Phi(A^*) = B^*$. Soare shows [5, Corollary 1.5] that this is enough. Soare also notes [5, p. 85] that all known constructions of $\Phi \in \text{Aut } \mathcal{E}^*$ are accomplished by constructing a permutation p of N inducing Φ (i.e., $p(C) = {}^*\Phi(C)$) and functions f and g so that

$$(1) \quad \Phi(W_n^*) = W_{f(n)}^* \quad \text{and} \quad \Phi^{-1}(W_n^*) = W_{g(n)}^* .$$

Of course p guarantees that Φ preserves inclusion and f and g guarantee that Φ and Φ^{-1} map \mathcal{E}^* to \mathcal{E}^* . To meet the two different requirements of (1), we will enumerate two recursive arrays $\{V_n\}_{n \in N}$ and $\{U_n\}_{n \in N}$ such that $U_n = {}^*V_n = {}^*W_n$ for all n and we will simultaneously enumerate recursive arrays $\{\hat{U}_n\}_{n \in N}$ and $\{\hat{V}_n\}_{n \in N}$ so that:

$$(2) \quad p(U_n) = {}^*\hat{U}_n \quad \text{and} \quad p^{-1}(V_n) = {}^*\hat{V}_n .$$

The requirements (2) can be naturally split into two parts corresponding to A and \bar{A} . We will satisfy for each n both

$$(3) \quad p(\bar{A} \cap U_n) = {}^*\bar{B} \cap \hat{U}_n \quad \text{and} \quad p^{-1}(\bar{B} \cap V_n) = {}^*(\bar{A} \cap \hat{V}_n) ,$$

and

$$(4) \quad p(A \cap U_n) = {}^*(B \cap \hat{U}_n) \quad \text{and} \quad p^{-1}(B \cap V_n) = {}^*(A \cap \hat{V}_n) .$$

Let $\hat{U}_n = \hat{U}_n^+ \cup \hat{U}_n^-$ where $\hat{U}_n^+ = \hat{U}_n \setminus B$ and $\hat{U}_n^- = B \setminus \hat{U}_n$. Similarly let $\hat{V}_n = \hat{V}_n^+ \cup \hat{V}_n^-$ where \hat{V}_n^+ and \hat{V}_n^- are defined with A in place of B . Condition (3) causes us only to enumerate elements in the sets \hat{U}_n^+ and \hat{V}_n^+ . Condition (4) should only require us to enumerate elements in \hat{U}_n^- and \hat{V}_n^- . However, these requirements are not quite independent as $\hat{U}_n^+ \setminus B \neq \emptyset$ and $\hat{V}_n^+ \setminus A \neq \emptyset$. To see how this might affect meeting (4) in our case, consider the following situation. Suppose $|V_n \cap \bar{B}|$ is large so that we must enumerate many elements of \bar{A} in \hat{V}_n^+ . Later, to make A dense simple, many of these elements must be enumerated in A . But then we must enumerate many elements of $V_n \cap \bar{B}$ into B to meet (4), but these might be restrained by the requirements to make B nonhypersimple. It is easy to see here that the positive requirement on B is $|V_n^+ \setminus A| = \infty \Rightarrow |V_n \setminus B| = \infty$.

In general, Soare gives necessary and sufficient conditions [5, Theorem 2.2, p. 91] on the enumeration of A and B so that the permutation p guaranteeing (4) exists. We will state these conditions precisely in §5. We will control the enumerations of A and B to meet Soare's conditions. Thus, (4) will be met by [5, Theorem

2.2]. We remark here that the requirements of Soare on the enumeration of A and B can be phrased as purely positive requirements. Thus they conflict with the requirements on B , which are negative, but only conflict with the requirements on A to make \bar{A} infinite.

We are thus left only with the requirements (3). Namely, we must construct a permutation $p|_{\bar{A}}$ and enumerate r.e. sets $\{\hat{U}_n^+\}_{n \in N}$ and $\{\hat{V}_n^+\}_{n \in N}$ so that

$$(5) \quad p(\bar{A} \cap U_n) = *(\bar{B} \cap \hat{U}_n^+) \quad \text{and} \quad p^{-1}(\bar{B} \cap V_n) = *(\bar{A} \cap \hat{V}_n^+).$$

Now these requirements are the same requirements that Soare had to meet in proving [5, Theorem 2.2] but for $A, B, \hat{U}_n, \hat{V}_n$ in place of $\bar{A}, \bar{B}, \hat{U}_n^+, \hat{V}_n^+$. Our construction of p is therefore similar to Soare's. There is one complication increasing the difficulty of our task, however. In Soare's case, since A and B are r.e., no elements "leave" the domain of p and p^{-1} . Here, however, we are constantly enumerating elements in A to make A dense simple and elements in both A and B to satisfy Soare's conditions for (4) to hold. The reader familiar with [5, Theorem 2.2] will see that this is the main new feature of our proof and that it adds considerable complexity to the construction.

The first example of an argument in which requirements (5) are met is in [6] where Soare proves that the lattice of supersets of a low r.e. set $A(A' \equiv_{\tau} \emptyset')$ is isomorphic to \mathcal{E} . The main technical device for meeting such requirements introduced by him is the assignment of certain priorities so that the domain of p settles down. It is this device which we use in a different way to meet the requirements. We have written this paper and [6] simultaneously, so as to develop a common framework for automorphism arguments which feature requirements of the form (5). While writing this paper we have benefitted from numerous suggestions of W. Maass.

3. The construction. We assume that we are given a recursive function g which enumerates simultaneously $\{W_n\}_{n \in N'}$, $\{\hat{W}_n\}_{n \in N'}$ and computations $\varphi_{i,s}(j)$, $i, j, s \in N$, where $\{W_n\}_{n \in N}$ and $\{\hat{W}_n\}_{n \in N}$ are standard enumerations of the r.e. sets. We will also assume for convenience that $W_0 = \hat{W}_0 = N$ and g enumerates $x \in W_0(\hat{W}_0)$ before enumerating $x \in W_e(\hat{W}_e)$ for any $e > 0$. Further, we assume that, if $x \in W_e$, x is enumerated in W_e at infinitely many different stages.

We will enumerate in stages recursive arrays $\{U_n\}_{n \in N'}$, $\{V_n\}_{n \in N'}$, $\{\hat{U}_n^+\}_{n \in N'}$, $\{\hat{V}_n^+\}_{n \in N'}$ and r.e. sets A and B and construct a permutation p so that all the requirements described in §2 are met.

We will, like Soare, present the proof as a construction on pinball machines. The *pinball machine* M is shown in Figure 1; \hat{M} is identical except that each symbol X is replaced by \hat{X} . Pocket \hat{A} will also be referred to as pocket B . Two copies of N , $\{n\}_{n \in N}$ and $\{\hat{n}\}_{n \in N}$ act as the *balls* of M and \hat{M} . A ball x (\hat{x}) enters M (\hat{M}) from hole H_1 (\hat{H}_1). It then proceeds along the *surface* of the machine, that portion of the machine covered by arrows, until it reaches a pocket. From a pocket, x (\hat{x}) may *re-enter* the machine from hole H_2, H_3, H_4 .

We now describe what the various parts of the machines are for. Balls entering pocket A (\hat{A}) represent numbers enumerated in A (\hat{B}). Balls elsewhere in the machine M (\hat{M}) are numbers in \bar{A} (\bar{B}). Thus no ball is allowed to leave pocket A (\hat{A}). When x (\hat{x}) is at gate G_2 (\hat{G}_2) we determine whether to enumerate $x \in A$ (\hat{B}) to meet the conditions of the Extension Theorem of Soare [5, Theorem 2.2]. Hole H_4 is used to place $x \in A$ to make A dense simple. The rest of the machine is used to guarantee (5). Before describing

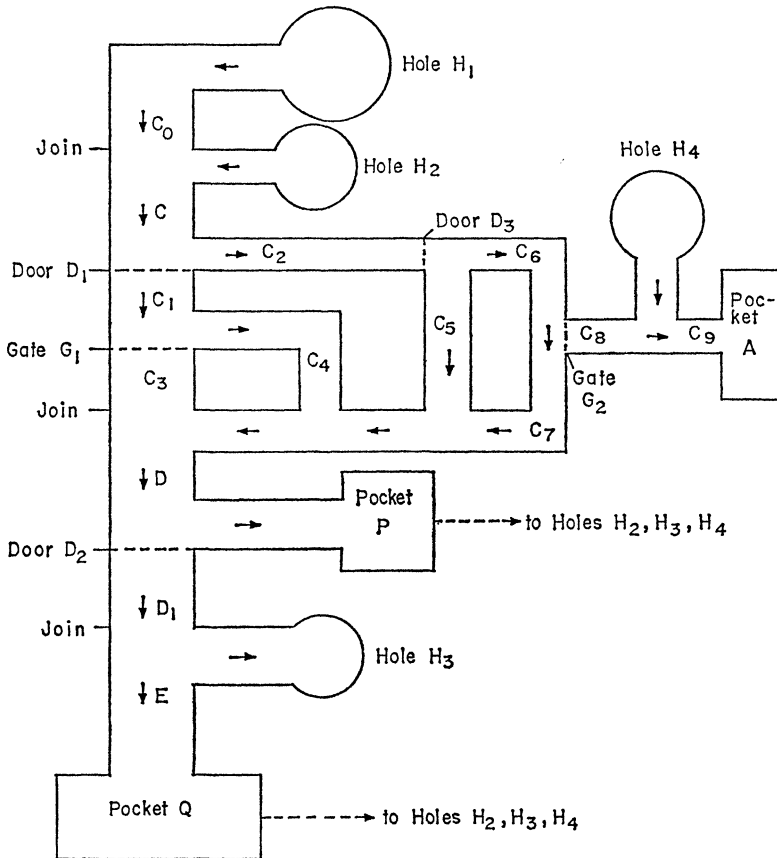


FIGURE 1

how this is done we need some notation and definitions.

If $x \in M_s$, the full e -state of x at stage s denoted $\nu(s, e, x)$ is the triple $\langle e, \sigma(s, e, x), \tau(s, e, x) \rangle$ where $\sigma(s, e, x)$ and $\tau(s, e, x)$ denote the e -states of x at stage s measured with respect to $\{U_n\}_{n \in N}$ and $\{\hat{V}_n^+\}_{n \in N}$ respectively. We will use e -states as finite sets of integers $\leq e$. If $\hat{x} \in \hat{M}_s$, $\nu(s, e, \hat{x})$ is $\langle e, \sigma(s, e, \hat{x}), \tau(s, e, \hat{x}) \rangle$ where $\sigma(s, e, \hat{x})$ and $\tau(s, e, \hat{x})$ are e -states measured with respect to $\{\hat{U}_n^+\}_{n \in N}$ and $\{V_n\}_{n \in N}$. if σ, τ are e -states and $\nu = \langle e, \sigma, \tau \rangle$ a full e -state, $[\sigma]_i = \sigma \cap \{0, 1, \dots, i\}$ and $[\nu]_i = \langle i, [\sigma]_i, [\tau]_i \rangle$. Let $\nu_{-1} = \langle -1, \phi, \phi \rangle$. Let $\nu = \langle e, \sigma, \tau \rangle$ and $\nu' = \langle e', \sigma', \tau' \rangle$. We say ν' extends ν , written $\nu \preceq \nu'$ if $e \leq e'$, $[\sigma']_e = \sigma$, and $[\tau']_e = \tau$. We say $\nu \leq \nu'$ if $e = e'$, $\sigma \subseteq \sigma'$, and $\tau \supseteq \tau'$. Let $\nu \leq^r \nu'$ ($\nu \leq^o \nu'$) denote $\nu \leq \nu'$ and $\tau = \tau'(\sigma = \sigma')$. If \mathcal{S} is a set of full e -states, let $\mathcal{S}[\nu_0] = \{\nu : \nu \in \mathcal{S} \text{ and } \nu_0 \preceq \nu\}$. If $\nu = \langle e, \sigma, \tau \rangle$, the length of ν , $|\nu|$, is e .

A track is a section of the surface of the machine between any two of the following: door, gate, pocket, or join. If X is a track of M , let $\mathcal{S}_s(X) = \{\nu(s, e, x) : e \leq x \text{ and } x \text{ enters track } X \text{ at stage } s\}$, arranged as a sequence in order of e . Let $\mathcal{S}(X)$ be the concatenation of the $\mathcal{S}_s(X)$, $s \in N$. Such a sequence of full e -states is called stream X . If $\nu \in \mathcal{S}(X)$ infinitely often, we write $\nu \in \mathcal{S}(X)$ i.o. $\mathcal{S}(\hat{X})$ for \hat{X} a track of \hat{M} is defined similarly. We say a stream X covers (dual covers, τ -exactly covers, exactly covers) ν if some $\nu' \in \mathcal{S}(\hat{X})$ i.o. where $\nu \leq \nu'$ ($\nu' \leq \nu$, $\nu \leq^r \nu'$, $\nu = \nu'$). Steam X covers steam Y if X covers every ν such that $\nu \in \mathcal{S}(Y)$ i.o. Streams X and Y are equivalent if each exactly covers the other.

We now describe the role of pockets P and \hat{Q}, \hat{P} and Q are similar. To construct a permutation satisfying (5), we would like to pair elements $x \in M$ and $\hat{y} \in \hat{M}$ which are in the same full e -state. Pocket \hat{Q} will consist of elements \hat{y} needing mates. We will attempt to choose mates for the elements of \hat{Q} from those elements x entering track D . We will choose such an x as a mate for \hat{y} at stage $s + 1$ if $\nu(s, d, x) = \nu(s, d, \hat{y})$ where d depends on s and \hat{y} . We allow d to be small when few x appear in any appropriate state making it easier to choose a mate for \hat{y} . For most (i.e., almost all) \hat{y} , however, we must have d be large so as to guarantee (5). Of course, to insure that x 's do appear that are appropriate mates for \hat{y} , we must, in general, enumerate $x(\hat{y})$ in some sets $V_n^+(U_n^+)$. For instance, suppose $\nu \in \mathcal{S}(\hat{E})$ i.o. That is, infinitely many \hat{y} appear on the track before pocket \hat{Q} in state ν . Suppose that $\nu' \in \mathcal{S}(C)$ i.o. where $\nu' \geq \nu$. Let $\nu' = \langle e, \sigma', \tau' \rangle$ and $\nu = \langle e, \sigma, \tau \rangle$. Then it is possible to match up the \hat{y} in state ν with the x in state ν' by enumerating the elements x into full e -state $\langle e, \sigma', \tau \rangle$ and the elements \hat{y} into $\langle e, \sigma', \tau \rangle$ since $\nu' \geq \nu$. Thus, we should try to arrange that C covers \hat{E} (and, in fact, every stream \hat{X}) and that D τ -

exactly covers \hat{E} . Further we should enumerate \hat{y} only into states ν which appear to be τ -exactly covered by D . With this motivation, we now continue describing the construction.

At the beginning of a stage, there is at most one ball on the surface of one machine. If there is such a ball x , the stage consists of moving the ball according to the *rules* (§ 4) down one *track*—a section of the surface between any two of the following: gates, doors, joins, or pockets. This may also cause us to enumerate $x \in A, B, \hat{U}_n^+, \hat{V}_n^+$ and to place other balls above holes. If there is no ball on the surface of the machine, we choose one ball above a hole and move it down the surface of the machine in successive stages. If at the beginning of a stage all balls are in pockets, we enumerate another element according to g . This may cause us to place some balls above holes.

CONSTRUCTION. Stage 0. Do nothing.

Stage $s + 1$. The action consists of three steps.

Step 1. Adopt the first case below which holds.

Case 1. Some x (\hat{x}) is on the surface of M (\hat{M}). There will be at most one such ball x (\hat{x}). Exactly one of the rules of § 4 will apply to x (\hat{x}). Move x (\hat{x}) according to that rule.

Case 2. Some x (\hat{x}) is above a hole. Choose the least such x (or least such \hat{x} if x doesn't exist) and place x (\hat{x}) on the surface of the machine at the end of the next track downward from the hole. (And so x (\hat{x}) will fall in Case 1 at the next stage.)

Case 3. Each x (\hat{x}) in M (\hat{M}) at stage $s + 1$ is in a pocket. Enumerate one more value of our simultaneous enumeration g and adopt the corresponding subcase below.

(a) g enumerates $x \in W_0$ ($x \in \hat{W}_0$). Place a ball marked x (\hat{x}) above hole H_1 (\hat{H}_1) unless x (\hat{x}) is in Pocket A (\hat{A}). Enumerate $x \in U_0$ (V_0).

(b) g enumerates $x \in W_e$ ($x \in \hat{W}_e$), $e > 0$. By the convention on the enumeration of W_0 (\hat{W}_0) and Subcase 3(a), x (\hat{x}) must be in some pocket of M (\hat{M}). Enumerate x (\hat{x}) in U_e (V_e) unless x (\hat{x}) is in pocket P . If x (\hat{x}) is in pocket P , Rule R_{10} (\hat{R}_{10}) will apply to x (\hat{x}). If x (\hat{x}) is in pocket A (\hat{A}) proceed to stage $s + 2$. If x (\hat{x}) is in pocket Q (\hat{Q}), Rule R_3 (\hat{R}_3) may apply to x (\hat{x}) and should be followed.

(c) g enumerates some computation $\varphi_i(j) = k$, $i \leq j$. Let $a_1^s < a_2^s < \dots$ be the elements of \bar{A}_s in increasing order. If $a_j^s \leq k$, let m' be the least m such that $a_m^s > k$. Remove those balls marked

$a_j^s, a_{j+1}^s, \dots, a_{m'-1}^s$ from pockets P and Q or from the store of balls not yet in M and place them above hole H_4 . (This may cause various other balls to be removed from pockets P (\hat{P}) according to the rules.)

Step 2. Apply Rule R_4 to every x in pocket Q in increasing order of x . Apply Rule \hat{R}_4 to every \hat{x} in pocket \hat{Q} in increasing order of x .

Step 3. Apply Rule R_{11} (\hat{R}_{11}) to every element in pocket P (\hat{P}).

4. The rules. We now give the rules and some of the properties of the construction. Rules with numbered subscripts are the same or substantially the same as the corresponding rule in the proof of the Extension Theorem [5, Theorem 2.2]. Rules with lettered subscripts are new to this construction.

Rule R_1 (\hat{R}_1) determines which balls enter tracks C_1 and C_2 (\hat{C}_1 and \hat{C}_2) when they reach door D_1 . We let \mathcal{R}_s denote a certain sequence defined by induction on s and containing (exactly once) each pair $\langle \nu, j \rangle$ for all $j \in \{1, 2\}$ and all full e -states ν , for $e < s$. Let $\mathcal{R}_0 = \{\langle \nu_{-1}, 1 \rangle, \langle \nu_{-1}, 2 \rangle\}$.

RULE R_1 . Suppose that sequence \mathcal{R}_s is given. If an element x enters track C at stage s , then at stage $s+1$ it enters either track C_1 or C_2 (with $\nu(s+1, x, x) = \nu(s, x, x)$) as follows. Let $\langle \nu', i' \rangle$ be the first pair $\langle \nu, i \rangle$ on the sequence \mathcal{R}_s such that $\nu \preceq \nu(s, x, x)$. Remove $\langle \nu', i' \rangle$ from its present position on \mathcal{R}_s , place it at the end of the sequence, and place x on track $C_{i'}$. In this case we say that $\langle \nu', i' \rangle$ is reset at stage $s+1$. Finally, whether an element x entered track C or not, add $\langle \nu, i \rangle$ at the end of the sequence (in any fixed effective order) for each $i \in \{1, 2\}$ and each full s -state ν . Let \mathcal{R}_{s+1} denote the resulting sequence.

RULE \hat{R}_1 . Like R_1 but with $\hat{C}, \hat{C}_1, \hat{C}_2$ in place of C, C_1, C_2 .

Rule R_A (\hat{R}_A) is similar and determines which balls enter tracks C_5 and C_6 (\hat{C}_5 and \hat{C}_6) when they reach door D_5 .

RULE R_A . Let \mathcal{R}'_s be a sequence of full e -states with the same properties as \mathcal{R}_s in Rule R_1 except that C, C_1 , and C_2 are replaced by C_5, C_6 , and C_6 . Rule R_A reads as Rule R_1 , but with $\mathcal{R}'_s, C_5, C_6, C_6$ replacing R, C_1, C_2, C .

RULE \hat{R}_A . Like Rule R_A but with \hat{C}_5, \hat{C}_6 , and \hat{C}_6 replacing C_5, C_6 , and C_6 .

LEMMA 4.1. *Streams C_1, C_2, C_5, C_6 ($\hat{C}_1, \hat{C}_2, \hat{C}_5, \hat{C}_6$) are each equivalent to C (\hat{C}).*

Proof. The proof follows by making the obvious modifications to [5, Lemma 4.1]. □

The next rules are new and concern gate G_2 . If X is a track, let $\mathcal{S}_{\leq s}(X)$ be the concatenation of the \mathcal{S}_t for $t \leq s$. The purpose of track C_6 (\hat{C}_6) is to arrange for A to cover \hat{A} (\hat{A} to dual cover A) to fulfill the hypothesis of the Extension Theorem of Soare. We will check members of $\mathcal{S}_{\leq s}(C_6)$ (\hat{C}_6) which are successfully covered. We must be careful to restrain our enumerations to make B nonhyper-simple and A coinfinite.

RULE R_B . Suppose x is on track C_6 at the beginning of stage $s + 1$. If there is a full e -state ν on $\mathcal{S}_{\leq s}(C_6)$ for some $e \leq x$ which satisfies

- (1) ν is unchecked,
- (2) $\nu \leq \nu(s, e, x)$;

place x on track C_6 at stage $s + 1$ and check all ν on $\mathcal{S}_{\leq s}(C_6)$ such that $\nu \leq \nu(s, e, x)$. Otherwise, place x on track C_7 .

RULE $R_{\hat{B}}$. Suppose \hat{x} is on track \hat{C}_6 at the beginning of stage $s + 1$ and $x \in D_{h(e)}$. If there is a full i -state ν on $\mathcal{S}_{\leq s}(C_6)$ for some $i \leq x$ which satisfies

- (1) ν is unchecked,
- (2) $\nu(s, i, \hat{x}) \leq \nu$,
- (3) the unchecked ν is one of the first e members of $\mathcal{S}_{\leq s}(C_6)$;

place \hat{x} on track \hat{C}_6 at stage $s + 1$ and check all ν on $\mathcal{S}_{\leq s}(C_6)$ such that $\nu(s, i, \hat{x}) \leq \nu$. Otherwise, place \hat{x} on track \hat{C}_7 .

Let A (B) be the set consisting of those elements x (\hat{x}) which reside in pocket A (\hat{A}) at the end of the construction. The following rules clearly guarantee that A and B are r.e.

RULE R_C (\hat{R}_C). No x (\hat{x}) may leave pocket A (\hat{A}).

As described previously, the pocket \hat{Q} (Q) will finally contain the elements of the domain of a finite-one map with the elements of P (\hat{P}) as range. The desired permutation p will be determined by this map. Rule R_2 governs the entry of elements into the pocket P . Roughly, an element enters pocket P if it is an appropriate "mate" for an element (or finite set of elements) of \hat{Q} , or if it is a more "desirable" mate for an element of \hat{Q} than the present mate.

Our Rule \hat{R}_2 is the same as Soare's, but Rule R_2 needs considerable modification. Roughly this is because now elements can leave pocket P to meet the positive requirements on A . Before stating Rule R_2 , we introduce some notation.

In Definition 4.2, we define uniformly recursively in s a sequence \mathcal{H}_s of full states which contains each full j -state, $j < s$, exactly once. (This is the same definition that Soare gives.)

DEFINITION 4.2. (a) If $\nu_1, \nu_2 \in \mathcal{H}_s$, we write $\nu_1 \leq_s^{\#} \nu_2$ if ν_1 precedes ν_2 in the sequence \mathcal{H}_s .

(b) Let $\mathcal{H}_0 = \{\nu_{-1}\}$. Given \mathcal{H}_s , let $\mathcal{H}_{s+1}^2 = \{\nu: \nu \in \mathcal{H}_s \text{ and } (\exists \nu') [\nu <^{\tau} \nu' \text{ and } \nu' \in \mathcal{S}_{s+1}(C) \cup \mathcal{S}_{s+1}(D)]\}$. Let $\mathcal{H}_{s+1}^1 = \mathcal{H}_s - \mathcal{H}_{s+1}^2$. Let \mathcal{H}_{s+1}^3 be a sequence of all full s -states arranged in some effective order (uniformly in s) such that $\nu' = \langle s, \sigma', \tau' \rangle$ precedes $\nu = \langle s, \sigma, \tau \rangle$ if $\tau' \subsetneq \tau$ or if $\tau = \tau'$ and $\sigma \subseteq \sigma'$. Let \mathcal{H}_{s+1} denote the concatenation of $\mathcal{H}_{s+1}^1, \mathcal{H}_{s+1}^2, \mathcal{H}_{s+1}^3$ in that order where $\mathcal{H}_{s+1}^1, \mathcal{H}_{s+1}^2$ have the orderings induced by $\leq_s^{\#}$.

(c) Define a function $r(s, \nu)$ for $s \in N$ and $\nu \in K_s$ as follows:

$$\begin{aligned} r(0, \nu_{-1}) &= 1, \quad r(0, \nu) \text{ is undefined if } \nu \neq \nu_{-1}; \\ r(s + 1, \nu) &= r(s, \nu) \text{ if } \nu \in \mathcal{H}_{s+1}^1; \\ r(s + 1, \nu) &= \max \{r(s, \nu): r(s, \nu) \text{ is defined}\} + k \text{ if} \\ &\quad \nu \in \mathcal{H}_{s+1}^2 \cup \mathcal{H}_{s+1}^3 \text{ and is the } k\text{th such} \\ &\quad \text{in the ordering } \leq_{s+1}^{\#}; \\ r(s + 1, \nu) &\text{ is undefined if } \nu \in \mathcal{H}_{s+1}^3. \end{aligned}$$

(d) Let \mathcal{H}_ω denote those ν which are in \mathcal{H}_{s+1}^2 for only finitely many s .

Think of the list \mathcal{H}_s as a list of full j -states, $j < s$, each occupying a row. At stage $s + 1$, certain full j -states (those in \mathcal{H}_{s+1}^2) are *reset* — they are removed from the list and placed on “fresh” rows below the list \mathcal{H}_s . Also at stage $s + 1$, the full s -states are added to \mathcal{H}_{s+1} in fresh rows below \mathcal{H}_{s+1}^2 . In this picture, $r(s, \nu)$ is the number of the row which ν occupies in \mathcal{H}_s . Note that $\nu \in \mathcal{H}_\omega$ iff $\lim_s r(s, \nu)$ exists. Also note that $r(s, \nu) \leq r(s, \nu')$ iff $\nu \leq_s^{\#} \nu'$.

If \mathcal{S} is a sequence of full e -states, we say that ν_0 is *maximal with respect to \mathcal{S}* if

$$(\forall \nu)[[\nu \in \mathcal{S} \text{ i.o. and } \nu_0 \leq^{\tau} \nu] \implies \nu = \nu_0].$$

Soare proves the following lemmas:

LEMMA 4.3 [5, Lemma 5.1]. *For all e -states σ, σ', τ and all $s > e$,*

$$\sigma' \supseteq \sigma \Rightarrow [\langle e, \sigma', \tau \rangle \leq^* \langle e, \sigma, \tau \rangle].$$

LEMMA 4.4 [5, Lemma 5.2]. *For all $\nu, \nu \in \mathcal{K}_\omega$ (and hence $\lim_s r(s, \nu)$ exists) just if ν is maximal with respect to $\mathcal{S}(D)$.*

Let P_s, Q_s, M_s ($\hat{P}_s, \hat{Q}_s, \hat{M}_s$) denote the set of elements in pocket P , pocket Q , machine M ($\hat{P}, \hat{Q}, \hat{M}$) at the end of stage s .

DEFINITION 4.5. Define (in increasing order of \prec) a function $\hat{q}(s, \nu)$ by $\hat{q}(s, \nu) = \mu \hat{y} \in \hat{Q}_s$ such that

- (1) $\nu \preceq \nu(s, y, \hat{y})$,
- (2) $\hat{q}(s, \nu') \neq \hat{y}$ for any $\nu' \prec \nu$.

If \hat{y} does not exist, then $\hat{q}(s, \nu)$ is undefined.

Clearly, if $\hat{y} \in \hat{Q}_s, \hat{y} = \hat{q}(s, \nu)$ for some ν . Roughly, the goal of Rule R_2 is to define a function $p(s, \nu)$ with values in P_s so that if $\hat{q}(\nu) = \lim_s \hat{q}(s, \nu)$ and $p(\nu) = \lim_s p(s, \nu)$, then $\hat{q}(\nu) \rightarrow p(\nu)$ is the intended piece of the permutation mapping \hat{Q} to P . The major difficulty is that $p(\nu)$ may not necessarily exist even if $\hat{q}(\nu)$ does. In this case, we will try to arrange that $\hat{q}(\nu')$ exists for only finitely many $\nu' \succ \nu$, thereby enabling us to construct at least a finite-one map. Our major concern is to control the cases in which $p(s, \nu)$ becomes undefined infinitely often for a fixed ν .

DEFINITION 4.6. If $x \in M_s$ and ν is a full e -state such that $\nu \preceq \nu(s, x, x)$, the ν -rank of x at stage s , denoted $\rho(s, \nu, x)$, is the minimum of the set $\{r(s, \nu') : \nu \preceq \nu' \preceq \nu(s, x, x)\}$.

Rule R_2 , which governs the selection of $p(s, \nu)$, will depend on the function ρ defined above as well as certain markers $A_{\langle k, \ell, \nu \rangle}$. Rule R_2 will choose $p(s, \nu)$ so that the value $\rho(s, \nu, p(s, \nu))$ is as small as possible. (The reason for this measure of "desirability" of candidates for $p(s, \nu)$ is not obvious but will come out in the proof of Lemma 5.17.) The marker $A_{\langle k, \ell, \nu \rangle}$ will be used to prevent the positive requirements on A from enumerating $p(s, \nu)$ into A infinitely often where $k = \rho(s, \nu, p(s, \nu))$ and $\ell = (\mu t)(\forall \nu' \preceq \nu)[\hat{q}(\ell, \nu') = \hat{q}(\nu')]$.

Let ν_k be the unique ν such that $(\exists s)[r(s, \nu) = k]$. For each $k, \ell \in N$, form triples $\langle k, \ell, \nu \rangle$ where $\nu \preceq \nu_k$. Linearly order these triples by

$$\begin{aligned} \langle k, \ell, \nu \rangle &< \langle k', \ell', \nu' \rangle \text{ if } 2^k 3^\ell < 2^{k'} 3^{\ell'} \text{ or} \\ &k = k', \ell = \ell', \text{ and } \nu < \nu'. \end{aligned}$$

We will suppose that these triples have been put in 1-1 order preserving correspondence with N and we will denote the integer cor-

responding to $\langle k, \ell, \nu \rangle$ also by $\langle k, \ell, \nu \rangle$. When an element x enters pocket P (necessarily by Rule R_2) we will *assign* a certain marker $A_{\langle k, \ell, \nu \rangle}$ to it. That marker remains *assigned* to x until x leaves pocket P , at which time the assignment will be *cancelled*.

RULE R_2 . Suppose that x is on track D at the end of stage s . Let ν' be the first ν in the ordering \preceq such that

- (1) $\nu \preceq \nu(s, x, x)$,
- (2) $\hat{q}(s, \nu)$ is defined,
- (3) either $p(s, \nu)$ is undefined or

$$\rho(s, \nu, x) < \rho(s, \nu, p(s, \nu)) .$$

If ν' exists, then at stage $s+1$ place x in pocket P , let $p(s+1, \nu') = x$ (unless x is immediately removed from pocket P by the remainder of this Rule), and remove from P all elements $p(s, \nu'')$ such that $\nu' \preceq \nu''$ (and place these according to Rule R_{12}). Assign marker $A_{\langle k, \ell, \nu' \rangle}$ to x where

$$k = \rho(s, \nu', x) \quad \text{and} \\ \ell = (\mu t)(\forall z)_{t \leq z \leq s} (\forall \nu'' \preceq \nu') [\hat{q}(s, \nu'') = \hat{q}(z, \nu'')] .$$

Let m be the integer such that $x = a_m^s$. If $m \leq \langle k, \ell, \nu' \rangle$ do nothing more, but if $\langle k, \ell, \nu' \rangle < m$, place above hole H_4 all a_i^s where $\langle k, \ell, \nu' \rangle \leq i < m$. (This may cause other balls to be removed from pocket P by the next rule, Rule R_D .) If ν' does not exist, place x on track D_1 .

RULE R_D . If $x = p(s, \nu)$ is placed above hole H_4 at stage $s+1$, remove each $p(s, \nu')$ for $\nu' \succeq \nu$ and place these balls according to Rule R_{12} .

RULE R_E . A ball x may only be placed above hole H_4 by Subcase 3(c) of Step 1 of the construction or Rule R_2 .

Rule \hat{R}_2 , the pockets \hat{P} and Q , and the functions \hat{p} and q are duals to the above except that the markers $A_{\langle k, \ell, \nu \rangle}$ are omitted. This is because there are no positive requirements on B requiring elements to be taken out of \hat{P} . Thus, there is no Rule \hat{R}_D , and \hat{R}_E reads as follows:

RULE R_E . No \hat{x} may be placed above hole \hat{H}_4 .

By induction on s , we now define recursively, uniformly in s , sequences of full j -states $\mathcal{M}_s, \mathcal{P}_s$ such that $\mathcal{M}_s \subseteq \mathcal{P}_s \subseteq \mathcal{H}_s$ for all s . Our main goal is to prove that D exactly covers \hat{Q} so that

\hat{Q} sees enough “mates” to achieve the desired piece of the permutation. (Here \hat{Q} can be considered a stream in the following sense: let $\mathcal{S}_{s+1}(\hat{Q}) = \{\nu(s+1, e, \hat{x}) : e \leq x, \hat{x} \in \hat{Q}_{s+1} - \hat{Q}_s \text{ or } \hat{x} \in \hat{Q}_{s+1} \cap \hat{Q}_s, \text{ and } \nu(+1, x, \hat{x}) \neq \nu(s, x, \hat{x})\}$.) We will do this by insuring that C covers every stream \hat{X} of \hat{M} and that D τ -exactly covers every stream that C covers.

DEFINITION 4.7. (a) Let $\mathcal{M}_0 = \{\nu_{-1}\}$. Given \mathcal{K}_{s+1} and \mathcal{M}_s we will define \mathcal{M}_{s+1} . Let $\nu = \langle e, \sigma, \tau \rangle$ and suppose we have already determined whether $\nu' \in \mathcal{M}_{s+1}$ if $|\nu'| < e$. We say ν is excluded from \mathcal{M}_{s+1} if one of the following conditions holds:

CONDITION 1. $(\exists \nu')[r(s, \nu') < r(s, \nu) \text{ and } \nu' \in \mathcal{K}_{s+1}^2]$.

CONDITION 2. $(\exists \nu')(\exists \hat{X})[\nu' \in \mathcal{S}_s(\hat{X}) - \mathcal{P}_s \text{ and } |\nu'| < e]$, where \hat{X} is any stream of \hat{M} except \hat{A} , including \hat{Q} .

CONDITION 3. There is a full j -state ν' and an element x such that

- (i) $x = p(s, \nu')$,
- (ii) x is placed above hole H_i at stage $s + 1$,
- (iii) $\rho(s, \nu', x) \leq r(s, \nu)$,
- (iv) $r(s, \nu) \geq \iota = (\mu t \leq s)(\forall z)_{t \leq z \leq s}(\forall \nu'' \preceq \nu')[\hat{q}(z, \nu'') = \hat{q}(s, \nu'')]$.

Define $\nu \in \mathcal{M}_{s+1}$ if ν is not excluded from \mathcal{M}_{s+1} and either $\nu \in \mathcal{M}_s$ or $\nu \in \mathcal{S}_{s+1}(D)$.

(b) Let $\mathcal{P}_{s+1} = \{\nu : (\exists \nu')[\nu' \in \mathcal{M}_{s+1} \text{ and } \nu \leq^r \nu']\}$.

(c) Let $\mathcal{M}_\omega = \{\nu : \nu \in \mathcal{M}_s \text{ for almost all } s\}$.

(d) Let $\mathcal{P}_\omega = \{\nu : \nu \in \mathcal{P}_s \text{ for almost all } s\}$.

(e) Let $s_{\hat{x}} = \mu t[\hat{x} \in \hat{M}_t]$. Define

$d(s, \hat{x}) = \max\{e : \nu(s, e, \hat{x}) \in \mathcal{P}_s \text{ and } (\forall t)[s_{\hat{x}} \leq t < s \implies e \leq d(t, \hat{x})]\}$. For other \hat{x} and s let $d(s, \hat{x}) = -1$. (Note that $\lim_s d(s, \hat{x})$ exists.)

(f) Let $\nu^*(s, \hat{x}) = [\nu(s, x, \hat{x})]_{d(s, \hat{x})}$ and $\nu^*(\hat{x}) = \lim_s \nu^*(s, \hat{x})$.

We now give some intuition as to these lists. First, \mathcal{K}_ω is simply the maximal full e -states with respect to $\mathcal{S}(D)$. \mathcal{M}_ω , except for the complications of conditions (2) and (3) of Definition 4.7(a), is an approximation to those states in \mathcal{K}_ω which appear infinitely often on $\mathcal{S}(D)$. Thus \mathcal{P}_ω is a recursive approximation to those full e -states that D τ -exactly covers. If we knew the full e -states that D does in fact τ -exactly cover, we would be able to guarantee that D τ -exactly covered \hat{Q}_ω by only enumerating elements of \hat{M} into these states. As it is, $\nu^*(s, \hat{x})$ is simply a “guess” as to the largest initial segment of $\nu(s, x, \hat{x})$ which D τ -exactly covers.

The next rule, Rule R_s , determines what happens when an element x arrives at gate G_1 . Here the element x may be enumerated

in certain sets \hat{V}_n^+ . The purpose of this rule is to insure that D τ -exactly covers any stream \hat{X} which C covers. In the next rule and associated lemmas, X (\hat{X}) ranges over streams of M (\hat{M}) except for A (\hat{A}).

Rule R_3 involves a certain r.e. sequence \mathcal{H} of full e -states, which is the concatenation of the finite sequences $\mathcal{H}_s, s \in N$, defined as follows. $\mathcal{H}_s = \phi$ unless there is some track \hat{X} of \hat{M} , some element \hat{y}_1 on track \hat{X} at stage s , and some full e_1 -state ν_1 such that

$$\nu_1 \in \mathcal{S}_s(\hat{X}) - \mathcal{P}_s$$

via \hat{y}_1 , in which case \mathcal{H}_s consists of the following full e_1 -states for each such \hat{y}_1 (in some effective order uniformly in s):

$$\{\nu: (\exists t)_{\leq s} [\hat{y}_1 \in \hat{M}_t \text{ and } \nu(e_1, t, \hat{y}_1) \leq^{\tau} \nu]\}.$$

Once added to \mathcal{H} , a given ν is never removed from \mathcal{H} or altered in position, although it may later be *checked* during an application of Rule R_3 . Let $\mathcal{H}_{\leq s}$ denote the sequence of elements added to \mathcal{H} by the end of stage s .

RULE R_3 . Suppose an element x enters track C_1 at stage s . Let $\nu_0 = \langle x, \sigma_0, \tau_0 \rangle$ denote $\nu(s, x, x)$. Let $\nu_1 = \langle e_1, \sigma_1, \tau_1 \rangle$ be the first member $\nu = \langle e, \sigma, \tau \rangle \in \mathcal{H}_{\leq s}$ such that:

- (i) ν has not been checked by the end of stage s ;
- (ii) $e \leq x$; and
- (iii) $\nu \leq^{\sigma} [\nu_0]_e$ (i.e., $\sigma = [\sigma_0]_e$, and $\tau \supseteq [\tau_0]_e$). If ν_1 exists, then at stage $s + 1$ check ν_1 , enumerate x in $\hat{V}_{n,s+1}^+$ for each $n \leq e_1$ such that $n \in \tau_1 - \tau_0$ (so that $\nu(s + 1, e_1, x) = \nu_1$), and place x on track C_3 . If ν_1 fails to exist, then at stage $s + 1$ place x on track C_1 , and let $\nu(s + 1, x, x) = \nu(s, x, x)$.

RULE \hat{R}_3 . Same as Rule R_3 , but with $C_1, C_3, C_4, \hat{C}, \hat{V}_n^+, \mathcal{H}$ replaced by $\hat{C}_1, \hat{C}_3, \hat{C}_4, C, \hat{U}_n^+$, and \mathcal{H}' (which is defined in the analogous way using \mathcal{P}'_s and $\mathcal{S}_s(X)$), and with the roles of σ and τ interchanged.

Rule \hat{R}_4 determines the enumeration of elements \hat{x} in sets \hat{U}_n^+ while \hat{x} is in pocket \hat{Q} . It is a combination of the Rules \hat{R}_4 and \hat{R}_5 of [5]. Thus there is no Rule \hat{R}_5 although we have kept the same numbering of the other rules to avoid confusion.

Rule \hat{R}_4 . Suppose that \hat{x} is in \hat{Q} at the end of step 1 of stage $s + 1$ of the construction. Then \hat{x} remains in pocket \hat{Q} through

the end of stage $s + 1$ and $\tau(s + 1, \hat{x}) = \tau(s, \hat{x})$, (else \hat{x} was removed at stage $s + 1$, step 1, under Rule \hat{R}_5).

Case 1. If $\nu^*(s, \hat{x}) \in \mathcal{M}_s$ do nothing.

Case 2. Otherwise perform the following enumeration on \hat{x} . Let $\mathcal{S}_{\leq t}(D)$ denote the sequence which is the concatenation of the sequences $\{\mathcal{S}_u(D) : u \leq t\}$. Let $\nu'(s + 1) = \langle e, \sigma', \tau' \rangle$ denote $\nu^*(s, \hat{x})$. Define

$$\mathcal{F}_{s+1} = \{\nu : \nu \in \mathcal{M}_s \text{ and } \nu'(s + 1) \leq^r \nu\}.$$

Note that $\nu'(s + 1) \in \mathcal{P}_s$ and thus $\mathcal{F}_{s+1} \neq \phi$, by the definitions of \mathcal{P}_s and d . Furthermore, for each $\nu \in \mathcal{M}_s, \nu \in \mathcal{S}_{\leq(s)}(D)$. Define $\nu''(s + 1) = \langle e, \sigma'', \tau' \rangle$ to be the last ν on the sequence $\mathcal{S}_{\leq(s)}(D)$ such that $\nu \in \mathcal{F}_{s+1}$. Enumerate \hat{x} in $\hat{U}_{n,s+1}^+$ for each $n \in \sigma'' - \sigma'$.

LEMMA 4.8. $(\forall \hat{x}) [\hat{x} \in \hat{Q}_\omega \Rightarrow \nu^*(\hat{x}) \in \mathcal{M}_\omega]$.

Proof. Let $e = \lim_s d(s, \hat{x})$. Let $\nu = \lim_s \nu(s, e, \hat{x})$. Then $\nu = \nu^*(\hat{x})$. Let s_0 be any stage such that, for $s \geq s_0, d(s, \hat{x}) = e, \nu(s, e, x) = \nu$, and $\hat{x} \in \hat{Q}_s$. By Rule \hat{R}_4 , if $\nu \notin \mathcal{M}_s$ for any such s, \hat{x} is enumerated in certain r.e. sets $\hat{U}_{n,s+1}^+$ for $n \leq e$ causing $\nu(s + 1, e, x) \neq \nu$, contrary to the hypothesis on stage s_0 . Thus $\nu = \nu^*(x) \in \mathcal{M}_s$ for each $s \geq s_0$. \square

Rules R_4 and a function \hat{d} are similar and are omitted.

We now give the rest of the rules of the machines. The duals of these rules are similar and are omitted.

RULE R_6 . An element $x \in \hat{V}_{n,s+1}^+ - \hat{V}_{n,s}^+$ only if at stage $s + 1$, Rule R_3 or R_4 applies to x .

RULE R_7 . If $x \in U_{n,s+1} - U_{n,s}$ and $x \in M_s$ then x is in some pocket at the beginning of stage $s + 1$.

RULE R_8 . If $x \in Q_s$ and $x \in U_{n,s+1} - U_{n,s}$ for some $n \leq x$ remove x from pocket Q at stage $s + 1$ and place x above hole H_2 .

RULE R_9 . If $x \in Q_s$ then $x \in Q_{s+1}$ unless x is removed at stage $s + 1$ from pocket Q under Rules R_2 or R_3 or Subcase 3(c) of the construction.

RULE R_{10} . If $x = p(s, \nu) \in P_s$ for some ν such that $|\nu| = e$ and x is enumerated in W_n at stage $s + 1$ for some $n \leq e$, then at stage $s + 1$ remove from pocket P all $p(s, \nu')$ such that $\nu \preceq \nu'$ and enumer-

ate x in U_n . If x is enumerated in W_n for some $n > e$, proceed to stage $s + 2$.

RULE R_{11} . If $\hat{q}(s, \nu) \neq \hat{q}(s + 1, \nu)$ or $\hat{q}(s + 1, \nu)$ is undefined, then remove from pocket P at stage $s + 1$ all $p(s, \nu')$ such that $\nu \preceq \nu'$.

RULE R_{12} . If $x \in P_s$, then $x \in P_{s+1}$ unless x is removed from P at stage $s + 1$ by the construction Subcase 3(c), or Rules R_2 , R_{10} , R_{11} , or R_D . If x is not placed above hole H_4 by Rule R_2 or Subcase 3(c) and x last entered pocket P at stage t , then place x above hole H_2 if $\nu(s + 1, x, x) \neq \nu(t, x, x)$ and place x above hole H_3 otherwise.

RULE R_F . If at the end of stage s , x is on any of the tracks $C_0, C_4, C_5, C_7, C_8, D_1$, or E , place x at stage $s + 1$ on the next track or pocket in the downward direction (the direction of the arrows).

5. The verification. The first two lemmas of this section guarantee that each ball eventually reaches a pocket where it resides for the rest of the construction.

LEMMA 5.1. *Each element x (\hat{x}) re-enters the surface of M (\hat{M}) at most finitely often.*

Proof. Suppose x is an element of M . (The case $\hat{x} \in \hat{M}$ is similar.) From hole H_4 , x can only reach pocket A and so x may be placed above hole H_4 at most once. Suppose x is placed above hole H_2 at stage $s + 1$. Then $x \in P_s$ or Q_s . If $x \in Q_s - Q_{s+1}$ then $\nu(s + 1, x, x) \neq \nu(s, x, x)$ by Rule R_9 , which can happen only finitely often. Similarly, if $x \in P_s - P_{s+1}$, $\nu(s + 1, x, x) \neq \nu(t, x, x)$ by Rule R_{12} where t is the last stage before s that x entered P . Thus x can be placed above hole H_2 only finitely often. If x is placed above hole H_3 , x can only re-enter M from pocket Q and hence via hole H_2 . \square

LEMMA 5.2. *Each element x (\hat{x}) is in some particular pocket for cofinitely many stages.*

Proof. By Lemma 5.1, each element x re-enters only finitely often. Once x is on the surface of M it will reach a pocket by the construction. The only possible obstacle then is for x to remain above a hole cofinitely often. Choose any stage s such that x is above a hole. At stage s , only finitely many elements are in M_s . By Lemma 5.1, each of the elements besides x can generate only

finitely many moves; eventually x will be the x of Case 2 of step 1 of the construction and will re-enter the surface of M .

We can actually prove already that B is not hypersimple.

LEMMA 5.3. *B is not hypersimple.*

Proof. That

$$B = \{x: \hat{x} \text{ is in pocket } B \text{ for cofinitely many stages}\}$$

is r.e. is already clear by Rule \hat{R}_C . By Rule \hat{R}_E , no \hat{x} enters pocket B except by track \hat{C}_e and hence by Rule \hat{R}_B . Since each $\hat{x} \in D_{h(e)}$ which enters B in this way checks one of the first e members of $\mathcal{S}_{\leq s}(C_e)$, $|D_{h(e)} \cap B| \leq e$. Of course $\{D_{h(e)}\}_{e \in \mathbb{N}}$ witnesses that B is not hypersimple since $|D_{h(e)}| = e + 1 \Rightarrow D_{h(e)} \cap \bar{B} \neq \phi$. \square

The next set of lemmas together will show that D τ -exactly covers every stream \hat{X} of \hat{M} . Recall that markers $A_{\langle k, \ell, \nu \rangle}$ are assigned to elements of pocket P so that each element of pocket P has exactly one marker assigned to it at stage s . Marker $A_{\langle k, \ell, \nu \rangle}$ was supposed to insure that if $\ell = (\mu t)(\forall \nu' \preceq \nu)[\hat{q}(t, \nu') = \hat{q}(\nu')]$ and $\rho(s, \nu, p(s, \nu)) = k$ for infinitely many s , then the positive requirements on A do not remove $p(s, \nu)$ from pocket P infinitely often. We arranged this by guaranteeing that, if $x = p(s, \nu)$ has marker $A_{\langle k, \ell, \nu \rangle}$ assigned to it at stage s and $\langle k, \ell, \nu \rangle = m$, then $x \leq a_m^*$. Thus only computations $\varphi_i(j)$ for $i, j \leq m$ can put such an x in A . Lemma 5.5 guarantees that the use of marker $A_{\langle k, \ell, \nu \rangle}$ accomplishes somewhat more than the above. This lemma depends heavily on the careful indexing of triples $\langle k, \ell, \nu \rangle$. We first prove a technical lemma which will be used in Lemma 5.5.

LEMMA 5.4. *Suppose that at stage $s + 1$ x is placed above hole H_2 . Then x is placed on track C at some stage $t > s + 1$.*

Proof. By the construction, x eventually enters track C unless x is placed above hole H_4 at some stage $t + 1 > s + 1$. By Rule R_E , x may only be placed above hole H_4 by Subcase 3(c) of the construction or by Rule R_2 . Subcase 3(c) of the construction never applies while x is above hole H_2 . If Rule R_2 placed x above hole H_4 at stage $t + 1$, then at stage t some y is on track D and $x < y$. But, by the order in which balls are released from holes, no $y > x$ is ever on any track while x is above hole H_2 . (This can be proved easily by induction on the stages that x is above hole H_2). \square

LEMMA 5.5. *Fix k, ℓ, ν . Then there are only finitely many*

stages $s + 1$ such that Rule R_2 assigns marker $A_{\langle k, \ell, \nu \rangle}$ at stage $s + 1$.

Proof. By induction on $m = \langle k, \ell, \nu \rangle$, we may assume that there is a stage s_0 such that if $n = \langle k', \ell', \nu' \rangle < m$, then Rule R_2 never assigns marker A_n at a stage $s + 1 \geq s_0$ and, if the last assignment of marker A_n is ever cancelled, it is cancelled before stage s_0 . Of course we may also assume that Subcase 3(c) of step 1 of the construction never applies to α_m^s if $s \geq s_0$. Let ν_k denote the unique ν such that $r(s, \nu) = k$ for some s . If Rule R_2 assigns $A_{\langle k, \ell, \nu \rangle}$ to x at stage $s + 1$, then

- (1) $x = p(s + 1, \nu)$,
- (2) $k = \rho(s, \nu, x)$,
- (3) $\ell = (\mu t \leq s)(\forall z)_{t \leq z \leq s}(\forall \nu' \preceq \nu)[\hat{q}(z, \nu') = \hat{q}(s, \nu')]$,
- (4) $x \leq \alpha_m^{s+1}$.

For the sake of a contradiction, assume that Rule R_2 assigns $A_{\langle k, \ell, \nu \rangle}$ at infinitely many stages $s + 1 > s_0$. Thus we also may assume that $\lim_s r(s, \nu_k) = k$ since $\rho(s, \nu, x) = k$ for any stage s such that $A_{\langle k, \ell, \nu \rangle}$ is assigned at stage $s + 1$ and $\rho(s, \nu, x) = r(s, \nu')$ for some $\nu' \preceq \nu$. Further, we may assume that $\hat{q}(\nu) = \lim_s \hat{q}(s, \nu)$ exists by (3). At a stage $s + 1 > s_0$ such that $A_{\langle k, \ell, \nu \rangle}$ is assigned by Rule R_2 at stage $s + 1$ the element x which newly arrives in pocket P has full x -state satisfying $\nu \preceq \nu_k \preceq \nu(s, x, x)$. Since this element was not preferred as $p(s + 1, \nu')$ for any $\nu' < \nu$, it must be the case that $\rho(s, \nu', p(s, \nu')) \leq k$ for any such $\nu' < \nu$. Now we note that $\rho(s, \nu', p(s, \nu'))$ does not decrease as long as $p(s, \nu)$ remains in pocket P since $p(s, \nu')$ is not enumerated in any r.e. set while in pocket P . Thus, at stage s , $p(s, \nu')$ had some marker $A_{\langle k', \ell', \nu' \rangle}$ still assigned to it where $k' \leq \rho(s, \nu', p(s, \nu')) \leq k$ and $\ell' \leq \ell$ (since $\nu' < \nu$). Thus $\langle k', \ell', \nu' \rangle < \langle k, \ell, \nu \rangle$ so that, by our induction hypothesis $p(s_0, \nu') = p(\nu')$ because $A_{\langle k', \ell', \nu' \rangle}$ is never assigned by Rule R_2 or cancelled after stage s_0 .

Suppose that the assignment of A_m is cancelled at some stage $s + 1 > s_0$. Then, by Rule R_{12} , $p(s, \nu)$ is taken out of pocket P at stage $s + 1$ by Rules R_2, R_{10}, R_{11} , or R_D or by Subcase 3(c) of the construction. It is easy to see that $p(s, \nu)$ could not be removed by Rule R_{11} or Subcase 3(c) of the construction by our hypotheses about stage s_0 . Also Rule R_D could not remove $p(s, \nu)$ at stage $s + 1$ because this would require some $p(s, \nu')$ to be placed above hole H_1 at stage $s + 1$ for some $\nu' < \nu$. If Rule R_2 removes $p(s, \nu)$ at stage $s + 1$, then either Rule R_2 defines $p(s + 1, \nu)$, Rule R_2 defines $p(s + 1, \nu')$ for some $\nu' < \nu$, or Rule R_2 assigns a marker A_n for $n < m$. The latter two cases cannot happen after stage s_0 , so suppose Rule R_2 defines $p(s + 1, \nu)$. Then $\rho(s, \nu, x) < \rho(s, \nu, p(s, \nu))$ for some x on track D at stage s . But $\rho(s, \nu, p(s, \nu)) = k$ since

$p(s, \nu)$ does not change state while in P nor is ν_k ever reset. Thus Rule R_2 assigns a marker $A_{\langle k', \ell, \nu \rangle}$ to x with $k' < k$. This contradicts the hypothesis about s_0 since $\langle k', \ell, \nu \rangle < \langle k, \ell, \nu \rangle$. Finally, we consider Rule R_{10} . If $x = p(s, \nu)$ is removed from pocket P at stage $s + 1$ by Rule R_{10} , $x \in U_{n, s+1} - U_{n, s}$ for some $n \leq |\nu| \leq |\nu_k|$. By Rule R_{12} , x is placed above hole H_2 . By Lemma 5.4, x is placed on track C at some stage $t + 1 > s + 1$. Then $\nu_k <^r [\nu(t + 1, x, x)]_{|\nu_k|}$ and so, at stage $t + 1$, $\nu_k \in \mathcal{K}_{t+1}^2$ and $r(t + 1, \nu_k) > k$, contradicting the assumption that $\lim_s r(s, \nu_k) = k$. This final contradiction establishes the lemma. \square

LEMMA 5.6. *Any full e -state $\nu \in \mathcal{K}_\omega$ is excluded from \mathcal{M}_{s+1} by condition 3 of Definition 4.7(a) only finitely often.*

Proof. Let $k = \lim_s r(s, \nu)$ which exists since $\nu \in \mathcal{K}_\omega$. If ν is excluded by condition 3 at stage $s + 1$, some element $x = p(s, \nu')$ is placed above hole H_4 at stage $s + 1$ where x was assigned a marker $A_{\langle k', \ell', \nu' \rangle}$ at stage s with $k' \leq k$ and $\ell' \leq k$. Since there only finitely many such markers, ν is excluded by condition 3 only finitely often because of Lemma 5.5. \square

The preceding lemma says that condition 3 exclusion was not used too often. In Lemma 5.17 we will see that condition 3 exclusion is used often enough.

Recall that \mathcal{M}_ω was our approximation to those full e -states that are in $\mathcal{S}(D)$ infinitely often and are the maximal such e -states with respect to \leq^r . The next lemma derives the crucial properties of \mathcal{M}_ω and \mathcal{P}_ω under a certain assumption that will be verified later (in Lemma 5.9).

LEMMA 5.7. *Suppose that ν is excluded only finitely often from \mathcal{M}_{s+1} under condition 2 of Definition 4.7(a). Then*

- (1) $\nu \in \mathcal{K}_\omega$ and $\nu \in \mathcal{S}(D)$ i.o. $\implies \nu \in \mathcal{M}_\omega$, and
- (2) $\nu \in \mathcal{P}_s$ i.o. $\implies \nu \in \mathcal{P}_\omega$.

Proof. Fix $\nu = \langle e, \sigma, \tau \rangle$ with $\nu \in \mathcal{K}_\omega$ and $\nu \in \mathcal{S}(D)$ i.o. Consider the conditions under which ν could be excluded from \mathcal{M}_{s+1} . Since $\nu \in \mathcal{K}_\omega$, ν is excluded by condition 1 only finitely often. By hypothesis, ν is excluded by condition 2 only finitely often. Finally, ν is excluded by condition 3 only finitely often by Lemma 5.6. Thus, $\nu \in \mathcal{M}_\omega$ since $\nu \in \mathcal{S}(D)$ i.o.

(2) This follows from (1) as in [5, Lemma 5.3, p. 107]. \square

LEMMA 5.8. *With the same hypothesis as Lemma 5.7 for ν_1 and*

ν_2 , let ν_1 and ν_2 , be full e -states. Suppose that D τ -exactly covers ν_1 and Rule \hat{R}_4 applies at infinitely many stages s such that $\nu'(s) = \nu_1$ and $\nu''(s) = \nu_2$. Then D τ -exactly covers ν_2 .

Proof. Fix $\nu_1 = \langle e, \sigma_1, \tau_1 \rangle$ and $\nu_2 = \langle e, \sigma_2, \tau_2 \rangle$ satisfying the hypotheses and assume that D does not τ -exactly cover ν_2 . There is a full e -state $\nu_3 \in \mathcal{S}(D)$ i.o. with $\nu_1 \leq^{\tau} \nu_3$ and $\nu_3 \in \mathcal{H}_\omega$ for D τ -exactly covers ν_1 . If ν_3 is excluded from \mathcal{M}_s under condition 2 of Definition 3.4.7(a), so is ν_2 since $|\nu_3| = |\nu_2| = e$. Thus ν_3 is excluded only finitely often from \mathcal{M}_s under condition 2. Thus, by Lemma 5.7(1), $\nu_3 \in \mathcal{M}_\omega$. But since $\nu_2 \in \mathcal{S}(D)$ i.o. and $\nu_3 \in \mathcal{S}(D)$ i.o., for almost all s we must have that some occurrence of ν_3 follows the last occurrence of ν_2 on $\mathcal{S}_{\leq s}(D)$. Thus, for almost all s , ν_3 is preferred to ν_2 as $\nu''(s)$ when $\nu'(s) = \nu_1$. \square

Of course, Lemmas 5.7 and 5.8 have duals for \hat{M} which we have not discussed. However, the dual lemmas can be proved just as in [5] since we defined the lists $\mathcal{H}'_s, \mathcal{M}'_s, \mathcal{P}'_s$ just as in [5] because we have no new rules removing elements of \hat{M} to hole \hat{H}_4 .

The next lemma establishes that Rule R_3 does not require too much enumeration. Let $\mathcal{S}^e(X) = \{\nu : \nu \in \mathcal{S}^e(X) \text{ and } |\nu| \leq e\}$.

LEMMA 5.9. *Fix e . If each ν such that $|\nu| < e$ is added to \mathcal{H} at most finitely often then \hat{C} dual covers $\mathcal{S}(C_3)$.*

Proof. Soare gives a proof [5, Lemma 5.4, p. 108] which depends only on Lemma 5.2 and Rule R_3 . \square

Nevertheless, Rule R_3 allows enough enumeration to prove the following key lemma.

LEMMA 5.10. *If \hat{X} is any stream of \hat{M} , and C covers \hat{X} , then D τ -exactly covers \hat{X} .*

Proof. (This is essentially the same proof as that of [5, Lemma 5.6, p. 109] which has similar content.)

Fix a stream \hat{X} of \hat{M} , and assume for a contradiction that some $\nu_1 \in \mathcal{S}(\hat{X})$ i.o. and C covers ν_1 but no $\nu \in \mathcal{S}(D)$ i.o., if $\nu_1 \leq^{\tau} \nu$. Let $\nu_1 = \langle e, \sigma_1, \tau_1 \rangle$. Then there exists s' such that no ν is added to $\mathcal{S}(D)$ at any stage $s \geq s'$ if $\nu_1 \leq^{\tau} [\nu]_e$. Choose $e' \geq e$ and $e' > |\nu|$ for any $\nu \in \mathcal{S}(D)$ such that $\nu_1 \leq^{\tau} [\nu]_e$. Replacing ν_1 by some extension if necessary we may assume that $\nu_1 \in \mathcal{S}(\hat{X})$ i.o., C covers ν_1 , and

$$(6) \quad (\forall \nu)[\nu_1 \leq^{\tau} [\nu]_e \implies \nu \in \mathcal{S}(D)].$$

$\nu_1 \notin \mathcal{P}_s$ for any s . Thus, by the definition of \mathcal{H} , each $\nu \in \mathcal{H}$ i.o. if $\nu_1 \leq^r \nu$.

Since C covers ν_1 , some $\nu_2 = \langle e, \sigma_2, \tau_0 \rangle \in \mathcal{S}(C)$ i.o. (and hence $\nu_2 \in \mathcal{S}(C_1)$ i.o.) where $\sigma_2 \supseteq \sigma_1$ and $\tau_0 \supseteq \tau_1$. Furthermore, $\nu_1 \in \mathcal{S}_s(\hat{X}) - \mathcal{P}_s$ for infinitely many s implies that $\nu_3 = \langle e, \sigma_2, \tau_1 \rangle \in \mathcal{H}$ i.o.

Now $\nu_1 \leq^r \nu_3$, and thus by (6) $\nu_3 \notin \mathcal{S}(D)$. Hence, $\nu_3 \notin \mathcal{S}(C_3)$ and ν_3 once added to \mathcal{H} is never checked under Rule R_3 . Choose s_0 such that no ν preceding ν_3 on the sequence \mathcal{H} is checked at any stage $s \geq s_0$. Choose $s_1 \geq s_0$ such that some $x \geq e$ enters track C_1 at stage s_1 , where $\nu(s_1, e, x) = \nu_2 = \langle e, \sigma_2, \tau_0 \rangle$. But $\tau_0 \subseteq \tau_1$, and hence at stage $s + 1$ by Rule R_3 , $\nu_3 = \langle e, \sigma_2, \tau_1 \rangle$ is checked, and x is placed on track C_3 with $\nu(s_1 + 1, e, x) = \nu_3$, contrary to (6). \square

LEMMA 5.11. For each track X of M (\hat{X} or \hat{M}) and each ν ,

- (1) $\nu \in \mathcal{S}(\hat{X})$ i.o. $= \nu \in \mathcal{P}_\omega$, and
- (2) $\nu \in \mathcal{S}(X)$ i.o. $= \nu \in \mathcal{P}'_\omega$.

The proof is by induction on the length of ν . Fix e and suppose that (1) and (2) hold for all ν with $|\nu| < e$. It suffices to prove (1) for all ν of length e since (2) is dual. By inductive hypothesis (2), each ν of length $< e$ is added to \mathcal{H}' only finitely often. Thus by the dual of Lemma 5.9, C covers $\mathcal{S}^e(\hat{C}_3)$.

Now by inductive hypothesis (1), each ν of length e is excluded from \mathcal{M}_{s+1} under Condition 2 of Definition 4.7(a) only finitely often. Thus for every ν of length e we have

$$\begin{aligned} \nu \in \mathcal{S}(D) \text{ i.o. and } \nu \in \mathcal{H}_\omega &\implies \nu \in \mathcal{M}_\omega \\ \nu \in \mathcal{P}_s \text{ for infinitely many } s &\implies \nu \in \mathcal{P}_\omega. \end{aligned}$$

Now assume for a contradiction that for some \hat{X} and $\nu_1, \nu_1 \in \mathcal{S}(\hat{X})$ i.o., but $\nu_1 \notin \mathcal{P}_\omega$. Let $\nu_1 = \langle e, \sigma_1, \tau_1 \rangle$ with σ_1 minimal for e , and τ_1 minimal for e and σ_1 . Since $\nu_1 \notin \mathcal{P}_\omega$, $\nu_1 \in \mathcal{P}_s$ for finitely many s , and thus

$$\nu_1 \in \mathcal{S}_s(\hat{X}) - \mathcal{P}_s \text{ for infinitely many } s.$$

But for each such s all ν' of length $> e$ are excluded from \mathcal{M}_{s+1} under Condition 2 and hence

$$\nu \in \mathcal{M}_\omega \implies |\nu| \leq e.$$

Now choose elements \hat{y}_j and corresponding stages $s_j + 1$ such that for all $j \in N$,

$$\nu_1 = \nu(s_j + 1, e, \hat{y}_j) \neq \nu(s_j, e, \hat{y}_j).$$

Note that s_j exists since $\nu(v_j, e, \hat{y}_j) = \langle e, \phi, \phi \rangle \in \mathcal{P}_\omega$ where v_j is the

stage that \hat{y}_j enters \hat{M} . For each $j \in N$, define the finite sequence of full e -states,

$$\mathcal{F}_j = \{ \nu : (\exists s)[v_j \leq s \leq s_j + 1 \text{ and } \nu = \nu(s, e, \hat{y}_j)] \},$$

where v_j is the stage when \hat{y}_j entered \hat{M} . Let \mathcal{F} be the concatenation of $\{\mathcal{F}_j : j \in N\}$. From the definition of \mathcal{F} , note that if $\nu \in \mathcal{F}$ i.o., then $\nu' \in \mathcal{H}$ i.o. for all ν' such that $\nu \leq^c \nu'$. Thus by Rule R_3 we have

$$\nu \in \mathcal{F} \text{ i.o. and } C \text{ covers } \nu \implies D \text{ } \tau\text{-exactly covers } \nu.$$

But $\nu_1 \notin \mathcal{P}_\omega$, so D does not τ -exactly cover ν_1 and hence

$$C \text{ does not cover } \nu_1.$$

On the other hand we shall get a contradiction from this by proving that

$$C \text{ covers } \mathcal{F}.$$

First note that by the minimality of σ_1 and τ_1 above we have,

$$\nu \in \mathcal{F} \text{ i.o. and } \nu \neq \nu_1 \implies \nu \in \mathcal{P}_\omega,$$

and thus,

$$(a.a.j) (\forall s)_{\leq s_j} [d(s, y_j) \geq e],$$

by the definition of $d(s, y)$ from \mathcal{P}_s , and by induction on t for $v_j \leq t \leq s_j$.

Assume for a contradiction that C does not cover \mathcal{F} , and choose $\nu_0 = \langle e, \sigma_0, \tau_0 \rangle$ such that

$$\nu_0 \in \mathcal{F} \text{ i.o. but } C \text{ does not cover } \nu_0,$$

where σ_0 and τ_0 are minimal. Choose an infinite set J and stages $t_j, t_j + 1 \leq s_j + 1$ such that

$$(\forall j \in J)[\nu_0 = \nu(t_j + 1, e, \hat{y}_j) \neq \nu(t_j, e, \hat{y}_j)].$$

(Of course, $t_j + 1 > v_j$ since \hat{y}_j changes e -state only when in machine \hat{M} .)

$$C \text{ covers the sequence } \{ \nu(t_j, e, \hat{y}_j) : j \in J \}.$$

Now by Rule \hat{R}_0 , for almost all $j \in J$, either Rule \hat{R}_3 or \hat{R}_4 applies to \hat{y}_j at stage $t_j + 1$. Now Rule \hat{R}_3 applies for at most finitely many $j \in J$ because if Rule \hat{R}_3 applies infinitely often, $\nu_0 \in \mathcal{S}(\hat{C}_3)$ i.o. but then C , which covers $\mathcal{S}(\hat{C}_3)$, would cover ν_0 .

Thus, for almost all $j \in J$, Rule \hat{R}_4 applies to \hat{y}_j at stage $t_j + 1$,

with $\nu_0 \preceq \nu''(t_j + 1)$. However, $\mathcal{M}_s[\nu_0] \subseteq \{\nu_0\}$ for almost all s since $\nu \in \mathcal{M}_\omega \Rightarrow |\nu| \leq e$ and D does not τ -exactly cover ν_0 . Hence, $\nu''(t_j + 1) = \nu_0$ for almost all $j \in J$. Fix any ν'_0 such that $\nu'_0 = \nu'(t_j + 1)$ in Rule \hat{R}_4 for infinitely many $j \in J$. C covers ν'_0 , and $\nu'_0 \in \mathcal{F}$ i.o., so D τ -exactly covers ν'_0 . Thus, by Lemma 5.8, D τ -exactly covers ν_0 , and therefore C covers ν_0 which is the desired contradiction. \square

LEMMA 5.12. C covers \hat{C}_3 and \hat{C} dual covers C_3 .

Proof. The proof in [5, Lemma 5.12, p. 113] depends only on Lemma 5.11 and Rule R_3 (\hat{R}_3). \square

LEMMA 5.13. $(\forall e)$ (a.a.s) (a.a. \hat{y}) [$\hat{y} \in \hat{M}_e - \hat{A}_e \Rightarrow d(s, \hat{y}) \geq e$].

Proof. [5, Lemma 5.13, p. 113]. (This uses only Lemma 5.11 and the definition of d .) \square

LEMMA 5.14. Given ν_1 and infinitely many elements $\hat{y}_j, j \in N$, such that for all $j \in N$, Rule \hat{R}_4 applies to \hat{y}_j at stage s_j with $\nu_1 \preceq \nu(s_j + 1, y_j, \hat{y}_j)$, then C covers ν_1 .

Proof. The proof [5, Lemma 5.14, p. 113] depends only on Lemma 5.13 and Rule \hat{R}_4 . \square

LEMMA 5.15. C covers every stream \hat{X} of \hat{M} , $\hat{X} \neq \hat{A}$.

Proof. [5, Lemma 5.15, p. 113]. The proof depends on Lemma 5.12 and Lemma 5.14. \square

LEMMA 5.16. D τ -exactly covers every stream \hat{X} of \hat{M} , $\hat{X} \neq \hat{A}$.

Proof. Lemmas 5.15 and 5.10. \square

Of course the dual Lemmas 5.13–5.16 have similar proofs.

The next lemma is crucial for constructing the permutation of \bar{A} to \bar{B} . The statement of the lemma is the same as [5, Lemma 6.1] but the proof is different due to the complications of our construction. It is here that the condition 3 exclusion of Definition 4.7(a) plays a crucial role. If $\hat{q}(\nu_0) = \lim_s \hat{q}(s, \nu_0)$ exists but $p(\nu_0)$ does not, we would like it to be the case that $\hat{q}(\nu)$ exists for only finitely many $\nu \succeq \nu_0$. This is done by arranging that $\mathcal{M}_\omega[\nu_0]$ is finite since if $\hat{x} \in \hat{Q}_\omega, \nu^*(\hat{x}) \in \mathcal{M}_\omega$. As in [5], condition 1 exclusion handles the cases where $p(s, \nu_0)$ becomes undefined due to changes in $|\nu_0|$ -state.

Condition 3 was designed to handle the case in which $p(s, \nu_0)$ becomes undefined because $p(s, \nu_0)$ enters A .

LEMMA 5.17. *Fix ν_0 and suppose that*

- (1) $(\forall \nu < \nu_0)$ [$\lim_s p(s, \nu)$ exists],
- (2) $(\forall \nu \preceq \nu_0)$ [$\lim_s \hat{q}(s, \nu)$ exists], and
- (3) $\lim_s p(s, \nu_0)$ does not exist.

Then $\mathcal{M}_\omega[\nu_0]$ is finite.

Proof. Suppose that $\nu_0 = \langle e_0, \sigma_0, \tau_0 \rangle$ and let s_0 be a stage such that

$$(\forall \nu < \nu_0)[p(s_0, \nu) = p(\nu)] , \text{ and}$$

$$(\forall \nu \preceq \nu_0)[\hat{q}(s_0, \nu) = \hat{q}(\nu)] .$$

We will show that, if $\nu_1 \in \mathcal{M}_{s_1}[\nu_0] - \mathcal{M}_{s_1-1}[\nu_0]$ for some $s_1 > s_0$ and $r(s_1, \nu_1) > s_0$, then ν_1 is excluded from \mathcal{M}_{t+1} for some stage $t \geq s_1$. (This clearly implies that $\mathcal{M}_\omega[\nu_0]$ is finite since $\mathcal{M}_\omega[\nu_0] \subseteq \mathcal{M}_{s_0}[\nu_0] \cup \{\nu_k : k \leq s_0\}$.) If ν_1 is never excluded from \mathcal{M}_{t+1} for $t \geq s_1$, by Condition 1 of Definition 4.7(a), it must be the case that $\lim_s r(s, \nu) = r(s_1, \nu)$ for all ν such that $r(s_1, \nu) \leq r(s_1, \nu_1)$.

Since $\nu_1 \in \mathcal{M}_{s_1} - \mathcal{M}_{s_1-1}$, some element x is on track D at stage s_1 in state $\nu(s_1, x, x)$ such that

$$\nu_0 \preceq \nu_1 \preceq \nu(s_1, x, x) .$$

By Rule R_2 , x is preferred as $p(s_1 + 1, \nu_0)$ unless $\rho(s_1 + 1, \nu_0, p(s_1, \nu_0)) \leq r(s_1 + 1, \nu_1)$. Thus, in any case,

$$\rho(s_1 + 1, \nu_0, p(s_1 + 1, \nu_0)) \leq r(s_1 + 1, \nu_1) .$$

Now $p(s, \nu_0) = p(s + 1, \nu_0)$ unless Rule R_2, R_{10}, R_{11}, R_D or Subcase 3(c) of the construction removes it. It is easy to see that Rules R_{11} and R_D never apply to $p(s, \nu_0)$ for any $s > s_0$ since $p(s_0, \nu') = p(\nu')$ for every $\nu' < \nu_0$. Rule R_2 can remove $p(s, \nu_0)$ from P_s in two ways after stage s_0 : by placing $p(s, \nu_0)$ above hole H_4 at stage $s + 1$ or by defining $p(s + 1, \nu_0)$ to be a new element. Suppose Rule R_2 is used to replace $p(s, \nu_0)$ at some state $s_2 + 1 > s_1 + 1$ before any of the other Rules are used to remove $p(s, \nu_0)$. Then, by induction on stages s such that $s_1 + 1 \leq s \leq s_2$,

$$\rho(s_2, \nu_0, p(s_2, \nu_0)) = \rho(s_1 + 1, \nu_0, p(s_1, \nu_0)) \leq r(s_1 + 1, \nu_1) .$$

This is true by Lemma 4.3, the fact that $p(s_2, \nu_0) = p(s_1, \nu_0)$ does not change in state while in pocket P , and the fact that $\lim_s r(s, \nu) = r(s_1, \nu)$ for all ν such that $r(s_1, \nu) \leq r(s_1, \nu_1)$. At stage $s_2 + 1, \rho(s_2, \nu_0, p(s_2 + 1, \nu_0)) < \rho(s_2, \nu_0, p(s_2, \nu_0))$. Thus, by induction on stages $s > s_1$,

$\rho(s, \nu_0, p(s, \nu_0))$ decreases with each application of Rule R_2 to $p(s, \nu_0)$ as long as no other rule intervenes.

Thus we have shown that there must be a stage $s + 1 > s_1 + 1$ such that $p(s, \nu_0)$ is removed from pocket P at stage $s + 1$ by being removed to hole H_4 (by Subcase 3(c) or Rule R_2) or by Rule R_{10} . Furthermore, $\rho(s, \nu_0, p(s, \nu_0)) \leq r(s_1 + 1, \nu_1)$.

If $p(s, \nu_0)$ is removed to hole H_4 , it is easy to see that ν_1 is excluded from \mathcal{M}_{s+1} by condition 3 of Definition 4.7(a) since

$$r(s_1, \nu_1) > s_0 \geq (\mu t \leq s)(\forall z)_{t \leq z \leq s}(\forall \nu' \simeq \nu_0)[\hat{q}(z, \nu') = \hat{q}(s, \nu')] .$$

and

$$r(s_1, \nu_1) \geq \rho(s, \nu_0, p(s, \nu_0)) .$$

Suppose, on the other hand, that $p(s, \nu_0)$ is removed from pocket P at stage $s + 1$ by Rule R_{10} . Then

$$\nu(s + 1, e_0, p(s, \nu_0)) \tau > \nu(s, e_0, p(s, \nu_0))$$

and, in fact, if the state corresponding to $\rho(s, \nu_0, x)$ is $\langle e_1, \sigma_1, \tau_1 \rangle$, $\nu(s + 1, e_1, p(s, \nu_0)) \tau > \nu(s, e_1, p(s, \nu_0))$ since $e_0 \leq e_1$. But then at some later stage $t + 1 > s + 1$, $p(s, \nu_0)$ enters track C in full e_1 -state $\nu(s + 1, e_1, p(s, \nu_0))$ by Lemma 5.4. This causes ν_1 to be excluded from \mathcal{M}_{s+1} by condition 1 and the fact that $r(s_1 + 1, \nu) \geq \rho(s, \nu_0, p(s, \nu_0))$ and $\nu(s, e_1, p(s, \nu_0)) \in \mathcal{N}_{t+1}^2$. This last contradiction shows that ν_1 must in fact be excluded from \mathcal{M}_s , for some $s > s_1$. □

LEMMA 5.18. *Fix ν and suppose that there are infinitely many $\hat{y}_j \in \hat{Q}_\omega, j \in N$, such that $\nu \simeq \nu(y_j, \hat{y}_j), j \in N$. Then $\lim_s p(s, \nu)$ exists.*

Proof. By the definition of ν^* and Lemma 5.13 it is clear that $\nu \simeq \nu^*(\hat{y}_j)$ for almost every j . Now by Lemma 5.14, if $\lim_s p(s, \nu)$ does not exist, then $\mathcal{M}_\omega[\nu]$ is finite. But if $\hat{y}_j \in \hat{Q}_\omega, \nu^*(\hat{y}_j) \in \mathcal{M}_\omega$ by Lemma 4.8. For any particular $\nu_0, \nu^*(\hat{y}_j) = \nu_0$ for only finitely many j . This is a contradiction. □

Recall that the desired permutation of \bar{A} to \bar{B} was the one that sent $\hat{q}(v)$ to $p(v)$ and $q(v)$ to $\hat{p}(v)$ for every full e -state ν . Of course, this is not quite right since $\hat{q}(v)$ may exist where $p(v)$ does not. In this case, however, Lemma 5.18 suggests a finite-one map which will suffice.

LEMMA 5.19. *There are finite-one maps $p_1: \hat{Q}_\omega \rightarrow P_\omega$ and $p_2: Q_\omega \rightarrow \hat{P}_\omega$ such that for every n*

$$(1) \quad p_1(\hat{Q}_\omega \cap \hat{U}_n^+) = {}^* P_\omega \cap U_n,$$

- (2) $p_1(\hat{Q}_\omega \cap V_n) = {}^* P_\omega \cap \hat{V}_n^+$,
- (3) $p_2(Q_\omega \cap \hat{V}_n^+) = {}^* \hat{P}_\omega \cap V_n$, and
- (4) $p_2(Q_\omega \cap U_n) = {}^* \hat{P}_\omega \cap \hat{U}_n^+$.

Proof. We define p_1 ; p_2 is similar. If $p(\nu)$ is defined, let $p_1(q(\nu)) = p(\nu)$. If $p(\nu)$ is undefined, let $\nu' \preceq \nu$ the maximal full e -state in the ordering \preceq such that $p(\nu')$ is defined and define $p_1(q(\nu)) = p(\nu')$. This map is finite-one by Lemma 5.18 and meets (1) and (2) since $q(\nu)$ and $p(\nu)$ are in the same full $|\nu|$ -state. □

Although we have not given a one-to-one map from \bar{A} to \bar{B} inducing an automorphism on \bar{A} , the details on constructing such a permutation are straightforward, given Lemma 5.19, and can be found in [5, Corollary 1.7, p. 85]. Notice that \bar{A} is infinite, since \bar{B} is. Also, note that $U_n = {}^* V_n = {}^* W_n$, since if $x \in W_n$, $x \in U_n(x \in V_n)$ unless $x = p(\nu)$ ($\hat{x} = \hat{p}(\nu)$) for some ν such that $|\nu| < n$. (Note that x cannot be in pocket P infinitely often without being $p(\nu)$ for some ν since x enters P only finitely often by Lemma 5.1.)

We can now show that the hypothesis of the Extension Theorem [5, Theorem 2.2] are met by our enumeration of A and B . It is easy to see that this hypothesis is simply

LEMMA 5.20. C_\circ covers \hat{C}_\circ ; \hat{C}_\circ dual covers C_\circ .

Proof. \hat{C} dual covers every stream X of M . Thus \hat{C}_\circ dual covers every stream X of M by Lemma 4.1. Suppose $\nu \in \mathcal{S}(C_\circ)$ i.o. Then there are infinitely many $\nu' \in \mathcal{S}(\hat{C}_\circ)$ such that $\nu' \leq \nu$. Suppose ν appears as the e th element of $\mathcal{S}_{\leq s}(C_\circ)$. Then at some stage $s_0 > s$ some x not in $\bigcup_{i \leq e} D_{h(i)}$ must enter \hat{C}_\circ in full e -state $\nu' \leq \nu$. Thus, x checks ν and enters \hat{C}_\circ by Rule \hat{R}_B . Thus \hat{C}_\circ dual covers C_\circ since any x checks only finitely many occurrences of ν on $\mathcal{S}_s(C_\circ)$ and each occurrence is eventually checked.

The proof that C_\circ covers \hat{C}_\circ is similar. Instead of using some x not in $\bigcup_{i \leq e} D_{h(i)}$ to cover the e th element on $\mathcal{S}(\hat{C}_\circ)$, we use any x . Since C_\circ covers \hat{C}_\circ , eventually some such x in the appropriate e -state appears on track C_\circ . □

LEMMA 5.21. A is dense simple.

Proof. Subcase 3(c) of Step 1 of the construction guarantees that $p_{\bar{A}}$ dominates every (partial) recursive function. □

Lemma 5.21 shows that A is dense simple, Lemma 5.3 that B is not hypersimple. Lemma 5.20 and the Extension Theorem of

Soare guarantee the existence of a permutation p and r.e. sets $\{\hat{U}_n^-\}_{n \in N}$ and $\{\hat{V}_n^-\}_{n \in N}$ extending our permutation of Lemma 5.19 to an automorphism of \mathcal{E}^* taking A^* to B^* .

6. Open questions. If B is any simple set, is there a dense simple set A and an automorphism Φ of \mathcal{E} such that $\Phi(A) = B$? It is known that A can be found such that $\mathcal{L}^*(B) \cong \mathcal{L}^*(A)$.

What is the class of r.e. sets A such that $\Phi(A)$ is hypersimple for every automorphism Φ of \mathcal{E} ? Only trivial results about this question are known.

Is the property of strong hypersimplicity invariant? (A is *strongly hypersimple* if there does not exist a recursive function f such that $\bigcup_n W_{f(n)} = N$, $W_{f(n)} \cap \bar{A} \neq \emptyset$ for each n , and the r.e. sets $W_{f(n)}$ are finite and pairwise disjoint.) This question may require yet another modification of the automorphism machinery to answer.

Much recent research has identified new properties of r.e. sets which are invariant under $\text{Aut}(\mathcal{E})$ (see [1], [2], and [4] for example). Can the methods now known for constructing automorphisms of \mathcal{E} be used to find some other orbit of $\text{Aut}(\mathcal{E})$ besides the maximal sets?

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