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NON-NORMAL BLASCHKE QUOTIENTS

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A quotient B_1/B_2 of two infinite Blaschke products B_1 and B_2 with no common zero is called a Blaschke quotient. The existence of a Blaschke quotient which is not normal in the open unit disk D, is well known. We shall show among other things, that, for each $p, 0 < p < \infty$, there exists a nonnormal Blaschke quotient f such that

$$\iint_D (1-|z|)^p |f'(z)|^2/(1+|f(z)|^2)^2 dxdy < \infty .$$

This might be of interest because, if g is meromorphic in D and if $\iint_D |g'(z)|^2/(1+|g(z)|^2)^2 dxdy < \infty$, then g is normal in D.

1. Introduction. By a Blaschke product we mean a holomorphic function in $D = \{|z| < 1\}$, denoted by

$$B(z; \{c_n\}) = \prod_{n=1}^{\infty} \frac{|c_n|}{c_n} \frac{c_n - z}{1 - \bar{c}_n z} ,$$

where $\{c_n\}$ is an infinite complex sequence satisfying $0 < |c_n| < 1$, $n = 1, 2, \cdots$, and $\sum (1 - |c_n|) < \infty$. By a Blaschke quotient we mean a meromorphic function in D, defined by

$$Q(z; \{c_n\}, \{c'_n\}) = B(z; \{c_n\})/B(z; \{c'_n\}) ,$$

where the Blaschke products in the right-hand side have no zero in common.

A meromorphic function f in D is called normal in D if $\sup_{z \in D} (1 - |z|)f^{\#}(z) < \infty$, where $f^{\#} = |f'|/(1 + |f|^2)$; see [5]. We shall construct nonnormal Blaschke quotients with some additional properties. It is easy to merely construct a nonnormal Blaschke quotient. For example, set $c_n = 1 - (2n)^{-\lambda}$ and $c'_n = 1 - (2n+1)^{-\lambda}$, $n = 1, 2, \cdots$, where $\lambda > 1$ is a constant. Then $Q(z) = Q(z; \{c_n\}, \{c'_n\})$ is not normal. Actually, let

$$\sigma(z_1, z_2) = \frac{1}{2} \log \frac{1 + \rho(z_1, z_2)}{1 - \rho(z_1, z_2)}$$

be the non-Euclidean distance between z_1 and z_2 in D, where

$$\rho(z_1, z_2) = |z_1 - z_2|/|1 - \bar{z}_1 z_2| .$$

Then, $Q(c_n) = 0$, $Q(c'_n) = \infty$, $n \geqq 1$, and $\lim_{n\to\infty} \sigma(c_n, c'_n) = 0$. Therefore, Q is not uniformly continuous from D, endowed with $\sigma(\cdot,\cdot)$, into the Riemann sphere, endowed with the spherical chordal distance. Consequently, Q is not normal in D. Accordingly, J. A. Cima [3, Theorem 4] proved the existence of a nonnormal Blaschke quotient $Q(z; \{c_n\}, \{c'_n\})$ with $\inf_{j,k\geqq 1}\sigma(c_j, c'_k) > 0$.

There is another way of finding nonnormal Blaschke quotients. Namely, if a Blaschke quotient Q has two asymptotic values at one boundary point of D, then Q is not normal in D by [5, Theorem 2]. Therefore, one can easily conclude that the Blaschke quotients found by D. A. Storvick [6, p. 37] and C. Tanaka [7, Theorem 2] both are not normal in D. A meromorphic function f in D is said to have the left angular limit w (possibly, ∞) at 1 if $f(z)\to w$ as $z\to 1$ within each triangular domain whose vertices are 1 and two points in $D^+ = \{z\in D \mid \operatorname{Im} z > 0\}$. Also, f is said to have the right angular limit w at 1 if $\overline{f(\bar z)}$ has the left angular limit $\bar w$ at 1 (convention: $\bar\infty = \infty$). A Blaschke quotient $Q(z) = Q(z; \{c_n\}, \{c'_n\})$ is called symmetric if $\bar c_n = c'_n$ for each n. If Q is symmetric, then $Q(z)\overline{Q(\bar z)} \equiv 1$ in D, so that Q has the left angular limit w at 1 if and only if Q has the right angular limit $1/\bar w$ (convention: $1/0 = \infty$, $1/\infty = 0$) at 1. Therefore, if Q is symmetric and if Q has the left angular limit 0 at 1, then Q is never normal in D because Q has 0 and ∞ as asymptotic values at 1.

Now, for f meromorphic in D, we set

$$S_p(f) = \iint_D (1-|z|)^p f^\#(z)^2 dxdy\ , \qquad z = x+iy\ , \quad 0 \leqq p < \infty\ .$$

It is familiar that if $S_0(f) < \infty$, then f is normal in D. It is not difficult to observe that $S_1(Q) < \infty$ for each Blaschke quotient Q. In effect, since Q is of bounded characteristic in the sense of R. Nevanlinna, it follows from

$$\int_0^1 \left[\iint_{|z|<r} Q^\#(z)^2 dxdy\right] dr < \infty\ ,$$

that $S_1(Q) < \infty$; see (2.10) in §2.

Our first result is

THEOREM 1. *Let $0 < p < 1$, and let $0 < q < \infty$. Then there exists a symmetric Blaschke quotient $Q(z) = Q(z; \{a_n\}, \{\bar a_n\})$ satisfying the following three conditions.*

(I) $\inf_{j,k\geqq 1}\sigma(a_j, \bar a_k) \geqq q$.

(II) *Q has 0 as the left angular limit at 1.*

(III) $S_p(Q) < \infty$.

If we restrict p in (III) of Theorem 1 as $1/2 < p < 1$, then we can construct Q with an additional property.

By a left horocyclic angle at 1 we mean a domain

$$\{z \in D^+ \mid 1 - x_1 < |z - x_1| \quad \text{and} \quad 1 - x_2 > |z - x_2|\} ,$$

where $0 < x_2 < x_1 < 1$. A meromorphic function f in D is said to have the left horocyclic angular limit w at 1 if $f(z) \to w$ as $z \to 1$ within each left horocyclic angle at 1; the notion is essentially due to F. Bagemihl [1]. Also, f is said to have the right horocyclic angular limit w at 1 if $\overline{f(\bar{z})}$ has $\bar{w}$ as the left horocyclic angular limit at 1. Again, a symmetric Blaschke quotient Q has the left horocyclic angular limit w at 1 if and only if Q has the right horocyclic angular limit $1/\bar{w}$ at 1. Therefore, if a symmetric Q has the left horocyclic angular limit 0 at 1, then Q is never normal in D.

THEOREM 2. *Let $1/2 < p < 1$, and let $0 < q < \infty$. Then there exists a symmetric Blaschke quotient $Q(z) = Q(z; \{a_n\}, \{\bar{a}_n\})$ satisfying the same conditions as* (I), (II), *and* (III) *in Theorem* 1, *together with*

(IIH) *Q has* 0 *as the left horocyclic angular limit at* 1.

Lastly in the present section, we remark that Cima and P. Colwell [4, Theorem 2] proposed a necessary and sufficient condition for a Blaschke quotient to be normal in D in terms of interpolating sequences.

2. Proof of Theorem 1. By the linear transformation $w = \varphi(z) \equiv (1 + z)/(1 - z)$, the disk D is mapped onto the right half-plane R, so that, $R^+ = \varphi(D^+)$ is the first quadrant in the w-plane. Furthermore, by φ, the chord $L(\theta) = \{z \in D \mid \arg(1 - z) = \theta\}$, $|\theta| < \pi/2$, is mapped onto the half-line:

$$\Lambda(\theta) = \{w = x + iy \in R \mid y = (-\tan\theta)(x + 1)\} .$$

By a simple calculation one obtains

$$(2.1) \qquad 1 - |z|^2 = 4 \operatorname{Re} w / |w + 1|^2 , \qquad w = \varphi(z) , \quad z \in D ,$$

and

$$(2.2) \qquad \rho(z_1, z_2) = |w_1 - w_2| / |\bar{w}_1 + w_2|$$

for $w_j = \varphi(z_j)$, $z_j \in D$, $j = 1, 2$.

To costruct Q we choose A, $0 < A < 1$, such that

$$(2.3) \qquad \frac{1}{2} \log \frac{1 + t}{1 - t} = q \quad \text{and} \quad t = A/(1 + A^2)^{1/2} .$$

Choose θ_0, $-\pi/2 < \theta_0 < 0$, so that $A = -\tan\theta_0$, and then choose $s > 1/p > 1$. Consider the sequence of points $b_n \in \Lambda(\theta_0)$ such that $b_n = x_n + iy_n = n^s + iA(n^s + 1)$, $n = 1, 2, \cdots$. Let $a_n = \varphi^{-1}(b_n)$, $n \geqq 1$. Then $\{a_n\} \subset L(\theta_0)$. We then set $Q(z) = Q(z; \{a_n\}, \{\bar{a}_n\})$. First of all, Q is well defined because, by (2.1),

$$\begin{aligned}\sum (1 - |a_n|) = \sum (1 - |\bar{a}_n|) \leqq \sum (1 - |a_n|)^p \\ \leqq \sum (1 - |a_n|^2)^p \leqq 4^p \sum n^{-sp} < \infty .\end{aligned} \tag{2.4}$$

Further, one observes that

$$|Q(z)| = g(w) \equiv \prod_{n=1}^{\infty} g_n(w) , \qquad w = \varphi(z) , \tag{2.5}$$

where $g_n(w) = |w^2 - b_n^2|/|w^2 - \bar{b}_n^2|$, $n \geqq 1$.

Proof of (I). Let $w = x + iy \in R$, $\zeta = \xi + i\eta \in R$, with $y \geqq A(x + 1)$, $\eta \geqq A(\xi + 1)$. Since

$$X \equiv (x + \xi)/(y + \eta) \leqq A^{-1} ,$$

it follows that

$$|w - \bar{\zeta}|/|w + \zeta| \geqq (X^2 + 1)^{-1/2} \geqq (1 + A^{-2})^{-1/2} = t .$$

In view of (2.2) one can now easily conclude that $\rho(a_j, \bar{a}_k) \geqq t$, so that $\sigma(a_j, \bar{a}_k) \geqq q$ for all $j, k \geqq 1$.

Proof of (II). To prove that

$$\lim_{\substack{z \to 1 \\ z \in L(\theta_0)}} Q(z) = 0 , \tag{2.6}$$

it suffices by (2.5) to show that

$$\lim_{\substack{w \to \infty \\ w \in \Lambda(\theta_0)}} g(w) = 0 . \tag{2.7}$$

Since $g_n(w) \leqq 1$ for all $w \in R^+$ and for all $n \geqq 1$, it follows that

$$g(w) \leqq g_n(w) \leqq 1 \quad \text{for all} \quad w \in R^+ \quad \text{and all} \quad n \geqq 1 . \tag{2.8}$$

Given $\varepsilon > 0$, one can find a natural number N such that $x_{n+1}/x_n - 1 < \varepsilon$ for all $n \geqq N$. Then, for each $w = x + iy \in \Lambda(\theta_0)$ with $x \geqq x_N$,

$$g(w) \leqq A_1\varepsilon , \qquad A_1 = \frac{1}{2}(A + A^{-1}) , \tag{2.9}$$

which proves (2.7). To make sure of (2.9), we first find $n \geqq N$ such that $x_n \leqq x \leqq x_{n+1}$. Then,

$$|w - b_n| = (1 + A^2)^{1/2}(x - x_n) \leqq (1 + A^2)^{1/2}(x_{n+1} - x_n) ,$$

$$|w + \bar{b}_n| \geqq x + x_n \geqq 2x_n ,$$

whence

$$|w - b_n|/|w + \bar{b}_n| \leqq \frac{1}{2}(1 + A^2)^{1/2}\varepsilon .$$

On the other hand,

$$|w + b_n|/|w - \bar{b}_n| \leqq [(x + x_n)^2 + A^2(x + x_n + 2)^2]^{1/2}/[A(x + x_n + 2)] \leqq (1 + A^{-2})^{1/2} ,$$

so that $g_n(w) \leqq A_1\varepsilon$. Therefore, in view of (2.8), one can assert (2.9).

Since $|Q(z)| = g(\varphi(z)) \leqq 1$ in D^+ by (2.8), and since (2.6) holds, it follows from E. Lindelöf's theorem [8, Theorem VIII. 10, p. 306], together with an obvious conformal homeomorphism from the upper half-disk onto D^+, mapping 0 to 1, that Q has the left angular limit zero at 1.

Proof of (III). We remember L. Carleson's family T_α of meromorphic functions h in D such that

$$I_\alpha(h) \equiv \int_0^1 (1 - r)^{-\alpha}\left[\iint_{|z|<r} h^\#(z)^2 dxdy\right]dr < \infty ,$$

where $0 \leqq \alpha < 1$; see [2, p. 19]. Letting $X_r(z)$ be the characteristic function of the disk $\{|z| < r\}$, one observes that

$$(2.10) \qquad \begin{aligned} I_\alpha(h) &= \int_0^1 (1 - r)^{-\alpha}\left[\iint_D X_r(z)h^\#(z)^2 dxdy\right]dr \\ &= \iint_D\left[\int_0^1 (1 - r)^{-\alpha}X_r(z)dr\right]h^\#(z)^2 dxdy = (1 - \alpha)^{-1}S_{1-\alpha}(h) . \end{aligned}$$

For a Blaschke quotient $Q_1(z) = Q(z; \{c_n\}, \{c'_n\})$ we assume that

$$\sum (1 - |c_n|)^{1-\alpha} < \infty \quad \text{and} \quad \sum (1 - |c'_n|)^{1-\alpha} < \infty .$$

Then it follows from [2, Theorem 2.2, p. 24] that $Q_1 \in T_\alpha$.

Returning to our Q, we can easily conclude from (2.4) that $Q \in T_{1-p}$, whence $S_p(Q) < \infty$ by (2.10).

REMARK. The Blaschke quotient Q, described in the second paragraph in § 1, satisfies $S_p(Q) < \infty$, for a p, $0 < p < 1$, provided that $\lambda < 1/p$.

3. Proof of Theorem 2. Let $\lambda > (1/2)(p^{-1} + 1)$ and $1/(2p) < \mu < 1$, and $y_{n,m} = n^\lambda m^\mu$ $(n, m = 1, 2, \cdots)$. Let t and A be as in (2.3).

Then, for each fixed $n \geqq 1$, we may find a natural number M_n such that

$$y_{n,m} \geqq A(n+1) \geqq A(n^{-1}+1) \quad \text{for all} \quad m \geqq M_n .$$

Then, for each fixed $n \geqq 1$, the points $b_{n,m} = n + iy_{n,m}$, $m \geqq M_n$, lie on the half-line $\Gamma(n) = \{w \in R^+ \mid \operatorname{Re} w = n\}$, so that $a_{n,m} = \varphi^{-1}(b_{n,m})$ $(m \geqq M_n)$ lie on the half-oricycle $C(n) = \varphi^{-1}(\Gamma(n))$. Similarly, for each fixed $n \geqq 2$, the points $b^*_{n,m} = n^{-1} + iy_{n,m}$, $m \geqq M_n$, lie on the half-line $\Gamma^*(n) = \{w \in R^+ \mid \operatorname{Re} w = n^{-1}\}$, so that $a^*_{n,m} = \varphi^{-1}(b^*_{n,m})(m \geqq M_n)$ lie on the half-oricycle $C^*(n) = \varphi^{-1}(\Gamma^*(n))$. Let $\{a_n\} = \{a_{n,m}\} \cup \{a^*_{n,m}\}$. Then $Q(z) = Q(z; \{a_n\}, \{\bar{a}_n\})$ is the requested. We first observe that, for $n \geqq 1$,

$$\beta_n \equiv \sum_{m \geqq M_n} [\operatorname{Re} b_{n,m}/|b_{n,m}+1|^2]^p \leqq n^{p(1-2\lambda)} \sum_{m=1}^{\infty} m^{-2p\mu} ,$$

and for $n \geqq 2$,

$$\beta^*_n \equiv \sum_{m \geqq M_n} [\operatorname{Re} b^*_{n,m}/|b^*_{n,m}+1|^2]^p \leqq n^{-p(1+2\lambda)} \sum_{m=1}^{\infty} m^{-2p\mu} .$$

Since $p(1+2\lambda) > p(2\lambda - 1) > 1$ and $2p\mu > 1$, it follows from (2.1) that

$$\begin{aligned} \sum (1-|a_n|) &\leqq \sum (1-|a_n|^2)^p \\ &\leqq 4^p\Big(\sum_{n=1}^{\infty} \beta_n + \sum_{n=2}^{\infty} \beta^*_n\Big) < \infty , \end{aligned} \tag{3.1}$$

so that Q is well defined. Now, one observes that

$$|Q(z)| = G(w) \equiv \prod_{n=1}^{\infty} G_n(w) \prod_{n=2}^{\infty} G^*_n(w) , \qquad w = \varphi(z) , \tag{3.2}$$

where

$$G_n = \prod_{m=M_n}^{\infty} g_{n,m} , \qquad G^*_n = \prod_{m=M_n}^{\infty} g^*_{n,m} ,$$

$$g_{n,m}(w) = |w^2 - b^2_{n,m}|/|w^2 - \bar{b}^2_{n,m}| ,$$

$$g^*_{n,m}(w) = |w^2 - b^{*2}_{n,m}|/|w^2 - \bar{b}^{*2}_{n,m}| .$$

Proof of (I). The same as that of (I) of Theorem 1.

Proofs of (II) *and* (IIH). We shall first show that

$$\lim_{\substack{z \to 1 \\ z \in C(n)}} Q(z) = 0 \quad \text{for all} \quad n \geqq 1 , \tag{3.3}$$

and

(3.4) $$\lim_{\substack{z\to 1\\ z\in C^*(n)}} Q(z) = 0 \quad \text{for all} \quad n \geqq 2 .$$

Since $g_{n,m}(w) \leqq 1$ and $g^*_{n,m}(w) \leqq 1$ for all $w \in R^+$ and for all possible pairs n, m, it follows that

(3.5) $$G(w) \leqq g_{n,m}(w) \leqq 1 , \qquad w \in R^+ , \qquad n \geqq 1 , \qquad m \geqq M_n ,$$

and

(3.6) $$G(w) \leqq g^*_{n,m}(w) \leqq 1 , \qquad w \in R^+ , \qquad n \geqq 2 , \qquad m \geqq M_n .$$

For the proof of (3.3), it suffices by (3.2) to show that

(3.7) $$\lim_{\substack{w\to\infty\\ w\in\Gamma(n)}} G(w) = 0 , \qquad n \geqq 1 .$$

Since $\mu < 1$, it follows that, for each $n \geqq 1$ and for a given $\varepsilon > 0$ there exists a natural number $M'_n \geqq M_n$ such that $y_{n,m+1} - y_{n,m} < \varepsilon$ for all $m \geqq M'_n$. Then, for each $w = n + iy \in \Gamma(n)$ with $y \geqq y_{n,M'_n}$, there exists $m \geqq M'_n$ such that $y_{n,m} \leqq y \leqq y_{n,m+1}$. Consequently,

$$|w - b_{n,m}|/|w + \bar{b}_{n,m}| \leqq (y_{n,m+1} - y_{n,m})/(2n)$$

and

$$\left|\frac{w + b_{n,m}}{w - \bar{b}_{n,m}}\right| \geqq \sqrt{1 + \frac{4n^2}{(2y_{n,m})^2}} \leqq \sqrt{1 + n^{2-2\lambda}} ,$$

so that, by (3.5), $G(w) \leqq g_{n,m}(w) \leqq k_n\varepsilon$, where k_n is a constant depending only on n. The proof of (3.7) is thus complete. Similarly we can prove, via (3.6), that

$$\lim_{\substack{w\to\infty\\ w\in\Gamma^*(n)}} G(w) = 0 , \qquad n \geqq 2 ,$$

which, together with (3.2), shows (3.4). By the Lindelöf theorem [8, Theorem VIII. 10, p. 306] again, (II) is established. For the proof of the horocyclic part, we first note that $|Q| \leqq 1$ in D^+. Set $\mathscr{C} = \{C(n) | n \geqq 1\} \cup \{C^*(n) | n \geqq 2\}$. Then for each left horocyclic angle H at 1, we may find members C_1 and C_2 of $\mathscr{C}$ such that the left horocyclic angle H_1 at 1, bounded by C_1 and C_2 and a line segment on the real axis, contains H. Since

$$\lim_{\substack{z\to 1\\ z\in C_j}} Q(z) = 0 , \qquad j = 1, 2 ,$$

by (3.3) and/or (3.4), it follows from another theorem of Lindelöf [8, Theorem VIII. 7, p. 304], via an obvious conformal homeomorphism, that $Q(z)$ has the limit 0 as $z \to 1$ within H_1 containing H. We have thus established (IIH).

Proof of (III). The same as that of (III) of Theorem 1.

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