PACIFIC JOURNAL OF MATHEMATICS
Vol. 101, No. 1, 1982

SCHAUDER BASES AND FIXED POINTS OF NONEXPANSIVE MAPPINGS

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A fixed point theorem is proved for nonexpansive mappings in Banach spaces which are isomorphic to spaces with certain boundedly complete bases.

1. Introduction. Suppose X and Y are isomorphic Banach spaces with $h\|\cdot\|_Y \leqq \|\cdot\|_X \leqq k\|\cdot\|_Y$, where $\|\cdot\|_Y$ and $\|\cdot\|_X$ denote the norms in Y and X respectively. Let $t = kh^{-1}$ (this notation will be kept fixed throughout the paper). Suppose also that every convex weakly compact (weak* compact, when X is a dual Banach space) subset K of X has the fixed point property with respect to nonexpansive mappings (i.e., mappings $T: K \to K$ such that $\|Tx - Ty\|_X \leqq \|x - y\|_X$, for all $x, y \in K$). It is not known in general whether, assuming t sufficiently close to 1, convex weakly compact (weak* compact) subsets of Y have the same property (but see Bynum [1]).

In this paper we answer in the affirmative this question when X has a Schauder basis (b_n) which satisfies a condition introduced by Gossez and Lami Dozo [2]. For every positive integer k and $x \in X$ set $U_k(x) = \sum_{n=1}^{k} f_n(x)b_n$, where (f_n) denotes the associated system of linear functionals. We shall always assume that there exists a strictly increasing sequence (k_n) with the following property:

for every $c > 0$ there is $\rho > 0$ such that whenever $x \in X$ and n satisfy

$$\|U_{k_n}(x)\|_X = 1$$
$$\|x - U_{k_n}(x)\|_X \geqq c$$

then $\|x\|_X \geqq 1 + \rho$.

It is easy to see (Lemma 1 below) that the above condition implies that the basis (b_n) is boundedly complete, so that X is a dual Banach space.

In the next sections it is proved that there exists $t_0 > 1$ such that for $t < t_0$ every weak* compact convex subset of Y has the fixed point property with respect to nonexpansive mappings. For $t = 1$ this follows easily from the results of Karlovitz [3], while for $t > 1$ it can not be deduced from [3]. As a remarkable consequence we obtain that, in every Banach space Y isomorphic to l^1 with $t < 2$, weak* compact convex subsets have the fixed point property with respect to nonexpansive mappings.

2. Properties of the space X.

LEMMA 1. *Suppose X is a Banach space with a Schauder basis (b_n) satisfying the assumptions of the above section. Then the basis (b_n) is boundedly complete and X is isomorphic to the dual of the Banach space generated by the system of the linear functionals (f_n).*

Proof. Suppose that (a_n) is a sequence of scalars such that $\sup_N \|\sum_{n=1}^N a_n b_n\|_X < \infty$. Then, the same argument as in [6, p. 290-291] implies that, for some subsequence k_{n_j}, $\sum_{n=1}^{k_{n_j}} a_n b_n$ converges to a point $x \in X$. Then, of course, $f_n(x) = a_n$ for every n, so that $\sum_{n=1}^\infty a_n b_n = x$. The second assertion is proved in [6, Th. II 6.2, 3)].

For every positive integer n and every real $c > 0$ we set $r_n(c) = \inf \|x\|_X - 1$, where the infimum is taken over all $x \in X$ such that $\|U_{k_n}(x)\|_X = 1$, $\|x - U_{k_n}(x)\|_X \geqq c$. We set also $r(c) = \inf_n r_n(c)$. Clearly $r(c) > 0$ for all positive c. We complete the definition of $r(c)$ by letting $r(0) = 0$. In the following we set $V_{k_n}(x) = x - U_{k_n}(x)$.

LEMMA 2. *$r(c)$ is a nondecreasing continuous function of c.*

Proof. Let $\varepsilon > 0$ be arbitrarily small and $c_2 > c_1 \geqq 0$. There exist $n > 0$ and $x \in X$ such that $\|U_{k_n}(x)\|_X = 1$, $\|V_{k_n}(x)\|_X \geqq c_2$ and $1 + r(c_2) + \varepsilon > \|x\|_X \geqq 1 + r(c_1)$. Hence $r(c_2) \geqq r(c_1) \geqq 0 = r(0)$ and $r(c)$ is nondecreasing.

Observe now that there exist a sequence of points $x_j \in X$ and a sequence of positive integers n_j such that

$$\|U_{k_{n_j}}(x_j)\|_X = 1\,, \qquad \|V_{k_{n_j}}(x_j)\|_X \geqq c_1 \quad \text{and} \quad 1 + r(c_1) + j^{-1} > \|x_j\|_X\,.$$

We set $v_j = \|V_{k_{n_j}}(x_j)\|_X$. After extracting a subsequence if necessary, we may suppose that $v = \lim_j v_j$ exists. If $v > c_2$, then, for large values of j, $1 + r(c_1) + j^{-1} > \|x_j\|_X \geqq 1 + r(c_2)$, so that, by what has been already proved, $r(c_1) = r(c_2)$, and we are done. Thus we may assume $c_1 \leqq v \leqq c_2$. Let $y_j = x_j + s_j V_{k_{n_j}}(x_j)$, where s_j is a scalar such that $(1 + s_j)v_j = c_2$. Clearly we must have $\|y_j\|_X \geqq 1 + r(c_2)$ and $\|x_j - y_j\|_X = |s_j| v_j$. Hence

$$\begin{aligned} 1 + r(c_1) + j^{-1} > \|x_j\|_X &\geqq \|y_j\|_X - |s_j| v_j \\ &\geqq 1 + r(c_2) - |s_j| v_j \end{aligned}$$

that is,

$$r(c_2) - r(c_1) \leqq |s_j| v_j + j^{-1}\,.$$

Now, if $v < c_2$, then $|s_j| = s_j = (c_2 - v_j)v_j^{-1} \leqq (c_2 - c_1)v_j^{-1}$ for j large enough. If $v = c_2$ then s_j tends to 0, so that, if j is large, $|s_j| <$

$(c_2 - c_1)v_j^{-1}$. In any case, for large values of j, we obtain $r(c_2) - r(c_1) \leqq (c_2 - c_1) + j^{-1}$, and the proof is ended.

LEMMA 3. *Suppose that* $(x_n) \subseteqq X$ *is a sequence of points converging in the weak* topology to a point* $z \in X$. *Let* $\gamma = \limsup_n \|x_n - z\|_X$. *Then, for every* $y \in X$, $y \neq z$

$$\limsup_n \|x_n - y\|_X \geqq \{1 + r(\gamma\|y - z\|_X^{-1})\}\|y - z\|_X .$$

Proof. Let $\varepsilon > 0$ be arbitrarily small. There exists $j = j(\varepsilon)$ such that $\|V_{k_j}(y - z)\|_X < \varepsilon$. Since $x_n - z$ converges weak* to 0 and the associated functionals f_n are weak* continuous (Lemma 1), for every fixed j we can find n_0 such that $\|U_{k_j}(x_n - z)\|_X < \varepsilon$ for n greater than n_0. Therefore, for $n > n_0$, we have by Lemma 2

$$\begin{aligned}
&\|y - x_n\|_X \\
&\quad \geqq -2\varepsilon + \|U_{k_j}(y - z) + V_{k_j}(z - x_n)\|_X \\
&\quad \geqq -2\varepsilon + \|U_{k_j}(y - z)\|_X\{1 + r(\|V_{k_j}(z - x_n)\|_X \cdot \|U_{k_j}(y - z)\|_X^{-1})\} \\
&\quad \geqq -2\varepsilon + (\|y - z\|_X - \varepsilon)\{1 + r((\|z - x_n\|_X - \varepsilon)(\|y - z\|_X + \varepsilon)^{-1})\} .
\end{aligned}$$

By Lemma 2 again

$$\begin{aligned}
&\limsup_n \|y - x_n\|_X \\
&\qquad \geqq (\|y - z\|_X - \varepsilon)\{1 + r((\gamma - \varepsilon)(\|y - z\|_X + \varepsilon)^{-1})\} - 2\varepsilon .
\end{aligned}$$

Since ε is arbitrary and r is continuous, the lemma follows.

3. Main results.

The following lemma is a variant of a result of [5].

LEMMA 4. *Suppose* Y *is a dual Banach space,* $K \subseteqq Y$ *is a convex weak* compact subset,* $T: K \to K$ *is a nonexpansive mapping. Then, for every* $x \in K$ *there is a closed convex subset* $H(x) \subseteqq K$ *which is invariant under* T *and satisfies*

(a) $\operatorname{diam} H(x) \leqq \sup_n \|x - T^n x\|_Y$

(b) $\sup_{y \in H(x)} \|x - y\|_Y \leqq 2 \sup_n \|x - T^n x\|_Y$.

Proof. For $x \in K$, set $d(x) = \sup_n \|x - T^n x\|_Y$ and denote by $O(x)$ the orbit of x (i.e., $O(x) = \{x, Tx, T^2x, \cdots, T^n x, \cdots\}$). Set also

$$A_0 = \mathrm{cl}^* \operatorname{co} O(x) \qquad A_{n+1} = \mathrm{cl}^* \operatorname{co} T(A_n) , \quad n = 0, 1, 2, \cdots$$

where $\mathrm{cl}^* \operatorname{co}$ denotes the weak* closure of the convex hull. Clearly $A_n \subseteqq K$, $O(T^{n+1}x) \subseteqq T(A_n) \subseteqq A_{n+1}$, $\operatorname{diam} A_n \leqq d(x)$. Since K is weak* compact, $B_k = \bigcap_{n \geqq k} A_n$ is nonvoid for every $k = 0, 1, 2, \cdots$. Moreover B_k is closed and convex, $\operatorname{diam} B_k \leqq d(x)$, $B_k \subseteqq B_{k+1}$, $T(B_k) \subseteqq B_{k+1}$.

It follows that $H(x) = \overline{\bigcup_{k=0}^{\infty} B_k}$ satisfies (a). Property (b) follows from the fact that $H(x)$ contains the set $\bigcap_{n=0}^{\infty} \mathrm{cl}^* O(T^n x)$. It is also clear that $H(x)$ is invariant.

The following theorem is our main result announced in § 1.

THEOREM. *Suppose X is a Banach space with a Schauder basis satisfying the assumptions of § 1. Let Y denote an isomorphic Banach space with $t < 1 + r(1)$. Then, every convex weak* compact subset K of Y has the fixed point property with respect to nonexpansive mappings.*

Proof. Suppose $T: K \to K$ is a nonexpansive mapping. There is a sequence $(x_n^0) \subseteq K$ such that $\lim_n \|x_n^0 - Tx_n^0\|_Y = 0$. After passing to a subsequence if necessary, we may assume that x_n^0 is weak* convergent to a point $z^0 \in K$, and that $\alpha_0 = \lim_n \|x_n^0 - z^0\|_Y$ exists. By nonexpansiveness, for every positive integer k we have $\|z^0 - T^k z^0\|_Y \leqq \limsup_n \|x_n^0 - T^k z^0\|_Y \leqq \alpha_0$. Thus $d(z^0) \leqq \alpha_0$. By Lemma 4 there is a closed convex invariant subset $H(z^0) \subseteq K$ such that $\operatorname{diam} H(z^0) \leqq \alpha_0$. Then there exists a sequence (x_n^1) contained in $H(z^0)$ such that $\|x_n^1 - Tx_n^1\|_Y$ tends to 0, x_n^1 converges weak* to $z^1 \in K$, $\alpha_1 = \lim_n \|x_n^1 - z^1\|_Y$ exists and also $\gamma_1 = \lim_n \|x_n^1 - z^1\|_X$ exists. We then have (recall the notation introduced in § 1) for every m

$$\begin{aligned}\alpha_0 \geqq \limsup_n \|x_m^1 - x_n^1\|_Y &\geqq k^{-1} \limsup_n \|x_m^1 - x_n^1\|_X \\ &\geqq k^{-1}\|x_m^1 - z^1\|_X\{1 + r(\gamma_1\|x_m^1 - z^1\|_X^{-1})\} \\ &\geqq k^{-1}h\|x_m^1 - z^1\|_Y\{1 + r(\gamma_1\|x_m^1 - z^1\|_X^{-1})\}\end{aligned}$$

by Lemma 3. Letting m tend to infinity we get

$$\begin{aligned}\alpha_0 &\geqq \limsup_m (\limsup_n \|x_m^1 - x_n^1\|_Y) \\ &\geqq t^{-1}\alpha_1(1 + r(1))\end{aligned}$$

that is,

$$\alpha_1 \leqq t(1 + r(1))^{-1}\alpha_0 .$$

Moreover, since z^1 belongs to the weak* closure $H(z^0)$, Lemma 4, (b) implies $\|z^0 - z^1\|_Y \leqq 2\alpha_0$.

Carrying on this process we produce a sequence of nonnegative numbers α_n such that $\alpha_{n+1} \leqq t(1 + r(1))^{-1}\alpha_n \leqq \{t(1 + r(1))^{-1}\}^{n+1}\alpha_0$, and a sequence of points $z^n \in K$ such that $\|z^{n+1} - z^n\|_Y \leqq 2\alpha_n$, $\|z^n - Tz^n\|_Y \leqq \alpha_n$. Hence z^n is strongly convergent to a fixed point of T.

If $X = l^p$, it is easy to see that $1 + r(1) = 2^{1/p}$. Therefore we have the following remarkable corollary.

COROLLARY. *Suppose Y is isomorphic to l^1 with $t < 2$. Then every weak* compact convex subset of Y has the fixed point property with respect to nonexpansive mappings.*

This corollary generalizes a result of Karlovitz ([3, Corollary]). In [4] an example was given of a space isomorphic to l^1 with $t = 2$, whose unit ball has not the fixed property with respect to nonexpansive mappings. Hence our corollary is the best result possible.

4. Concluding remarks and comparison with previous results. If X is reflexive, then the above theorem can be proved in a much simpler way. This case however is not new, because it is easily seen that, under our assumption on Y, every convex weakly compact subset of Y has normal structure. If X is not reflexive, we were not able to decide whether every weak* compact convex subset of Y has normal structure (of course when $t < 1 + r(1)$). Recall that a weak* closed convex subset $C \subseteqq Y$ has normal structure if every weak* compact convex subset $K \subseteqq C$ (containing more than one point) has a nondiametral point (see ([4])). A sufficient condition for C to admit normal structure was also given in [4]. The condition is as follows.

Suppose there exists a functions $\delta: (R^+)^2 \to R^+$ such that

(i) for each fixed s, $\delta(r, s)$ is continuous and strictly increasing

(ii) $\delta(s, s) > s$ for all s

(iii) if x_n tends to 0 weak* and $\|x_n\|_Y$ tends to s, then, for all $y \in K$, $\|y - x_n\|_Y$ tends to $\delta(\|y\|_Y, s)$.

It is easy to see that this condition is not satisfied in the space Y obtained by renorming l^1 with the norm $\|y\|_Y = \max(\|y\|_{l^\infty}, t^{-1}\|y\|_{l^1})$, where $1 < t < 2$. Indeed, if (b_n) is the natural basis of l^1, take $y = b_1$. Assume that the condition of [4] is satisfied, say, for the unit ball of Y. We have $\|y\|_Y = 1$. Set $x_n = (t-1)b_n$. Then $\|x_n\|_Y = t - 1$, $\|y - x_n\|_Y = 1$, so that, by (iii), $\delta(1, (t-1)) = 1$. On the other hand, if we choose $z = b_1 + (t-1)b_2$, we have $\|z\|_Y = 1$ and $\|z - x_n\|_Y = t^{-1}\|z - x_n\|_{l^1} = t^{-1}(2t-1)$. Hence, by (iii) we should have $\delta(1, t-1) = 2 - t^{-1}$, a contradiction.

Analogous arguments show also that the relation $\perp$ is not approximately uniformly symmetric in Y (in the sense of [3]) and our result cannot be deduced from [3].

For other examples concerning spaces X satisfying our assmuptions, we refer to [2] and [6].

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Received August 17, 1980.

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