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A HOMOGENEOUS EBERLEIN COMPACT SPACE WHICH IS NOT METRIZABLE

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We give an example of a first countable, hereditarily normal, homogeneous Eberlein compact space which is not metrizable. This answers a question of A. V. Arhangel'skiĭ.

1. Introduction. A compact Hausdorff space is called *Eberlein compact*, if it is homeomorphic to a weakly compact subset of a Banach space. For information concerning Eberlein compact spaces, see [1], [3], [5] and [7].

If X is Eberlein compact, then X is metrizable if X satisfies the countable chain condition, [2], [5], or if X is linearly orderable, [4]. In view of these facts, the following question due to Arhangel'skiĭ [3, p. 91 problem 5], is quite natural: *is there a nonmetrizable homogeneous*[1] *Eberlein compact space*? The aim of this paper is to construct such an example which in addition is zero-dimensional, first countable and hereditarily normal. The symbol "$X \approx Y$" means that X and Y are homeomorphic spaces. I am indebted to Mary Ellen Rudin for spotting some inaccuracies in an earlier version of this paper.

2. Preliminaries. A family $\mathscr{F}$ of subsets of a topological space X is called *separating* provided that for any distinct $x, y \in X$ there is an $F \in \mathscr{F}$ such that *either* $x \in F$ and $y \notin F$ *or* $y \in F$ and $x \notin F$. The family $\mathscr{F}$ is called *point-finite* if each $x \in X$ belongs to at most finitely many elements of $\mathscr{F}$. It is called *σ-point-finite* if $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$, where each $\mathscr{F}_n$ is point-finite.

The following purely topological characterization of Eberlein compacta, due to Rosenthal [8], is convenient for topologists.

THEOREM 2.1. *A compact Hausdorff space is Eberlein compact iff it has a σ-point-finite separating family of open F_σ-subsets.*

Let C denote the usual Cantor set in $[0, 1]$ (notice that $0 \in C$) and let X be any space. Topologize $X \times C$ in the following way:

(a) *a basic neighborhood of a point $\langle x, 0\rangle$ has the form*

$$(U \times C) - (\{x\} \times D),$$

where $U \subset X$ is open, contains x and $D \subset C - \{0\}$ is compact;

[1] A space X is called *homogeneous* provided that for any two points $x, y \in X$ there is an autohomeomorphism $h: X \to X$ with $h(x) = y$.

(b) *a basic neighborhood of a point* $\langle x, c\rangle$ *where* $c > 0$ *has the form*

$$\{x\} \times U ,$$

where $U \subset C - \{0\}$ *is an open neighborhood of* c.

The topological space we obtain in this way will be denoted by $X(C)$. Observe that the projection $\pi: X(C) \to X$ onto the first coordinate is continuous. In addition, the function $f: X \to X(C)$ defined by $f(x) = \langle x, 0\rangle$ is an embedding.

LEMMA 2.2. (1) *$X(C)$ is compact Hausdorff iff X is compact Hausdorff,*

(2) *$X(C)$ is first countable iff X is first countable,*

(3) *$X(C)$ is Eberlein compact iff X is Eberlein compact.*

Proof. (1) We only need to show that $X(C)$ is compact if X is. Let $\mathscr{U}$ be an open cover of $X(C)$ by basis elements. Finitely many elements of $\mathscr{U}$ cover $X \times \{0\}$ and the remaining part of $X(C)$ consists of finitely many compact sets. We conclude that $\mathscr{U}$ has a finite subcover.

Observe that (2) is trivial and that for (3) we only need to show that $X(C)$ is Eberlein compact if X is Eberlein compact (closed subsets of Eberlein compacta are Eberlein compact). To this end, let $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$ be a separating family of open $\mathscr{F}_\sigma$-subsets of X such that for all n the family $\mathscr{F}_n$ is point-finite. In addition, let $\{C_m: m = 1, 2, \cdots\}$ be a countable basis for $C - \{0\}$ consisting of compact open sets. For all $n, m \in N$ define

$$\widetilde{\mathscr{F}}_n = \{F \times C: F \in \mathscr{F}_n\} ,$$

and

$$\mathscr{C}_m = \{\{x\} \times C_m: x \in X\}$$

respectively. Observe that both $\widetilde{\mathscr{F}}_n$ and $\mathscr{C}_m$ are point-finite, that $\widetilde{F}_n$ consists of open F_σ-subsets of $X(C)$ and that $\mathscr{C}_m$ consists of clopen (= closed and open) subsets of $X(C)$. Since trivially,

$$\mathscr{G} = \bigcup_{n=1}^{\infty} \widetilde{\mathscr{F}}_n \cup \bigcup_{m=1}^{\infty} \mathscr{C}_m$$

is separating, Theorem 2.1 implies that $X(C)$ is Eberlein compact. □

3. The example. Let X be any space. Define $X_1 = X$ and $X_{n+1} = X_n(C)$. The projection from X_{n+1} onto its first coordinate is denoted by $f_{n,X}$. Put

$$\tilde{X} = \varprojlim (X_n, f_{n,X})$$

(i.e., $\tilde{X} = \{x \in \prod_{n=1}^{\infty} X_n : f_{n,X}(x_{n+1}) = x_n \text{ for all } n \in N\}$). Let $\pi_{n,X}: \tilde{X} \to X_n$ be the projection. Observe that

$$\pi_{n,X} = f_{n,X} \circ \pi_{n+1,X} .$$

LEMMA 3.1. (1) *If $A \subset X$, then $\tilde{A} \approx \pi_{1,X}^{-1}(A) \subset \tilde{X}$.*
(2) *$\tilde{X}_n \approx \tilde{X}$ for all $n \in N$.*

Proof. Obvious. □

We claim that $\tilde{C}$ is homogeneous, nonmetrizable and Eberlein compact. By a repeated application of Lemma 2.2(3) it follows that each C_n is Eberlein compact. Consequently, by Theorem 2.1,

$$\prod_{n=1}^{\infty} C_n$$

is Eberlein compact which implies that $\tilde{C}$ is Eberlein compact, being a closed subspace of $\prod_{n=1}^{\infty} C_n$. Similarly, each C_n is first countable and consequently, $\tilde{C}$ is first countable. It is clear that $\tilde{C}$ is not metrizable, since it maps onto the nonmetrizable space C_2 (C_2 is not metrizable since it contains an uncountable family of pairwise disjoint nonempty open subsets). Obviously, $\tilde{C}$ is zerodimensional.

THEOREM 3.2. *$\tilde{C}$ has the property that all of its nonempty clopen subspaces are homeomorphic (hence $\tilde{C}$ is strongly homogeneous in the sense of* [8]).

Proof. By induction on n we will show that $\pi_{n,C}^{-1}(U) \approx \tilde{C}$ for all nonempty clopen $U \subset C_n$. This is clearly true for $n = 1$ since all nonempty clopen subsets of C are homeomorphic to C which implies that

$$\pi_{1,C}^{-1}(U) \approx \tilde{U} \approx \tilde{C}$$

for all clopen $U \subset C_1$ (Lemma 3.1(1)). Now suppose the statement to be true for n and take a nonempty clopen $U \subset C_{n+1}$ arbitrarily. If $U \cap (C_n \times \{0\}) = \varnothing$ then U is homeomorphic to C by definition of the topology of C_{n+1}. Consequently, by Lemma 3.1(1), (2) it then follows that

$$\pi_{n+1,C}^{-1}(U) \approx \tilde{U} \approx \tilde{C} .$$

Therefore assume that $U \cap (C_n \times \{0\}) \neq \varnothing$. By definition of the topology of C_{n+1} there is a finite $F \subset C_n$ and for each $x \in F$ a clopen $S_x \subset C$ not containing 0 such that $F \times \{0\} \subset V = U \cap (C_n \times \{0\})$ while moreover

$$E = (V \times C) - (\bigcup_{x \in F} \{x\} \times S_x) \subset U .$$

For each $x \in F$ let $h_x: C - S_x \to C$ be a homeomorphism such that in a fixed neighborhood of 0 each h_x is the identity. Define $h: E \to V \times C$ by

$$\begin{cases} h(\langle a, b\rangle) = \langle a, b\rangle & \text{if} \quad a \notin F , \\ h(\langle a, b\rangle) = \langle a, h_a(b)\rangle & \text{if} \quad a \in F . \end{cases}$$

Clearly, h is a homeomorphism. Therefore

$$\pi^{-1}_{n+1,C}(E) \approx \pi^{-1}_n(V) \approx \tilde{C} ,$$

by induction hypothesis. Put $G = U - E$. Then G is a clopen subset of C_{n+1} which misses $C_n \times \{0\}$. If $G = \varnothing$ then we are done, and if $G \neq \varnothing$ then observe that

$$\pi^{-1}_{n+1,C}(G) \approx \tilde{C}$$

since $G \approx C$ (cf. the above remarks). Consequently, $\pi^{-1}_{n+1,C}(U)$ is the disjoint union of two clopen copies of $\tilde{C}$, hence is itself homeomorphic to $\tilde{C}$ since C is the disjoint union of two clopen copies of itself. This completes the induction.

Now let $A \subset \tilde{C}$ be clopen and nonempty. There is clearly an index $n \in N$ and a nonempty clopen $B \subset C_n$ such that

$$\pi^{-1}_{n,C}(B) = A .$$

Therefore $A = \pi^{-1}_{n,C}(B) \approx \tilde{C}$. □

The above theorem shows that $\tilde{C}$ is homogeneous, for any zero-dimensional strongly homogeneous first countable space X is homogeneous. This is well-known and for completeness sake we will include the trivial proof. Take $x, y \in X$. Since X is first countable, there is a clopen neighborhood basis $\{V_n: n \in N\}$ for x and a clopen neighborhood basis $\{W_n: n \in N\}$ for y such that

(1) $V_1 = W_1 = X$,

(2) V_{n+1} is properly contained in V_n, and

(3) W_{n+1} is properly contained in W_n.

For each $n \in N$ let $h_n: V_n - V_{n+1} \to W_n - W_{n+1}$ be any homeomorphism. The function $h: X \to X$ defined by

$$\begin{cases} h(x) = y , \\ h(a) = h_n(a) \quad \text{if} \quad a \in V_n - V_{n+1} \end{cases}$$

is clearly a homeomorphism mapping x onto y.

REMARK 3.3. It is not by accident that our example is first countable. By [5, 4.3] every Eberlein compact space is first countable at a dense set of points, consequently, a homogeneous Eberlein compact space must be first countable. Notice however that we used the first countability of $\widetilde{C}$ to show it is homogeneous.

4. $\widetilde{C}$ is hereditarily normal. In this section we will show that $\widetilde{C}$ is a continuous image of a compact linearly orderable topological space. This implies that $\widetilde{C}$ is hereditarily normal (even monotonically normal).

Let $L_1 = C$ and let $L_2 = C \times C$ with topology generated by the lexicographical ordering. Let $g_1: L_2 \to L_1$ be the projection onto the first coordinate. Observe that g_1 is order preserving. Let $\psi_1: L_1 \to C_1$ be the identity and let $h: C \to C$ be an arbitrary onto map such that

$$h(0) = 0 \quad \text{and} \quad h(1) = 0 .$$

Define $\psi_2: L_2 \to C_2$ by

$$\psi_2(\langle a, b\rangle) = \langle a, h(b)\rangle .$$

Because $h(0) = 0 = h(1)$, ψ_2 is continuous.

It is easily seen that the diagram

$$\begin{array}{ccc} L_1 & \xleftarrow{g_1} & L_2 \\ {\scriptstyle\psi_1}\downarrow & & \downarrow{\scriptstyle\psi_2} \\ C_1 & \xleftarrow[f_{1,C}]{} & C_2 \end{array}$$

commutes. Suppose that we have defined L_n and ψ_n. Let $L_{n+1} = L_n \times C$ with topology generated by the lexicographical ordering and let $g_n: L_{n+1} \to L_n$ be the projection. Define $\psi_{n+1}: L_{n+1} \to C_{n+1}$ by $\psi_{n+1}(\langle a, b\rangle) = \langle a, h(b)\rangle$, where h is defined as above. Observe that g_n is order preserving and that

$$(1) \qquad f_{n,C} \circ \psi_{n+1} = \psi_n \circ g_n .$$

Put $L = \varprojlim (L_n, g_n)$. Since the maps g_n are all order preserving, L can be ordered in a natural way (It is easy to describe the ordering of L. Alternatively, the orderability theorems given in [6] or [9] are also easily applied.). By (1), the space L maps onto $\widetilde{C}$ so that $\widetilde{C}$ is hereditarily normal.

As was pointed out to me by Dave Lutzer, it is also easily seen that $\tilde{C}$ is hereditarily paracompact, since L is a first countable compact LOTS and L maps onto $\tilde{C}$.

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