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TOTALLY BOUNDED GROUP TOPOLOGIES AND CLOSED SUBGROUPS

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Let (G, J) be an infinite compact totally disconnected abelian group. Finer totally bounded group topologies J' such that every J'-closed subgroup is J-closed are studied. Necessary and sufficient conditions for the existence of such a $J' \neq J$ are given.

Introduction. Throughout this paper all topologies are Hausdorff topological group topologies and all the groups are written in the additive notation.

A topological group G is called totally bounded if for every identity neighbhurhood V there is a finite subset F of G with $G = F + V$. This is tantamount to saying that G is embedded algebraically and topologically into its Bohr compactification under the natural map $G \to \alpha G$. We recall that for abelian G we have $\alpha G = ((G^\wedge)_d)^\wedge$ and that $(G^\wedge)_d = (\alpha G)^\wedge$.

Now let G be a compact abelian group with topology J and let G' be the same underlying group with a possibly finer totally bounded topology J'. Then $G^\wedge \subseteqq (G')^\wedge = (G'^\wedge)_d \subseteqq (G_d)^\wedge$; and conversely, any group H of (not necessarily continuous) characters of G with $G^\wedge \subseteqq H \subseteqq (G_d)^\wedge$ induces on G a coarsest topology J' making all characters of H continuous, and then the group G' with the topology J' is totally bounded such that $(\alpha G')^\wedge = G_d'^\wedge = H$. Thus there is a lattice isomorphism between the lattice of totally bounded topologies J' on G refining J and the lattice of subgroups of $(G_d)^\wedge$ containing $G^\wedge$. (These nice results are proved by W. W. Comfort and K. A. Ross in [1].) Furthermore, the diagram

$$\begin{array}{ccc} G' & \longrightarrow & \alpha G' \\ e\Big\downarrow & & \Big\downarrow \alpha e \\ G & \xrightarrow[1]{} & \alpha G = G \end{array}$$

shows that $\alpha G'$ is algebraically the direct sum of the image of G' in $\alpha G'$ and of $\ker \alpha e$.

The problem we are interested in studying is the following:

(P) Determine all those totally bounded topologies J' containing J such that every J'-closed subgroup of G' is a J-closed subgroup of G.

In view of the isomorphism of lattices mentioned before, this is tantamount to the following problem:

(P′) Determine all those intermediate groups H with $G^\frown \subsetneqq H \subsetneqq (G_d)^\frown$ for which the associated topology $J' = J_H$ has the same closed subgroups as $J = J_{\hat{G}}$.

In a totally bounded group the smallest closed subgroup containing a subset S is its bipolar $S^{\perp\perp}$; hence a subgroup S is closed if and only if it agrees with its bipolar if and only if it is the intersection of a collection of kernels of continuous characters. As a consequence, the J_H-closed subgroups are precisely the intersections of families of groups $\ker f$ with $f \in H$. Consequently problem (P′) is equivalent to (P″): Determine all those groups H with $G^\frown \subsetneqq H \subsetneqq (G_d)^\frown$ such that $\ker f$ is $J = J_{\hat{G}}$ — closed for all $f \in H$.

In this paper we consider only the case (G, J) is totally disconnected *ie* $G^\frown$ is a torsion group [5, p. 385]. We show that if H is a subgroup "admissible" in the sense of problem (P″) then $G^\frown$ is the torsion subgroup of H [Lemma 1.3]. In particular $\ker \alpha e$ is always connected in this case. Next for any $f \in (G_d)^\frown$ whose $\ker f$ is J-closed, $G^\frown + \langle f \rangle$ is admissible [Lemma 2.3]. We then prove that (G, J) has an admissible $H \supsetneqq G^\frown$ if and only if G has a direct factor which is p-adic integer group Δ_p or an infinite product of cyclic groups of prime power order for infinitely many different primes. [Theorem 2.5].

It is also shown that if there are admissible groups properly containing $G^\frown$ then there is no largest admissible H [Theorem 2.10].

That one can never expect pseudocompact $J' \neq J$ (whether or not J is totally disconnected) and existence of maximal admissible subgroups H is dealt with in a paper by W. W. Comfort and the second author [3].

The authors conclude the paper with a few remarks on the nonabelian case and a remark on Galois theory (in § 3).

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1. Preliminaries. Throughout this paper (G, J) denotes an infinite compact totally disconnected abelian group, $G^\frown$ is the group of all continuous characters of (G, J), G_d is the group G endowed with the discrete topology and $(G_d)^\frown$ is the group of all characters on G.

DEFINITION 1.1. A subgroup H of $(G_d)^\frown$ is said to be admissible if H contains $G^\wedge$ and ker f is a J-closed subgroup of G for all $f \in H$.

PROPOSITION 1.2. *If G is of finite exponent then the only admissible subgroup is $G^\wedge$.*

Proof. Let $f \in (G_d)^\wedge$ with ker f a J-closed subgroup of G. Since G is of finite exponent say m, $mx = 0$ for all $x \in G$ yields $f(G)$ is of finite exponent in $T = \boldsymbol{R}/\boldsymbol{Z}$. Hence $f(G)$ is a finite subgroup of T. Thus ker f is of finite index in G. Already it is J-closed. Hence ker f is J-open and so we get f is continuous and hence $f \in G^\wedge$. The proposition now easily follows.

LEMMA 1.3. *Let H be an admissible subgroup. Then $G^\wedge$ is the torsion subgroup of H.*

Proof. It is enough to show that if $f \in H$ is of finite order then $f \in G^\wedge$. Let $mf = 0$. Then $mf(G) = 0$. Hence $f(G)$ is of finite exponent in T and so $f(G)$ is a finite group. Hence ker f is of finite index in G. It is J-closed implies now that it is J-open. Hence we get f is continuous and $f \in G^\wedge$.

2. In this section we prove the main theorem.

LEMMA 2.1. *Let (A, τ) be an abelian totally bounded topological group and B a closed subgroup of (A, τ). If $B^\perp$ is the set of all continuous homomorphisms of (A, τ) into T which map B to 0, then $B = \bigcap_{f \in B^\perp} \ker f$.*

Proof. Let αA be the compact topological group in which (A, τ) is densely embedded. Then αA is also abelian. Let $\bar{B}$ be the closure of B in αA. We have $\bar{B} \cap A = B$. By Pontrjagin-van Kampen duality theory we have $\bar{B} = \bigcap_{f \in \bar{B}^\perp} \ker f$. By taking the restrictions of the $f \in \bar{B}^\perp$ to A, the lemma follows.

LEMMA 2.2. *Let A be an abelian group and f, g be two homomorphisms of A into T such that g is of finite order. Let n be any integer. Then* ker $(g + nf)$ *contains* (ker $f \cap$ ker g) *as a subgroup of finite index.*

Proof. Let g be of order m. Then for each x in A, $g(mx) = mg(x) = (mg)(x) = 0$. Hence $g(A)$ is of finite exponent and so is a finite subgroup of T. Consequently ker $f \cap$ ker g is of finite index in ker f. Easily ker $f \cap$ ker g is a subgroup of ker $(g + nf)$. Let $S = \ker(g + nf)$. Then for every $x \in S$ we have $0 = (g + nf)(mx) =$

$g(mx) + nf(mx) = mg(x) + nmf(x) = mnf(x)$. Now let B be the finite subgroup of order mn in T. Then clearly $S \subset f^{-1}(B)$. Also $\ker f$ is of finite index in $f^{-1}(B)$. Already $\ker f \cap \ker g$ is of finite index in $\ker f$. Hence $\ker f \cap \ker g$ is of finite index in $f^{-1}(B)$. Since $(\ker f \cap \ker g) \subset S \subset f^{-1}(B)$, the lemma follows.

LEMMA 2.3. *For any* $f \in (G_d)^{\wedge} \backslash G^{\wedge}$ *such that* $\ker f$ *is a* J*-closed subgroup of* G, $G^{\wedge} + \langle f \rangle$ *is an admissible subgroup.*

Proof. It is enough to show that $\ker h$ is a J-closed subgroup for all $h \in G^{\wedge} + \langle f \rangle$. Now let $h \in G^{\wedge} + \langle f \rangle$. Then $h = g + nf$ with $g \in G^{\wedge}$ and n an integer. As $G^{\wedge}$ is a torsion abelian group g is of finite order. Hence by Lemma 2.2, $\ker g \cap \ker f$ is of finite index in $\ker h$. Now $\ker g$ is J-closed since $g \in G^{\wedge}$ and $\ker f$ is J-closed by hypothesis. Hence $(\ker f \cap \ker g)$ is a J-closed subgroup and $\ker h$ is a finite union of cosets of $(\ker f \cap \ker g)$. Hence $\ker h$ is J-closed.

PROPOSITION 2.4. *Let* (G, J) *be one of the following two groups;*

(1) Δ_p, *the topological group of all the* p*-adic integers with the usual topology,* p *a prime.*

(2) $\prod_{p_i \in I} Z(p_i^{n_i})$, *the product of cyclic groups of prime power order* $p_i^{n_i}$, *with the product topology, where* I *is an infinite set of primes. (We shall denote this compact group by* $C\ (p_i, n_i)$.)

Then there is an admissible subgroup $H \neq G^{\wedge}$.

Proof. (1) Algebraically, Δ_p is a torsion free abelian group of cardinality c. Now $T = \sum Z(p^\infty) \oplus R$ algebraically where the sum is extended over all primes [4, p. 105]. R being a torsion free divisible group of cardinality c, we can find an algebraic monomorphism $f: \Delta_p \to T$. Clearly $\ker f = 0$ is a J-closed subgroup. Also $f \notin \Delta_p^{\wedge}$ since $mf = 0$ will imply $f(mx) = 0$ for all $x \in \Delta_p$ contradicting that $\ker f = 0$. Now Lemma 2.3 completes the proof.

(2) For this case we use a product decomposition. Algebraically $T = \prod Z(p^\infty)$ (see [4, p. 105]) the product extending over all primes. Again we have an algebraic monomorphism $f: G \to T$; with $\ker f = 0$, a J-closed subgroup. Since I is infinite G has elements of infinite order. Hence $mf = 0$ will yield $f(mx) = 0$ and contradict $\ker f = 0$. Hence $f \notin G^{\wedge}$. Now Lemma 2.3 completes the proof.

THEOREM 2.5. *Let* (G, J) *be an infinite compact totally disconnected abelian topological group. Then the following statements are equivalent.*

(1) *There exists a totally bounded group topology* J' *containing*

J properly such that every J'-closed subgroup is J-closed.

(2) *G has an infinite monothetic factor group.*

(3) *G has a direct factor M which is either a p-adic group Δ_p or a group $C(p_i, n_i)$.*

(4) *G has an infinite procyclic direct factor.*

Proof. (1) $\Rightarrow$ (2). Suppose there exists a totally bounded group topology J' on G containing J properly such that every J'-closed subgroup is J-closed. Let $H = \{f: f$ is a continuous homomorphism of (G, J') into $T\}$. Hence there is an $f \in H \backslash G^\wedge$. Clearly then $f \in (G_d)^\wedge \backslash G^\wedge$. Now $f \in H$ implies that $\ker f$ is J'-closed and so is J-closed by hypothesis. Thus f is a discontinuous character for (G, J) with $\ker f$ being J-closed. Let $\bar{G} = G/\ker f$ and $\bar{J}$ be the quotient topology on $\bar{G}$ obtained from J. We now have a monomorphism $\bar{f}: \bar{G} \to T$ i.e., $(\bar{G})_d$ can be injected into T. Hence by [5, p. 407] the torsion free rank of $(\bar{G})_d$ is at most c and the p-rank of the torsion subgroup of $(\bar{G})_d$ is at most 1 for all p. Also $(\bar{G}, \bar{J})$ is an infinite compact totally disconnected abelian group (since $f \notin G^\wedge$, $\ker f$ is not J-open and hence $\bar{G}$ cannot be finite). Since $\bar{G}^\wedge$ is a torsion abelian group let $\bar{G}^\wedge = \sum \bar{G}_p^\wedge$, $\bar{G}_p^\wedge$ being the p-primary part of $\bar{G}^\wedge$. If for some p, $\bar{G}_p^\wedge$ contains $Z(p^\infty)$ then we get $(\bar{G}, \bar{J})$ has a factor Δ_p and hence (G, J) has a factor group Δ_p which is an infinite compact monothetic group and we are done. Otherwise $\bar{G}_p^\wedge$ is a reduced group for each p. We claim now $\bar{G}_p^\wedge$ is cyclic of prime power order. Otherwise by [4, p. 117] we can have $\bar{G}_p^\wedge = Z(p^r) \oplus Z(p^s) \oplus B$ and consequently by duality $(\bar{G}, \bar{J})$ will have a direct factor $Z(p^r) \oplus Z(p^s)$ contradicting that p-rank of $(\bar{G})_d$ is at most 1. Thus each $\bar{G}_p^\wedge$ is cyclic of prime power order and so $\bar{G}^\wedge$ is isomorphic to a subgroup of T. (See [5, p. 407].) Hence $(\bar{G}, \bar{J})$ is monothetic and (2) follows.

(2) $\Rightarrow$ (3). Let $(\bar{G}, \bar{J})$ be an infinite monothetic factor of (G, J). Then $\bar{G}^\wedge$ is a torsion subgroup of T (see [5, p. 385]).

We consider now $G^\wedge$. If $G^\wedge$ contains a $Z(p^\infty)$ for some p then (G, J) will have a direct factor Δ_p and we are done. Otherwise $G^\wedge$ is a reduced group. Now $\bar{G}^\wedge$ is a subgroup of $G^\wedge$. Since $\bar{G}^\wedge$ is infinite and a subgroup of T and $\bar{G}^\wedge$ also has to be reduced we get $\bar{G}_p^\wedge \neq 0$ for infinitely many p. Hence $G_p^\wedge \neq 0$ for infinitely many p. Now applying [4, p. 117] to each of these $G_p^\wedge$ we easily get $\sum Z(p_i^{n_i})$ is a direct summand of $G^\wedge$. Hence (G, J) has a direct factor $C(p_i, n_i)$. Hence (3) follows.

(3) $\Rightarrow$ (1). Case (i): Let $(G, J) = N \oplus M$, N a J-closed subgroup, M is topologically isomorphic to Δ_p and the sum direct. Then by the proof of (1) in 2.4 we get easily a homomorphism $f: G \to T$ such

that $\ker f = N$ and f is injective on M. Surely order of f is infinite and hence $f \notin G^\wedge$. Also $\ker f$ is J-closed. Hence Lemma 2.3 shows that $G^\wedge + \langle f \rangle$ is admissible. Thus a J' exists by the equivalence in the introduction.

Case (ii): Let $(G, J) = N \oplus M$, N a J-closed subgroup, M is isomorphic to $C(p_i, n_i)$ and the sum direct. Then by the proof (2) in 2.4, we get easily a homomorphism $f\colon G \to T$ such that $\ker f = N$ and f is injective on M. Since $C(p_i, n_i)$ has torsion free elements and f is injective on M, order of f has to be infinite and so $f \notin G^\wedge$. Also $\ker f = N$ is J-closed. Thus Lemma 2.3 yields $G^\wedge + \langle f \rangle$ is admissible. Hence J'-exists by the equivalence in the introduction. Thus (1) follows.

(3) $\Rightarrow$ (4). Easy.

(4) $\Rightarrow$ (2). Let P be an infinite procyclic direct factor of (G, J). Then by duality $P^\wedge$ is a torsion group which is a direct limit of finite cyclic groups. By [4, p. 58] $\hat{p}$ is locally cyclic. Hence for each prime p the p-rank of $P^\wedge$ is atmost one. So each $P_p^\wedge$ is isomorphic to a $\boldsymbol{Z}(p^s)$, $s = 0, 1, \cdots, \infty$. This yields that $P^\wedge$ is isomorphic to a subgroup of T and hence P is monothetic. Thus (2) holds.

Now Theorem 2.5 follows.

We now proceed to discuss the existence of a largest admissible subgroup.

LEMMA 2.6. *There exists a largest admissible subgroup L if and only if the set of all $f \in (G_d)^\wedge$ such that* $\ker f$ *is J-closed form a group. In this case L consists precisely of these.*

Proof. Let L be a largest admissible subgroup. Let $f, g \in (G_d)^\wedge$ such that $\ker f$, $\ker g$ are J-closed. Then by Lemma 2.3 $G^\wedge + \langle f \rangle$ and $G^\wedge + \langle g \rangle$ are admissible subgroups; they will both be subgroups of L and hence, $f, g, f - g \in L$. Clearly then all such f's will form a group.

Conversely let $L = \{f \in (G_d)^\wedge \mid \ker f \text{ is } J\text{-closed}\}$ form a group. Then clearly L is admissible, and by definition any other admissible group should be a subgroup of L. Hence the lemma is proved.

PROPOSITION 2.7. *In $((\Delta_p)_d)^\wedge$ there is no largest admissible subgroup.*

Proof. Since Δ_p is a torsion free abelian group of cardinal c it has a maximal independent set B of cardinal c. Hence $\Delta_p/\langle B \rangle$ is a torsion abelian group.

Now $T = \Sigma Z(p^\infty) \oplus R$ (see [4, p. 105]). We can write $R = \Sigma Q$, c copies and then write $R = B_1 + B_2$ such that $B_1 \cap B_2 = Q$, B_1, B_2 each isomorphic to ΣQ, c copies. Now easily we can get embeddings h_1, h_2 of Δ_p into R, such that $h_1(\Delta_p) \subset B_1$, $h_2(\Delta_p) \subset B_2$ and $h_1(1) = h_2(1) = 1 \in Q = B_1 \cap B_2$, h_1, h_2 being obtained by mapping B to the corresponding independent sets. It is easy to see that $\ker(h_1 - h_2)$ is a countable subgroup, of $\Delta_p (= \{n/m; (p, m) = 1\})$. Clearly $\ker(h_1 - h_2)$ is not J-closed. Now Lemma 2.6 completes the proof.

PROPOSITION 2.8. *There is no largest admissible subgroup in* $(C(p_i, n_i)_d)\hat{}$.

Proof. We note $C(p_i, n_i) = \prod Z(p_{p_i}^{n_i})$ algebraically and also that $T = \prod Z(p^\infty)$, p varies over all primes [4, p. 105]. Hence there is an embedding $i: C(p_i, n_i) \to T$, with $\ker(i) = 0$; which is J-closed. Since $T \cong \sum Z(p^\infty) \oplus R$, there is an automorphism $g: T \to T$ such that $g(x) = x$ for elements of finite order and $g(x) = \sqrt{2}x$ for x in R. Then $g \circ i$ gives another embedding of $C(p_i, n_i)$. Now $\ker(i - g \circ i)$ is a countable subgroup namely $\sum Z(p_i^{n_i})$. Thus we get two embeddings $f_1, g_1: C(p_i, n_i) \to T$ such that $\mathrm{Ker}(f_1 - g_1)$ is countable and hence not J-closed. So Lemma 2.6 completes the proof.

DEFINITION 2.9. We say a topology J' is admissible if it satisfies the condition of (P).

THEOREM 2.10. *The following are equivalent:*
(1) *G has a largest admissible topology J_L,*
(2) *G has no admissible topology $J' \neq J$,*
(3) *J is the largest admissible topology.*

Proof. (1) $\Rightarrow$ (2). Suppose G has an admissible topology $J' \neq J$. Then G has a topological decomposition $G = A \oplus B$, A a closed subgroup and B is isomorphic Δ_p or $C(p_i, n_i)$. Then by Propositions 2.7 and 2.8, we have two embeddings of $f, g: B \to T$ such that $\ker(f - g)$ is countable and not J-closed. Hence we easily get two homomorphisms $F_1, G_1: G \to T$ such that $\ker F_1 = \ker G_1 = A$ is J-closed but $\ker(F_1 - G_1)$ is not J-closed. This contradicts Lemma 2.6. Hence (2) follows. (2) $\Rightarrow$ (3) is easy as also (3) $\Rightarrow$ (1).

PROPOSITION 2.11. *On Δ_p, there is an admissible topology J' having* $|(\Delta_p, J')\hat{}| = c$.

Proof. We note $T \cong \sum Z(p^\infty) \oplus R$ and $R = \sum Q$, c copies. Now we can write $R = \sum B_\alpha$, $\alpha \in I$; $|I| = c$ and each B_α is a torsion free divisible abelian group of cardinality c. This is possible as $c.c = c$. For each $\alpha \in I$, we can have an embedding $h_\alpha = \Delta_p \to B_\alpha$. Correspondingly we get embeddings $g_\alpha: \Delta_p \to T$ such that for each $x \neq 0$, the $g_\alpha(x)$ are independent. Let now H be the subgroup of $((\Delta_p)_d)^\wedge$ generated by $\Delta_p^\wedge$ and all these g_α. Surely $|H| = c$. Let J' be the totally bounded group topology determined by H. J' is finer than J, the usual topology. We claim J' is admissible. We have only to show that $\ker(h)$ is J-closed for each $h \in H$, since $H = (\Delta_p, J')^\wedge$, (see [5]). Now $h = f + \sum_1^k n_i g_{\alpha_i}$, $f \in \Delta_p^\wedge$, n_i are integers k finite. If all the n_i are 0, then there is nothing to prove. Let some $n_i \neq 0$. Since f is of finite order by Lemma 2.2, we have only to prove $\ker f \cap \ker(\sum_1^k n_i g_{\alpha_i})$ is J-closed. We claim $\ker(\sum_1^k n_i g_{\alpha_i}) = 0$. Let if possible $x \neq 0$ be in the kernel. $\sum n_i g_{\alpha_i}(x) = 0$ implies $\sum g_{\alpha_i}(n_i x) = 0$ and by independence $g_{\alpha_i}(n_i x) = 0$ for each i and each g_{α_i} being an embedding we get $n_i x = 0$ for each i, so $x = 0$. Thus $\ker f \cap \ker(\sum n_i g_{\alpha_i}) = 0$, a J-closed subgroup. Hence the results follows.

3. We now assume (G, τ) is a noncommutative compact totally disconnected group and make a few remarks on totally bounded group topologies τ' containing τ and such that each τ' closed subgroup is τ-closed. We shall again call such a τ' an admissible topology.

REMARK 3.1. If G is of finite exponent then $\tau' = \tau$.

Proof. Let $\alpha G'$ be a compact topological group in which (G, τ') is embedded as a dense subgroup. From hypothesis it now follows easily that for each $x \in \alpha G'$, $mx = 0$ (since x is limit of a net, from (G, τ')). Now $\bigcap_{n=1}^\infty n(\alpha G') = 0$, since $m\alpha G' = 0$. Hence by a theorem of Mycielski [8], $\alpha G'$ is totally disconnected and hence by [7, p. 56] has a basis of open subgroups of finite index at 0.

Hence (G, τ') has a basis $\{G'_\alpha\}$ of open subgroups of finite index at 0. Each of these G'_α is now τ-closed and hence τ-open. Hence we get τ is finer than τ'. Since τ is compact and τ' is Hausdorff we get $\tau = \tau'$.

REMARK 3.2. Let (K, τ) be a compact group of finite exponent. Then $K \times \Delta_p$ has an admissible topology different from the product topology.

Proof. Let $mx = 0$ for each $x \in K$. Let J_1 be an admissible topology on Δ_p; $J_1 \neq$ the usual topology J of Δ_p. Let J' be the product of τ and J_1 on $K \times \Delta_p$. Since $(K \times \Delta_p, J') \subset (K, \tau) \times (\alpha\Delta_p, \alpha J_1)$ where $\alpha\Delta_p$ is the compact group in which Δ_p is densely embedded, we get J' is totally bounded. Also J' is finer than the product topology $\tau \times J$. We have only to show that any J'-closed subgroup S is $\tau \times J$ closed. If $S \subset (K \times O)$ then we easily get the result. Suppose $S \not\subset K \times 0$. If $(x, y) \in S \backslash (K \times 0)$ then $m(x, y) = (mx, my) = (0, my) \in 0 \times \Delta_p$. Let $S \cap (0 \times \Delta_p) = M \neq (0, 0)$ and $S \cap (K \times 0) = M_1$. M is a J'-closed subgroup of Δ_p and hence J-closed. So $M = 0 \times p^n \Delta_p$ for some n. M_1 is J'-closed and hence J-closed since $K \times 0$ is J'-closed and $J' = \tau \times J$ on $K \times 0$. Now $M_1 \times M$ is J-closed and $\subset S$. We claim $M_1 \times M$ is of finite index in S. Let $p_1: S \rightarrow \Delta_p$ be the projection. Then $p_1(S) \supset M$. Hence M is of finite index in $p_1(S)$ (since M is of finite index p^n in Δ_p itself). Let $p_1(S) = M \cup (a_2 + M) \cup \cdots \cup (a_k + M)$ where $(y_i, a_i) \in S$, $i = 1, 2, \cdots, k$. We claim now $S = U_1^k((y_i, a_i) + (M_1 \times M))$. Let $(x, y) \in S$. Then $p_1(x, y) = y = a_i + t$ for some i and $t \in M$. Also $(0, t) \in M_1 \times M \subset S$. Hence we can assume $p_1(x, y) = a_i$. Also $p_1(y_i, a_i) = a_i$. Hence $(-(y_i, a_i) + (x, y)) = (-y_i + x, 0) \in M_1 \subset M_1 \times M$. Hence $(x, y) \in (y_i, a_i) + (M_1 \times M)$. Hence S is a finite union of cosets of $M_1 \times M$ and so we get S is $\tau \times J$-closed. That $\tau \times J_1$ is an admissible topology follows now easily.

REMARK 3.3. If E is an infinite algebraic separable normal extension of a field F and G is the Galois group of E over F then W. Krull [6] has shown that one can introduce a topology τ on G (the Krull topology) such that there is a $1-1$ Galois correspondence between all intermediate fields of E over F and all τ-closed subgroups of G. Furthermore (G, τ) is a compact totally disconnected group. It might be of some interest that if τ' is any other admissible topology on G then again there is a $1-1$ Galois correspondence between all intermediate fields of E over F and all τ'-closed subgroups of G.

REFERENCES

1. W. W. Comfort and K. A. Ross, *Topologies induced by groups of characters*, Fund. Math., **55** (1964), 283-291.

2. W. W. Comfort and V. Saks, *Countably compact groups and finest totally bounded topologies*, Pacific J. Math., **49** (1973), 33-44.

3. W. W. Comfort and T. Soundararajan, *Pseudocompact group topologies and totally dense subgroups* (submitted to the Pacific Journal of Mathematics).

4. L. Fuchs, *Infinite Abelian Groups, I*, Academic Press, New York, 1970.

5. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, I*, Springer-Verlag, New York, 1963.

6. W. Krull, *Galoissche Theorie der unendlichen algebraischen Erweiterungen*, Math, Ann., **100** (1928), 687-698.

7. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, 1955.

8. J. Mycielski, *Some properties of connected compact groups*, Colloq. Math., **5** (1957), 162-166.

9. K. A. Ross, *Closed subgroups of locally compact Abelian groups*, Fund. Math., **56**

(1965), 241-244.
10. A. Weil, *L' intégration dans les groups topologiques et ses applications*, Hermann, Paris, 1951.

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