POLYNOMIAL FORMS ON AFFINE MANIFOLDS

WILLIAM M. GOLDMAN AND MORRIS W. HIRSCH

An affine manifold is a differentiable manifold without boundary together with a maximal atlas of coordinate charts such that all coordinate changes extend to affine automorphisms of R^n. These distinguished charts are called affine coordinate systems.

Throughout this paper M denotes a connected affine manifold of dimension $n \geq 1$. We write E for R^n.

A tensor (field) on M is called *polynomial* if in all affine coordinate systems its coefficients are polynomial functions in n variables. In particular a real-valued function on M may be polynomial.

It is unknown whether there exists any compact affine manifold admitting a nonconstant polynomial function. The main purpose of this paper is to prove that for certain classes of affine manifolds there is no such function. These results are then applied to demonstrate that certain polynomial forms must also vanish. For related results, see Fried, Goldman, and Hirsch [2], Fried [1], [6], and [5].

1. **Development, holonomy, and polynomial tensors.** Let $p: \tilde{M} \to M$ be a universal covering space. There is an immersion $D: \tilde{M} \to E$, called the developing map, with the following properties (see e.g., [2]):

(1) D is affine, i.e., in affine coordinates D appears as an affine map;

(2) D is unique up to composition with an affine automorphism of E.

We call $D(\tilde{M})$ the *developing image*.

Let π denote the group of deck transformations of $\tilde{M}$. It follows from (2) that there is a homomorphism $\alpha: \pi \to \mathrm{Aff}\,(E)$, the group of affine automorphisms of E, such that D is equivariant respecting α, that is:

$$D \circ g = \alpha(g) \circ D \quad \text{for all } g \in \pi \ .$$

We call α the *affine holonomy*. The composition

$$\lambda: \pi \xrightarrow{\ \alpha\ } \mathrm{Aff}\,(E) \xrightarrow{\ \beta\ } GL(E) \ ,$$

where β is the natural homomorphism, is called the *linear holonomy*.

If $\alpha(\pi)$ fixes a point $p \in E$ then M is called a *radiant* manifold. In this case we can compose D with translation by $-p$ to obtain a new developing map whose corresponding affine holonomy fixes

the origin. When M is radiant we always assume D is such that $\alpha(\pi)$ fixes the origin. Then the affine and linear holonomy coincide.

The simplest example of a radiant manifold is a *Hopf manifold*. Here M is the quotient of $\mathbf{R}^n - \{0\}$ by the cyclic group generated by a linear expansion (a linear map all of whose eigenvalues have norm greater than one). One obtains more radiant manifolds by taking products of Hopf manifolds.

If $D: \widetilde{M} \to E$ is a homeomorphism then M is said to be *complete*. In this case M is the orbit space $E/\alpha(\pi)$.

LEMMA 1. *Suppose that M admits an affine immersion $f: M \to E$. Then each polynomial tensor T on M is f-related to a unique polynomial tensor S on E.*

Proof. Let f map a nonempty open set $U \subset M$ diffeomorphically onto an open set $V \subset E$. There is a unique tensor S_0 on V which is f-related to $T|_U$. Since f is affine, S_0 is polynomial; hence S_0 extends to a unique polynomial tensor S on E. The pullback f^*S is a polynomial tensor on M which agrees with T in U; hence it equals T because M is connected. The uniqueness of S is obvious. □

Now let T be a polynomial tensor on M and let $\widetilde{T} = p^*T$ be the pullback of T by $p: \widetilde{M} \to M$. Then $\widetilde{T}$ is a polynomial tensor on $\widetilde{M}$, where $\widetilde{M}$ is given the affine structure induced from M by p. Since the developing map $D: \widetilde{M} \to E$ is an affine immersion, Lemma 1 implies that $\widetilde{T}$ is D-related to a unique polynomial tensor S on E. The equivariance of D and the fact that $\widetilde{T}$ is π-invariant shows that S is $\alpha(\pi)$-invariant.

Conversely, starting with an $\alpha(\pi)$-invariant polynomial tensor S on E, there is a unique polynomial tensor T on M such that p^*T is D-related to S.

We have established a *natural one-to-one correspondence between polynomial tensors on M and $\alpha(\pi)$-invariant polynomial tensors on E.*

2. Polynomial functions. Let $D: \widetilde{M} \to E$ be the developing map of the affine manifold M. Let $\Gamma = \alpha(\pi) \subset \mathrm{Aff}\,(E)$ be the *affine holonomy group* of M.

Let $f: M \to \mathbf{R}$ be a polynomial function. In §1 we showed that there is a unique polynomial map $g: E \to \mathbf{R}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\ D\ } & E \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\
M & \xrightarrow{\ f\ } & \mathbf{R}
\end{array}
$$

and g is Γ-invariant.

Suppose f is bounded on M. Then clearly g is bounded on the developing image $D(\tilde{M})$. It follows that *if $L \subset E$ is a line such that $L \cap D(\tilde{M})$ is unbounded, then $g|_L$ is constant.* This implies that g, and hence also f, is constant if M is compact and complete. More generally:

THEOREM 2.1. *If $D(\tilde{M})$ contains a nonempty open cone then every bounded polynomial function on M is constant.*

Proof. Let $C \subset D(\tilde{M})$ be the open cone, with vertex $q \in E$. For every $x \in C$ the line xq meets C, and hence $D(\tilde{M})$, in a half line. Therefore g is constant on xq. Since C is open, the set of vertices q' of open cones C' contained in C is open. It follows that g is constant on a nonempty open set. Thus g, and also f, is constant. $\square$

THEOREM 2.2. *Let $D(\tilde{M}) \supset H(C)$ where $C \subset E$ is a nonempty open cone and $H: E \to E$ is a polynomial map whose Jacobian is not identically zero. Then every bounded polynomial function on M is constant.*

Proof. As in the proof of 2.1, one sees that $g \circ H: C \to E$ is constant. It follows that $g \circ H$ is constant. Since H has nonzero Jacobian at some point, the inverse function theorem implies that g is constant on an open set. Hence g and f are constant. $\square$

In Goldman [4] there is an example of a nonradiant compact affine 3-manifold with solvable fundamental group whose developing image is the convex set $\{(x, y, z) \in \mathbf{R}^3: y > x^2\}$. This set contains no open cone, but is equivalent to an open half-space by a polynomial automorphism of $\mathbf{R}^3$.

THEOREM 2.3. *Every polynomial function on a compact radiant manifold M is constant.*

Proof. We may assume the affine holonomy group Γ is contained in $GL(E)$. Therefore it preserves the vector field on E defined by the identity map $I(x) = x$, for $x \in E$. We consider I as a Γ-invariant tensor; therefore I corresponds to a polynomial vector field R on M called the *radiant vector field*.

Let $\{\phi_t\}_{t \in R}$ be the flow of R. Let $\tilde{R} = p^*R$ be the radiant vector field of $\tilde{M}$. Let $\tilde{\phi}_t$ be the flow of $\tilde{R}$. Then $D: \tilde{M} \to E$ is equivariant respecting $\tilde{\phi}_t$ and the flow of I, which is $x \to e^t x$. Therefore $D(\tilde{M})$ is invariant under multiplication by positive scalars, and hence an

open cone.   Thus 2.2 implies 2.3.   □

Let $A(\Gamma) \subset \text{Aff}(E)$ be the smallest algebraic subgroup of Aff (E) containing Γ.   We think of $A(\Gamma)$ as acting on E by affine automorphisms.

THEOREM 2.4.   *Suppose that $A(\Gamma)$ has an open orbit in E.   Then every polynomial function on M is constant.*

Proof.   Obvious, since Γ-invariant implies $A(\Gamma)$-invariant.

A condition generalizing radiance is *decomposability*:   We say that $\Gamma \subset \text{Aff}(E)$ preserves a decomposition of E if there is a direct-sum decomposition $E = E_1 \oplus F$, where E_1 is Γ-invariant and F is $\beta(\Gamma)$-invariant, where $\beta: \text{Aff}(E) \to GL(E)$ is the natural homomorphism.   If M has decomposable holonomy, then the radiant vector field on F defines a C^∞ vector field R_F as in § 6 of [2].   The sets $D^{-1}(a \times F)$, $a \in E$, are leaves of a foliation $\widehat{\mathscr{F}}$ of $\widetilde{M}$.   Being π-invariant, $\widehat{\mathscr{F}}$ comes from a foliation $\mathscr{F}$ of M.   Each leaf L of $\mathscr{F}$, in its manifold topology, has a natural affine structure, which is radiant, having R_F as its radiant vector field.

THEOREM 2.5.   *If M is compact, has decomposable holonomy as above, and $\dim E_1 = 1$, then every polynomial function on M is constant.*

Proof.   Since M is compact, the F-radiant vector field R_F on M is integrable; it then follows from 2.1 that on each leaf of $\mathscr{F}$ every polynomial function $f: M \to \mathbf{R}$ is constant.   Then the corresponding polynomial $g: E_1 \oplus F \to \mathbf{R}$ is constant on the cosets of F and g factors through a polynomial h as

$$E_1 \oplus F \longrightarrow E_1 \xrightarrow{\ h\ } \mathbf{R} \ ,$$

that is, $g(x, y) = h(x)$.

Since g is Γ-invariant, h is invariant under the affine action of Γ on E_1.   Passing to a double covering of M we may assume that Γ preserves orientation on E_1.   Now $E_1 \cong \mathbf{R}$, so every $\gamma \in \Gamma$ which acts nontrivially on E_1 has infinite orbit.   But a polynomial invariant under such a γ is constant.   Hence either $g|_{E_1}$ is constant, forcing g and f to be constant, or else Γ acts trivially on E_1.   But then Γ fixes the origin of E, so M is radiant and the theorem follows from 2.3.

One important case of decomposable holonomy occurs when M

is compact and Γ is a nilpotent group. Then the holonomy of M decomposes $E = E_1 \oplus F$, in such a way that under the projection $E \to E_1$ the developing image $D(\tilde{M})$ is a fibration *onto* E (see [2], 6.8). It then follows (see also [1]):

THEOREM 2.6. *A compact affine manifold whose affine holonomy group contains a nilpotent subgroup of finite index cannot admit a nonconstant polynomial function.*

Another proof of 2.6 follows from the fact that if M is a compact affine manifold with nilpotent affine holonomy group Γ, then either $E_1 = E$ and M is complete or else every Γ-invariant algebraic subset of E contains E_1 [6]. Since the level sets of an invariant polynomial $g : E \to R$ are Γ-invariant and disjoint, this contradiction proves 2.6.

In a forthcoming paper [1], D. Fried proves that on all compact affine manifolds of dimension three there are no nonconstant polynomial functions.

In [6] we prove that on all compact affine manifolds with "parallel volume," i.e., the affine structure is volume-preserving, there are no nonconstant polynomial functions.

3. Polynomial symmetric and exterior forms. A polynomial tensor T on an affine manifold has a well-defined *degree*. If the degree is 0 we call the tensor *parallel*. There is then a constant tensor S on E such that in every affine chart the components of T appear as S.

A parallel symmetric k-form on M corresponds to a Γ-invariant symmetric k-linear map $E \times \cdots \times E \to R$.

THEOREM 3.1. *On a compact radiant manifold every parallel symmetric k-form $(k \geq 1)$ is zero.*

Proof. Let T be the symmetric k-form and R the radiant vector field. Define a function $f : M \to R$ by $f(x) = T(R(x), \cdots, R(x))$. Then f is a polynomial function; by 2.3 f is constant.

Let $S : E^k \to R$ be the symmetric k-linear map corresponding to T. The radiant vector field corresponds to the identity vector field. Consider the map $g : E \to R$, $g(x) = S(x, \cdots, x)$. Since g is a polynomial function on E which corresponds to f, we see that g is constant. But the kth derivative of g at 0, considered as a symmetric k-linear map, equals $k!\, S$. Hence $S = 0$. $\qquad\square$

THEOREM 3.2. *On a compact radiant manifold every polynomial closed 1-form is zero.*

Proof. Let ω be such a 1-form and let R be the radiant vector field. Then $\omega(R)$ is a polynomial function, hence constant by 2.3.

The corresponding closed 1-form ω' on E is exact; thus $\omega' = du$ where $u: E \to R$ is a polynomial function. Now $\omega' = \Sigma(\partial u/\partial x_i)dx_i$; so the constancy of $\omega'(R')$ means that $\Sigma x_i(\partial u/\partial x_i)$ is constant. Since it vanishes at 0, it must vanish identically.

This says the derivative of u along any ray through the origin vanishes identically. Thus u is constant along such rays; so $u(x) = u(0)$ for all x. This shows $\omega' = du = 0$; hence $\omega = 0$.   □

Theorem 3.2 cannot be extended to polynomial 1-forms which are not closed, nor does it extend to nonradiant compact affine manifolds. In [2], § 6 there is an example of a compact radiant 3-manifold M which admits a parallel exterior 2-form ω. There are linear coordinates in R^3 so that $R = x(\partial/\partial x) + y(\partial/\partial y) + z(\partial/\partial z)$ is the radiant vector field and $\omega = dx \wedge dy$. The interior product $\iota_R\omega = x dy - y dx$ is a nonzero polynomial 1-form on M.

In general for an affine manifold M we may consider the polynomial de Rham complex $A^*_{\text{poly}}(M)$ consisting of all polynomial exterior differential forms on M, and form its cohomology $H^*_{\text{poly}}(M)$, the *polynomial de Rham cohomology* of M. There is a map $H^*_{\text{poly}}(M) \to H^*(M; R)$ induced by the inclusion $A^*_{\text{poly}}(M) \to A^*(M)$ of chain complexes.

Conjecture 3.3. If M is a complete affine manifold, then the natural mapping $H^*_{\text{poly}}(M) \to H^*(M; R)$ is an isomorphism.

In [2], § 8 this conjecture is proved when M is compact and $\pi_1(M)$ is nilpotent. In [3] this is extended to M compact and $\pi_1(M)$ virtually solvable.

In contrast to the complete case, we have the following result:

THEOREM 3.4. *Let M be a radiant affine manifold. Then for $k > 0$, $H^k_{\text{poly}}(M) = 0$, i.e., every closed polynomial k-form is exact.*

Proof. Let ω be a polynomial exterior k-form on M. Let ω' denote the corresponding form on E. Decompose ω' into its components $\omega' = \omega'_m + \cdots + \omega'_1 + \omega'_0$ where ω'_p is an exterior k-form whose coefficients are homogeneous polynomials of degree p. Since M is radiant each ω'_p defines a k-form ω_p invariantly on M. It is

easy to see that ω is closed if and only if each ω_p is closed. Hence to prove the theorem it suffices to assume that the coefficients of ω are homogeneous polynomials are some degree m.

Let R' (resp. R) denote the radiant vector field on E (resp. M) and let ξ'_t (resp. ξ_t) be the corresponding flow. It is easy to see that $\xi_t^* \omega = e^{(k+m)t}\omega$ so we may take the Lie derivative $\mathscr{L}_R \omega = (k+m)\omega$. Now $\omega = (k+m)^{-1}\mathscr{L}_R \omega = d((k+m)^{-1}\iota_R \omega)$ is exact. $\square$

In a special case, there is a converse to 3.4:

THEOREM 3.5. *Let M be a compact affine manifold whose holonomy group is nilpotent. If $H^k_{\mathrm{poly}}(M) = 0$ for all $k > 0$, then M is radiant.*

Proof. It is proved in [7] (see [6] for another application) that unless M is radiant there is a parallel exterior k-form ω with nonzero cohomology class $[\omega] \in H^k(M)$. Since ω is parallel, it defines a cohomology class $\{\omega\} \in H^k_{\mathrm{poly}}(M)$. Since the image of $\{\omega\}$ in $H^k(M)$ is nonzero, $\{\omega\}$ defines a nonzero element of $H^k_{\mathrm{poly}}(M)$. $\square$

REFERENCES

1. D. Fried, *Polynomials on affine manifolds*, (to appear in Comment. Math. Helv.).
2. D. Fried, W. Goldman, and M. Hirsch, *Affine manifolds with nilpotent holonomy*, Comm. Math. Helv., **56** (1981), 487–523.
3. W. Goldman, *On the polynomial cohomology of affine manifolds*, (in preparation).
4. ———, *Two examples of affine manifolds*, (to appear in Pacific J. Math.).
5. ———, *Rational functions on affine 3-manifolds*, (in preparation).
6. W. Goldman and M. Hirsch, *Parallel characteristic classes of affine manifolds*, (in preparation).
7. W. Goldman, M. Hirsch, and G. Levitt, *Invariant measures for affine foliations*, Technical report 34, Center for Pure and Applied Math., U. California at Berkeley, 1980,

UNIVERSITY OF COLORADO
BOULDER, CO 80309
AND
UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720