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SEMI-GROUPS OF QUASINORMAL OPERATORS

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Strongly continuous semi-groups $\{Q_t\}$ of quasinormal operators on Hilbert space are characterized as follows: there exist Hilbert spaces $\mathscr{L}$ and $\mathscr{K}$, a strongly continuous normal semi-group $\{N_t\}$ on $\mathscr{L}$ and a strongly continuous self-adjoint semi-group $\{h(t)\}$ on $\mathscr{K}$ such that $\{Q_t\}$ is unitarily equivalent to $\{N_t\}\oplus\{\overline{h(t)}L_t\}$ on $\mathscr{L}\oplus\mathscr{L}^2(\mathscr{K})$, where $\{L_t\}$ is the forward translation semi-group on $\mathscr{L}^2(\mathscr{K})$ and $(\overline{h(t)}f)(x)=h(t)f(x)$ a.e. for each f in $\mathscr{L}^2(\mathscr{K})$.

1. **Preliminaries.** In this paper we characterize one parameter strongly continuous semi-groups of quasinormal operators. The major result, found in Theorem 6, bears a marked resemblance to the characterization of quasinormal operators given by Brown in [2]. He showed that an operator A is *quasinormal* (A commutes with A^*A) if and only if there exist Hilbert spaces $\mathscr{L}$ and $\mathscr{K}$, a normal operator N on $\mathscr{L}$ and a positive operator P on $\mathscr{K}$ such that A is unitarily equivalent to $N\oplus S\bar{P}$ on $\mathscr{L}\oplus \ell^2(\mathscr{K})$ where S is the unilateral shift on $\ell^2(\mathscr{K})$ and $(\bar{P}x)_k = Px_k$ whenever $\{x_k\}\in \ell^2(\mathscr{K})$.

We shall use the following notation and conventions. $\mathscr{H}$ is a separable Hilbert space and $\mathscr{B}(\mathscr{H})$ is the space of continuous linear operators on $\mathscr{H}$. $\ell^2(\mathscr{H})$ is the Hilbert space of all sequences $\{x_n\}$ where $x_n\in\mathscr{H}$ and $\Sigma\|x_n\|^2<\infty$. In particular, $\ell^2=\ell^2(\mathscr{C})$, where $\mathscr{C}$ is the set of complex numbers. $\mathscr{R}_+$ denotes the set of non-negative real numbers. $\mathscr{L}^2(\mathscr{H})$ will stand for the Hilbert space of (equivalence classes) of weakly measurable functions from $\mathscr{R}_+$ into $\mathscr{H}$ such that

$$\int_0^\infty \|f(x)\|^2dx<\infty . \quad \text{In particular, } \mathscr{L}^2=\mathscr{L}^2(\mathscr{C}) .$$

An operator A on $\mathscr{H}$ is *self-adjoint* if $A=A^*$, *normal* if $AA^*=A^*A$, *subnormal* if A is the restriction of a normal operator to an invariant subspace, an *isometry* if $A^*A=I$ where I is the identity operator on $\mathscr{H}$, a *partial isometry* if $(A^*A)^2=A^*A$, and *unitary* if A is a normal isometry.

We use [3] as a general reference on semi-groups of operators. The set $\{S_t\}=\{S_t\colon t\in\mathscr{R}_+\}$ is a *semi-group* of elements of $\mathscr{B}(\mathscr{H})$ if $S_{t+r}=S_tS_r$ for all t and r in $\mathscr{R}_+$ and $S_0=I$. We say that $\{S_t\}$ has a certain property (for example, is quasinormal) if each of the operators S_t has that property. A semi-group $\{S_t\}$ is *strongly con-*

tinuous if $\lim_{t\to 0} \|S_t f - f\| = 0$ for each f in $\mathscr{H}$ and *uniformly continuous* if $\lim_{t\to 0} \|S_t - I\| = 0$. The *generator* of a strongly continuous semi-group $\{S_t\}$ is the (not necessarily bounded) linear transformation S defined by $Sf = \lim_{t\to 0} (S_t f - f)/t$, whenever this limit exists in the strong topology.

One semi-group which will play a prominent part in the development of ideas is the forward translation semi-group $\{L_t\}$ on $\mathscr{L}^2(\mathscr{H})$ defined for each f in $\mathscr{L}^2(\mathscr{H})$ by $(L_t f)(x) = f(x - t)$ if $x \geqq t$ and zero otherwise. It is well-known that $\{L_t\}$ is a strongly continuous semi-group and the infinitesimal generator of $\{L_t\}$ is defined by $f \to -f'$ for all f in $\mathscr{L}^2(\mathscr{H})$ for which f is absolutely continuous, $f' \in \mathscr{L}^2(\mathscr{H})$ and $f(0) = 0$. We shall denote this unbounded operator by $-D$. The semi-group of adjoints $\{L_t^*\}$ is the backward translation semi-group and for each f in $\mathscr{L}^2(\mathscr{H})$, $(L_t^* f)(x) = f(x + t)$. The generator of $\{L_t^*\}$ is defined by $f \to f'$ for all f in $\mathscr{L}^2(\mathscr{H})$ for which f is absolutely continuous and $f' \in \mathscr{L}^2(\mathscr{H})$.

The isometric semi-groups ($U_t^* U_t = I$) are obviously quasinormal. In [5] Cooper characterizes them as follows: a strongly continuous semi-group $\{U_t\}$ is isometric if and only if there exist Hilbert spaces $\mathscr{L}$ and $\mathscr{K}$ and a unitary semi-group $\{W_t\}$ on $\mathscr{L}$ such that $\{U_t\}$ is unitarily equivalent to $\{W_t\} \oplus \{L_t\}$ on $\mathscr{L} \oplus \mathscr{L}^2(\mathscr{K})$. In § 2 we show that $\{Q_t\}$ can be factored into an isometric semi-group and a self-adjoint semi-group, each of which is strongly continuous and which commute with one another. This reduces the general problem of characterizing quasinormal semi-groups to that of characterizing those semi-groups of the form $\{H_t L_t\}$ where $\{H_t\}$ is a self-adjoint semi-group commuting with $\{L_t\}$. In § 3 we complete the characterization.

In § 4 we investigate the properties of the infinitesimal generator of a quasinormal semi-group and give an explicit representation for it in terms of the characterization of the semi-group.

2. Factoring semi-groups. Let ϕ be a continuous, almost every where nonzero function from $\mathscr{R}_+$ into $\mathscr{C}$ and define $(S_t f)(x) = (\phi(x)/\phi(x - t)) f(x - t)$ if $x \geqq t$ and zero otherwise for f in $\mathscr{L}^2(\mathscr{K})$. Under suitable boundedness conditions on ϕ, $\{S_t\}$ is a strongly continuous semi-group in $\mathscr{B}(\mathscr{L}^2)$ [7, p. 334] and is called a *weighted translation semi-group*. Such a semi-group is quasinormal exactly when ϕ is a multiple of an exponential: $\phi(x) = Me^{ax}$ [7, p. 340–341]. A straightforward computation shows that $\{S_t^* S_t\}$ is a semi-group exactly when $\phi(x + t + s)\phi(x) = \phi(x + t)\phi(x + s)$ for all x, t, s, or equivalently, when ϕ is a multiple of an exponential. Therefore $\{S_t\}$ is quasinormal exactly when $\{S_t^* S_t\}$ is a semi-group. In Lemma 1 we show that this equivalence always occurs.

LEMMA 1. *Let* $\{Q_t\}$ *be a strongly continuous semi-group of operators.* $\{Q_t\}$ *is quasinormal if and only if* $\{Q_t^*Q_t\}$ *is a semi-group. Moreover in this case* $\{Q_t^*Q_t\}$ *is strongly continuous and* Q_r *commutes with* $\{Q_t^*Q_t\}$ *for each* r *and* t.

Proof. Assume first that $\{Q_t\}$ is quasinormal. Every quasinormal operator is subnormal [9] and every strongly continuous semi-group of subnormal operators has a normal extension as a semi-group [10]. That is, there exists a strongly continuous normal semi-group $\{N_t\}$ of operators on a Hilbert space $\mathscr{K}$, containing $\mathscr{H}$, with $N_t/\mathscr{H} = Q_t$. Since Q_t is quasinormal, then $\mathscr{H}$ is invariant under $N_t^*N_t$ [4] and since $\{N_t\}$ is a strongly continuous normal semi-group, it follows that $\{N_t^*N_t\}$ is a strongly continuous semi-group and N_r commutes with $N_t^*N_t$ for each r and t. Consequently, $\{Q_t^*Q_t\}$ inherits the same properties.

On the other hand if we assume that $\{Q_t^*Q_t\}$ is a semi-group, then for each t and each nonnegative integer n, $(Q_t^*)^n(Q_t)^n = Q_{nt}^*Q_{nt} = (Q_t^*Q_t)^n$, which is sufficient to imply that each Q_t is quasinormal [6].

By the *polar decomposition* of an operator A we mean the unique representation $A = UP$ where P is the unique square root of A^*A and U is a partial isometry such that $\ker U = \ker P = \ker A$. A necessary and sufficient condition that A be quasinormal is that U and P commute [2]. It is not difficult to show that when A is quasinormal, the polar decomposition of A^n is U^nP^n. The continuous analogues of these assertions are found in the following theorem.

THEOREM 2. *For each* t *in* $\mathscr{R}_+$ *let* U_tP_t *be the polar decomposition of* Q_t. *Then* $\{Q_t\}$ *is a strongly continuous quasinormal semi-group if and only if*

(i) $\{P_t\}$ *is a strongly continuous self-adjoint semi-group,*

(ii) $\{U_t\}$ *is a strongly continuous isometric semi-group, and*

(iii) P_r *commutes with* U_t *for each* r *and* t.

Proof. Obviously, if conditions (i), (ii), and (iii) are true, then $\{Q_t\}$ is a quasinormal semi-group. Moreover, in this case $\{Q_t\}$ is the product of strongly continuous semi-groups and is, itself, strongly continuous.

Assume now that $\{Q_t\}$ is a strongly continuous quasinormal semi-group. P_t is the positive square root of $Q_t^*Q_t$. Therefore since P_t^2 and P_r^2 commute, so do P_t and P_r for all t and r. This implies that $(P_{t+r})^2 = (P_tP_r)^2$. Since the positive square roots are unique, $P_{t+r} = P_tP_r$ and $\{P_t\}$ is a semi-group of self-adjoint operators. Moreover, since $P_t - I = (P_t + I)^{-1}(P_t^2 - I)$ and $\{P_t^2\}$ is strongly continuous by Lemma 1, then so is $\{P_t\}$. (We use here the fact that

$\|(P_t + I)^{-1}\| \leqq 1$ since P_t is positive.)

To show that U_t is an isometry, we only need show that $\ker P_t = \{0\}$. But if $P_t f = 0$, then $P_{(1/2)t} f = 0$ since P_t is positive. Thus by induction there is a sequence $t_n \to 0$ such that $P_{t_n} f = 0$. Using the strong continuity of $\{P_t\}$ we arrive at $f = 0$.

Since $\ker P_t = \{0\}$, any operator commuting with Q_t and P_t also commutes with U_t. Also, Q_r commutes with P_t for each r and t by Lemma 1. Therefore since each of $\{P_t\}$ and $\{Q_t\}$ is commutative, U_r commutes with P_t and U_t for each r and t. Also $U_t U_s P_{t+s} = U_t P_t U_s P_s = Q_t Q_s = Q_{t+s} = U_{t+s} P_{t+s}$ so that $U_t U_s = U_{t+s}$ on the range of P_{t+s} which is a dense subset of $\mathscr{H}$. We have shown that $\{U_t\}$ is an isometric semi-group.

To show that $\{U_t\}$ is strongly continuous we argue as follows: For f and g in $\mathscr{H}$

$$\begin{aligned} |\langle f - U_t f, g\rangle| &= |\langle f - Q_t f, g\rangle + \langle P_t f - f, U_t^* g\rangle| \\ &\leqq (\|f - Q_t f\| + \|P_t f - f\|)\|g\| , \end{aligned}$$

and consequently

$$\|f - U_t f\| \leqq \|f - Q_t f\| + \|P_t f - f\| .$$

Strong continuity of $\{Q_t\}$ and $\{P_t\}$ now implies strong continuity of $\{U_t\}$.

REMARK 1. We note that $\{Q_t\}$ is normal if and only if $\{U_t\}$ is unitary. This follows from Theorem 2(ii) and the fact that a quasinormal operator is normal if and only if the partial isometry in the polar decomposition of Q is normal.

In view of the nice behavior of the sets $\{U_t\}$ and $\{P_t\}$ when $\{Q_t\}$ is quasinormal, we shall write $\{Q_t\} = \{U_t\}\{P_t\}$ and call $\{U_t\}$ the *isometric factor* of $\{Q_t\}$ and $\{P_t\}$ the *positive factor*.

3. A characterization of quasinormal semi-groups.

THEOREM 3. *Let $\{Q_t\}$ be a strongly continuous quasinormal semi-group. There exist Hilbert spaces $\mathscr{L}$ and $\mathscr{K}$, a strongly continuous normal semi-group $\{N_t\}$ on $\mathscr{L}$ and a strongly continuous self-adjoint semi-group $\{H_t\}$ on $\mathscr{L}_2(\mathscr{K})$ commuting with $\{L_t\}$, such that $\{Q_t\}$ is unitarily equivalent to $\{N_t\} \oplus \{H_t L_t\}$ on $\mathscr{L} \oplus \mathscr{L}^2(\mathscr{K})$. Conversely, any semi-group constructed in this fashion is a strongly continuous quasinormal semi-group.*

Proof. The converse is immediate since $\{N_t\}$ is trivially quasinormal and $\{H_t L_t\}$ is a strongly continuous quasinormal semi-group by Theorem 2.

Assume that $\{Q_t\}$ is a strongly continuous quasinormal semi-group. By Theorem 2 $\{Q_t\} = \{P_t\}\{U_t\}$ where $\{P_t\}$ is self-adjoint and commutes with the isometric semi-group $\{U_t\}$. Cooper's theorem [5] tells us that $\{U_t\}$ is unitarily equivalent to $\{W_t\} \oplus \{V_t\}$ where $\{W_t\}$ is unitary and defined on $\mathscr{L}$, and $\mathscr{L}$ is the range of the projection $\lim_{t\to\infty} U_t U_t^*$. Moreover $\{V_t\}$ is unitarily equivalent to the forward translation semi-group $\{L_t\}$ on $\mathscr{L}^2(\mathscr{K})$ for some Hilbert space $\mathscr{K}$.

Since by Theorem 2 P_r commutes with U_t for each r and t, then $\mathscr{L}$ reduces $\{P_t\}$. Thus we have $\{P_t\}$ unitarily equivalent to $\{K_t\} \oplus \{H_t\}$ where $\{K_t\}$ is self-adjoint and commutes with $\{W_t\}$ on $\mathscr{L}$ and $\{H_t\}$ is self-adjoint and commutes with $\{L_t\}$ on $\mathscr{L}^2(\mathscr{K})$. Thus $\{Q_t\}$ is unitarily equivalent to $\{K_t W_t\} \oplus \{H_t L_t\}$ on $\mathscr{L} \oplus \mathscr{L}^2(\mathscr{K})$, and $\{K_t W_t\}$ is normal since $\{W_t\}$ is unitary and commutes with $\{K_t\}$.

The semi-group $\{H_t L_t\}$ is completely nonnormal in the sense that there exists no subspace which reduces $\{H_t L_t\}$ and on which $\{H_t L_t\}$ is normal. The last step in characterizing quasinormal semi-groups is to characterize the self-adjoint semi-groups commuting with $\{L_t\}$ on $\mathscr{L}^2(\mathscr{K})$.

Each h in $\mathscr{B}(\mathscr{K})$ *induces* an operator $\bar{h}$ in $\mathscr{B}(\mathscr{L}^2(\mathscr{K}))$ by $(\bar{h}f)(x) = hf(x)$ a.e. whenever $f \in \mathscr{L}^2(\mathscr{K})$. Each such induced operator $\bar{h}$ commutes with $\{L_t\}$ and if $\{h(t)\}$ is a (self-adjoint) semi-group in $\mathscr{B}(\mathscr{K})$, then $\{\overline{h(t)}\}$ is a (self-adjoint) semi-group in $\mathscr{B}(\mathscr{L}^2(\mathscr{K}))$. (We shall show in Theorem 5 that the strong continuity of either implies strong continuity of the other.) All of this leads to the following: $\{\overline{h(t)}\}$ is a strongly continuous self-adjoint semi-group, commuting with $\{L_t\}$ whenever $\{h(t)\}$ is a strongly continuous self-adjoint semi-group on $\mathscr{K}$. In Theorem 5 we shall show that this is the only way to construct a positive factor for a quasinormal semi-group with isometric factor $\{L_t\}$. The key to this result lies in the following lemma concerning the commutant of $\{L_t\}$.

The *commutant* of a collection $\mathscr{A}$ of operators on $\mathscr{K}$ is the algebra $\mathscr{A}' = \{T\colon T \in \mathscr{B}(\mathscr{K})$ and $TA = AT$ for all A in $\mathscr{A}\}$.

LEMMA 4. *Let $\{L_t\}$ be the forward translation semi-group on $\mathscr{L}^2(\mathscr{K})$. Then $\{L_t\}' \cap \{L_t^*\}' = \{\bar{h}\colon h \in \mathscr{B}(\mathscr{K})\}$.*

Proof. We have already observed that each $\bar{h}$ is in $\{L_t\}'$. Since $(L_t^* f)(x) = f(x + t)$, a quick check shows that each $\bar{h}$ is also in $\{L_t^*\}'$.

Now assume that H commutes with $\{L_t\}$ and $\{L_t^*\}$. Without loss of generality we may assume that H is self-adjoint since each of Re H and Im H commutes with $\{L_t\}$ and $\{L_t^*\}$. Let $\{e_n: n \in \Delta\}$ be a complete orthonormal basis of the separable Hilbert space $\mathscr{K}$ and identify $\mathscr{L}^2(\mathscr{K})$ with $\Sigma_n \oplus \mathscr{L}^2(\mathscr{C})$ in the usual fashion [8, p. 32]. The coordinate functions of each element f of $\mathscr{L}^2(\mathscr{K})$ are defined by $f_n(x) = \langle f(x), e_n \rangle$ and the matrix $[T_{nm}]$ of an operator T on $\mathscr{L}^2(\mathscr{K})$ is defined by $T_{nm}f = (T(fe_m))_n$ whenever $f \in \mathscr{L}^2$. (fe_m is the element of $\mathscr{L}^2(\mathscr{K})$ whose value at x is $f(x)e_m$ a.e.) Straightforward computations show the following:

(1) $[(L_t)_{nm}]$ is diagonal and $(L_t)_{nn} = L_t^{(0)}$, the forward translation by t on $\mathscr{L}^2 = \mathscr{L}^2(\mathscr{C})$;

(2) $H_{nm}^* = H_{mn}$ for each n and m since H is self-adjoint;

(3) H_{nm} commutes with $L_t^{(0)}$ for each n and m since H commutes with L_t and the matrix of L_t is diagonal.

But the forward translation semi-group on $\mathscr{L}^2$ is irreducible [1, p. 76]. Thus the self-adjoint operators on $\mathscr{L}^2$ commuting with $\{L_t^{(0)}\}$ are the scalar multiples of the identity operator I on $\mathscr{L}^2$. It now follows from (2) and (3) that Re H_{nm}, Im H_{nm} and consequently H_{nm} are scalar multiples of I. Let $H_{nm} = h_{nm}I$. For each f in $\mathscr{L}^2(\mathscr{K})$ and each n

(1) $(Hf)_n = \sum_{m \in \Delta} H_{nm} f_m = \sum_{m \in \Delta} h_{nm} f_m$.

Let $k \in \mathscr{K}$ and define $f(x) = k$ for x in $[0, 1]$ and 0 elsewhere. Then $(Hf)_n(x) = \sum_{m \in \Delta} h_{nm} k_m$ for x in $[0, 1]$ and 0 elsewhere. Also $\|f\| = \|k\|$ and $\sum_{n \in \Delta} |\sum_{m \in \Delta} h_{nm} k_m|^2 = \sum_{n \in \Delta} \int_0^1 |(Hf)_n(x)|^2 dx = \|Hf\|^2$. Thus the matrix $[h_{nm}]$ defines a (bounded) operator h on $\mathscr{K}$. Finally, we see from equation (1) that for each f in $\mathscr{L}^2(\mathscr{K})$, $(Hf)(x) = hf(x)$ a.e. so that $H = \bar{h}$.

Lemma 4 is the continuous analogue of the fact that $\{A\}' \cap \{A^*\}' = \{\bar{m}: m \in \mathscr{K}\}$ when A is the unilateral shift on $\mathscr{H}^2(\mathscr{K})$ [8, § 4]. The connection between the unilateral shift on $\mathscr{H}^2(\mathscr{K})$ and the forward translation semi-group on $\mathscr{L}^2(\mathscr{R}_+, \mathscr{K})$ is discussed in [11, p. 29–31].

Theorem 5. *The strongly continuous self-adjoint semi-groups on $\mathscr{L}^2(\mathscr{K})$, commuting $\{L_t\}$, are induced by the strongly continuous self-adjoint semi-groups on $\mathscr{K}$.*

Proof. First let $\{h(t)\}$ be a strongly continuous self-adjoint semi-group on $\mathscr{K}$. We have already noted that $\overline{\{h(t)\}}$ is a self-adjoint semi-group on $\mathscr{L}^2(\mathscr{K})$, commuting with $\{L_t\}$. We need to show that $\overline{\{h(t)\}}$ is strongly continuous. Let f be an element of $\mathscr{L}^2(\mathscr{K})$. Then for each x, $\lim_{t \to 0} h(t)f(x) = f(x)$, since $\{h(t)\}$ is strongly continuous on $\mathscr{K}$. Moreover $\{h(t)\}$ is bounded on finite intervals [3, p. 8]. Hence for t in $[0, 1]$ $\|h(t)f(x)\| \leqq M\|f(x)\|$ and consequently by the Lebesgue Dominated Convergence Theorem, $\|\overline{h(t)}f - f\| \to 0$, showing that $\{\overline{h(t)}\}$ is strongly continuous.

Secondly, assume that $\{H_t\}$ is a strongly continuous self-adjoint semi-group, commuting with $\{L_t\}$ on $\mathscr{L}^2(\mathscr{K})$. By Lemma 4, $H_t = \overline{h(t)}$ for some $h(t)$ in $\mathscr{B}(\mathscr{K})$. To verify that $\{h(t)\}$ has the desired properties we proceed as follows: Let $k \in \mathscr{K}$ and define f by $f(x) = k$ if $x \in [0, 1]$ and 0 otherwise. Then $f \in \mathscr{L}^2(\mathscr{K})$ and

(1) $h(t + s)k = (H_{t+s}f)(x) = (H_tH_sf)(x) = h(t)(H_sf)(x) = h(t)h(s)k,$

(2) $\langle H_tf,\ f\rangle = \int_0^\infty \langle h(t)f(x), f(x)\rangle dx = \langle h(t)k, k\rangle,$

(3) $\|H_tf - f\|^2 = \int_0^\infty \|h(t)f(x) - f(x)\|^2 dx = \|h(t)k - k\|^2.$

Thus $\{h(t)\}$ is (1) a semi-group, (2) self-adjoint, and (3) strongly continuous.

We combine the results of Theorems 3 and 5 to arrive at the continuous analogue of Brown's characterization of quasinormal operators.

THEOREM 6. *$\{Q_t\}$ is a strongly continuous quasinormal semi-group if and only if there exist Hilbert spaces $\mathscr{L}$ and $\mathscr{K}$, a strongly continuous normal semi-group $\{N_t\}$ on $\mathscr{L}$ and a strongly continuous self-adjoint semi-group $\{h(t)\}$ on $\mathscr{K}$ such that $\{Q_t\}$ is unitarily equivalent to $\{N_t\} \oplus \{\overline{h(t)}L_t\}$ on $\mathscr{L} \oplus \mathscr{L}^2(\mathscr{K})$.*

COROLLARY 7. *Let $\mathscr{K}$ and $\{h(t)\}$ be as in Theorem 6. If $\mathscr{K}$ is finite n-dimensional, then there exist real numbers $a_1, \cdots, a_n$ such that $\{\overline{h(t)}L_t\}$ is unitarily equivalent to $e^{a_1t}L_t^{(0)} \oplus \cdots \oplus e^{a_nt}L_t^{(0)}$, where $\{L_t^{(0)}\}$ is the forward translation semi-group on $\mathscr{L}^2(\mathscr{C})$.*

Proof. Since $\mathscr{K}$ is finite dimensional, the generator h of $\{h(t)\}$ is bounded, and since h is self-adjoint, h is diagonal. Let $\{e_k\}$ be a basis of $\mathscr{K}$ such that the matrix of h is diagonal with diagonal elements $a_1, \cdots, a_n$. Then $\{h(t)\}$ is diagonal with diagonal elements $e^{ta_1}, \cdots, e^{ta_n}$. Recall from the proof of Lemma 4 that $[(L_t)_{nm}]$ is diagonal and $(L_t)_{kk} = L_t^{(0)}$. Thus the matrix of $\overline{h(t)}L_t$ is diagonal with $(h(t)L_t)_{kk} = (e^{ta_k})L_t^{(0)}$, as desired.

We see now that the quasinormal weighted translation semi-groups introduced at the beginning of § 2 were quite typical. By Corollary 7 each quasinormal semi-group is a finite direct sum of quasinormal weighted translation semi-groups whenever the auxiliary space $\mathscr{K}$ is finite dimensional. We can go a little farther: if $\{h(t)\}$ is uniformly continuous and if the infinitesimal generator of $\{h(t)\}$ is a diagonal operator on $\mathscr{K}$, then the proof of Corollary 7 is valid whether $\mathscr{K}$ is finite or infinite dimensional. Consequently we can conclude that $\{\overline{h(t)}L_t\}$ is unitarily equivalent to a direct sum of quasinormal semi-groups of the form $\{e^{at}L_t^{(0)}\}$. However, if $\mathscr{K}$ is infinite dimensional and we choose a self-adjoint operator h on $\mathscr{K}$ with no point spectrum, then the induced operator $\bar{h}$ on $\mathscr{L}^2(\mathscr{K})$ also fails to have point spectrum and consequently $\{e^{t\bar{h}}L_t\}$ is not unitarily equivalent to a direct sum of quasinormal weighted translation semi-groups.

4. The generator of a quasinormal semi-group. Recall that the (infinitesimal) generator of a strongly continuous semi-group $\{S_t\}$ is the operator S (not necessarily bounded) defined by $Sf = \lim_{t\to 0}(S_tf - f)/t$, whenever this limit exists in the strong topology. We shall denote the domain of S by $\mathscr{D}(S)$. In general if $\{S_t\}$ is the product of two strongly continuous semi-groups $\{R_t\}$ and $\{T_t\}$, the most one can show is that $R + T \subset S$ in the sense that $\mathscr{D}(R) \cap \mathscr{D}(T) \subset \mathscr{D}(S)$ and that $R + T = S$ on $\mathscr{D}(R) \cap \mathscr{D}(T)$. However quite a bit more can be said about the generators of a quasinormal semi-group and its isometric and positive factors.

THEOREM 8. *Let $\{Q_t\} = \{U_t\}\{P_t\}$ be a strongly continuous quasinormal semi-group and let Q, U, and P be the generators of $\{Q_t\}$, $\{U_t\}$ and $\{P_t\}$, respectively. Then*

(i) $\mathscr{D}(Q) \subset \mathscr{D}(Q^*)$

(ii) $\mathscr{D}(Q) = \mathscr{D}(P) \cap \mathscr{D}(U)$

(iii) $Q = P + U$ *and* $Q^* = P - U$ *on* $\mathscr{D}(Q)$ *and*

(iv) $Q^*(\mathscr{D}(Q^2)) \subset \mathscr{D}(Q)$ *and* $QQ^* = Q^*Q$ *on* $\mathscr{D}(Q^2)$.

Proof. Assertion (i) follows from the fact that $\| Q_t^*f - f \| \leqq \| Q_tf - f \|$ for all f and t. Moreover $Q^*f = \lim_{t\to 0}(Q_t^*f - f)/t$ on $\mathscr{D}(Q)$.

To prove (ii) and (iii) we first prove that $\mathscr{D}(Q) \subset \mathscr{D}(P)$ and $P = (1/2)(Q + Q^*)$ on $\mathscr{D}(Q)$. For each f in $\mathscr{H}$ and each $t > 0$, $P_tf - f = (P_t + I)^{-1}[Q_t^*(Q_tf - f) + (Q_t^*f - f)]$. But as $t \to 0$, $(P_t + I)^{-1}$ converges strongly to $(1/2)I$, Q_t^* converges strongly to I, and if $f \in \mathscr{D}(Q)$, $(Q_tf - f)/t$ converges to Qf and $(Q_t^*f - f)/t$ converges to Q^*f. Therefore $\lim_{t\to 0}(P_tf - f)/t = (1/2)(Qf + Q^*f)$, so that $f \in \mathscr{D}(P)$ and $Pf = (1/2)(Qf + Q^*f)$.

Now observe that for each f and t

(2) $Q_tf - f = U_t(P_tf - f) + (U_tf - f)$.

Equation (2) immediately implies that $\mathscr{D}(P) \cap \mathscr{D}(U) \subset \mathscr{D}(Q)$ and $\mathscr{D}(Q) \cap \mathscr{D}(P) \subset \mathscr{D}(U)$. We have already shown $\mathscr{D}(Q) \subset \mathscr{D}(P)$.

These three set inclusions yield $\mathscr{D}(Q) = \mathscr{D}(P) \cap \mathscr{D}(U)$. Therefore, equation (2) can be used to conclude that $Qf = Pf + Uf$ for all f in $\mathscr{D}(Q)$. Finally since $Pf = (1/2)(Qf + Q^*f)$ for all f in $\mathscr{D}(Q)$, we also have $Q^*f = Pf - Uf$ for all f in $\mathscr{D}(Q)$.

Note now that if $f \in \mathscr{D}(Q^2)$, then by definition $f \in \mathscr{D}(Q)$ and $Qf \in \mathscr{D}(Q)$. But then $f \in \mathscr{D}(P)$ and $Qf \in \mathscr{D}(P)$ by (ii). Consequently $(P_t f - f)/t \to Pf$ and since P_t commutes with Q, $Q(P_t f - f)/t \to PQf$. Every generator is closed [3, p. 10] so that $Pf \in \mathscr{D}(Q)$ and $QPf = PQf$. Similarly $Vf \in \mathscr{D}(Q)$ and $QVf = VQf$. Finally, since $Q^*f = Pf - Vf$, we know that $Q^*f \in \mathscr{D}(Q)$. Moreover $QQ^*f = Q(Pf - Vf) = PQf - VQf = Q^*Qf$ by (iii) since $Qf \in \mathscr{D}(Q)$.

The fourth conclusion in Theorem 3 indicates that the generator Q behaves very much like a normal operator. In general it is not true that $Q^*(\mathscr{D}(Q)) \subset \mathscr{D}(Q)$ (for example, if $Q = -D$, the generator of the forward translation semi-group on $\mathscr{L}^2$). Thus the assertion $QQ^* = Q^*Q$ on $\mathscr{D}(Q)$ is not meaningful. We also note that the first conclusion of Theorem 3 cannot in general be strengthened.

Although we have not been able to verify it we conjecture that if Q is the generator of a strongly continuous semi-group $\{Q_t\}$ and Q satisfies conditions (i)–(iv) of Theorem 8, then $\{Q_t\}$ is quasinormal.

REMARK 2. Since a generator is closed and densely defined [3, p. 10], it is bounded if and only if it is everywhere defined. It follows now from Theorem 8(ii) that Q is bounded if and only if both U and P are bounded. But this is equivalent to $\{Q_t\}$ being uniformly continuous [3, p. 13] and normal, the normality resulting from each of the quasinormal operators Q_t being invertible (and hence normal) when Q is bounded.

It is well-known that the generator of a normal semi-group $\{N_t\}$ is normal. Applying Theorem 8 we note that the generator of $\{N_t\}$ is the sum of the generators of the unitary factor $\{W_t\}$ and the positive factor $\{K_t\}$ of $\{N_t\}$. The generator of $\{W_t\}$ is iT, where T is self-adjoint [8, p. 93] and the generator of $\{K_t\}$ is self-adjoint. To complete our analysis of the generator of a quasinormal semi-group we need to determine the generator of $\{\overline{h(t)L_t}\}$, the completely nonnormal part of $\{Q_t\}$.

COROLLARY 9. *Let $\{h(t)\}$ be a strongly continuous self-adjoint semi-group on $\mathscr{K}$ with generator h. The generator of $\{\overline{h(t)L_t}\}$ is $\bar{h} + (-D)$, where $-D$ is the generator of $\{L_t\}$ on $\mathscr{L}^2(\mathscr{K})$ and $\bar{h}$ is defined by $(\bar{h}f)(x) = hf(x)$ for all f in $\mathscr{L}^2(\mathscr{K})$ such that $f(x) \in \mathscr{D}(h)$*

a.e. *and* $(hf)(\cdot) \in \mathscr{L}^2(\mathscr{K})$.

Proof. By Theorem 8 we know that the generator of $\{\overline{h(t)L_t}\}$ is $H + (-D)$, where H is the generator of $\{\overline{h(t)}\}$. We need to show that $\mathscr{D}(H) = \mathscr{D}(\bar{h})$ and if $f \in \mathscr{D}(H)$, then $(Hf)(x) = hf(x)$ a.e.

First let $f \in \mathscr{D}(\bar{h})$. Then $\lim_{t\to 0}(h(t)f(x) - f(x))/t = hf(x)$ a.e. and $hf(\cdot) \in \mathscr{L}^2(\mathscr{K})$. But $\|(h(t)f(x)-f(x))/t\| \leqq \sup_{0\leqq t\leqq 1}\|h(t)\|\,\|hf(x)\|$ [3, p. 88] for all t in [0, 1] and once again the Lebesgue Dominated Convergence Theorem applies. The result is that $(\overline{h(t)}f - f)/t \to \bar{h}f$ in the $\mathscr{L}^2(\mathscr{K})$ norm. Consequently $f \in \mathscr{D}(H)$ and $Hf = \bar{h}f$.

Now let $f \in \mathscr{D}(H)$. By [3, p. 10] $\overline{h(t)}f - f = \int_0^t \overline{h(s)}Hfds$. Consequently, for almost all x, $h(t)f(x) - f(x) = \int_0^t h(s)(Hf)(x)ds$. But since $\{h(s)\}$ is strongly continuous, $\lim_{t\to 0} 1/t\int_0^t h(s)kds = h(0)k = k$ for all k in $\mathscr{K}$. Therefore $\lim_{t\to 0}(h(t)f(x) - f(x))/t = (Hf)(x)$ for almost all x. But then $f(x) \in \mathscr{D}(h)$ a.e. and $hf(x) = (Hf)(x)$. Thus $f \in \mathscr{D}(\bar{h})$ and $\bar{h}f = Hf$, completing the proof.

Using Corollary 9 it is now easy to construct a quasinormal semi-group such that neither the isometric nor the positive factor is uniformly continuous. We let $\{L_t\}$ on $\mathscr{L}^2(\ell^2)$ be the isometric factor. The Hille-Yosida theorem [3, p. 36] guarantees that the unbounded diagonal operator with diagonal $(-1, -2, \cdots, -n, \cdots)$ is the generator of a strongly continuous semi-group $\{h(t)\}$ on $\mathscr{L}^2$. The induced semi-group $\{\overline{h(t)}\}$ on $\mathscr{L}^2(\ell^2)$ is self-adjoint and strongly, but not uniformly, continuous. Thus neither factor of $\{\overline{h(t)L_t}\}$ is uniformly continuous.

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