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# LEFT THICK TO LEFT LUMPY—A GUIDED TOUR

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**We are concerned with locally compact semitopological semigroups, with the variations for such semigroups of the notions of left amenability and left thickness, and with systematizing the many results which generalize a theorem of T. Mitchell for discrete semigroups: A subset $T$ of $S$ is large enough to support a left-invariant mean on $S$ if and only if $T$ is left thick; that is, for each finite subset $F$ of $S$ there is a $v$ in $S$ such that $\{fv \mid f \in F\}$ is a subset of $T$.**

**In Part I: The textures of left thickness, we list many variations of left thickness which have already been used, place them in a pattern of $90 = 5 \times 3 \times 3 \times 2$ such conditions, and show that these fall into not more than six equivalence classes. In Part II: The flavors of left-amenability, we list various kinds of amenability that have already been used and try to match them with appropriate thickness conditions; that is we try to find what thickness a set $T$ in $S$ must have to support a given kind of left amenability, supposing, of course, that $S$ itself supports that much amenability. In this part we are also concerned with the thickness which a subsemigroup $S'$ of $S$ needs in order that some kind of amenability of $S'$ forces the same property on $S$.**

**1. Preliminaries.** T. Mitchell [6] invented the notion of left thickness for a discrete semigroup $S$ in order to characterize in $S$ itself those subsets $T$ of $S$ large enough to support a mean $\mu$ which is left-invariant under $S$. When the same issue is raised for locally compact semitopological semigroups the simplicity of Mitchell's characterization vanishes into a fog of alternative formulations; many of these [3, 4, 8, 9 and 10], have been chosen to suit one or another form of left-invariance that has been found useful by someone at some time.

Always $S$ is a locally compact (Hausdorff) semitopological semigroup; that is, multiplication is associative and is separately or jointly continuous. $M$ is the space of all regular Borel measures on $S$ and $P$ is the subset of $M$ consisting of all probability measures. $P_c$ is the subset of $P$ consisting of measures with compact support. $\delta(S)$ is the subset of $P$ containing all the evaluation functionals $\{\delta_s \mid s \in S\}$. We say that a probability measure $\nu$ is *on* a set $T$ if $\nu(T) = 1$, and that $\nu$ is *supported on* $T$ if the support of $\nu$ is contained in $T$.

We alse need to recall from Wong [8] and B. Johnson [5] that even when multiplication $\pi(s, t) = st$ is only separately continuous in

$S$, nevertheless for each Borel function $f$ in $S$ the composite functions $f \circ \pi$ is also Borel on $S \times S$.

## PART I. THE TEXTURES OF LEFT-THICKNESS

### 2.

DEFINITIONS. Mitchell [6] gave for discrete semigroups $S$ the definition that a subset $T$ of $S$ is called *left thick* if it satisfies

(LT) For each finite subset $F$ of $S$ there is an element $v$ of $S$ such that $Fv$ $(= \{fv \mid f \text{ in } F\})$ is a subset of $T$.

Mitchell noted that $v$ can be chosen in $T$; let $u$ be arbitrary in $S$, apply (LT) to $F_1 = Fu \cup \{u\}$ to get $v$, and then let $t = uv$; then $t \in T \cap uS$.

In a locally compact topological or semitopological semigroup $S$ several generalizations of this have been defined for Borel subsets $T$ of $S$. In increasing order of restriction on $T$, a Borel subset $T$ of $S$ has been called:

(LT) *Left thick*. As above.

(TLL) *Topologically left lumpy* (Day [4]). For each $\varepsilon > 0$ and each $\nu$ in $P_c$ there is $v$ in $S$ such that $[\nu * \delta_v](T) > 1 - \varepsilon$.

(TLT) *Topologically left thick* (Wong [9]). For each $\varepsilon > 0$ and each compact $K \subseteqq S$ there is $\mu$ in $P$ such that for each $\nu$ in $P$ which is supported on $K[\nu * \mu](T) > 1 - \varepsilon$.

(TLS) *Topologically left substantial* (Wong [8]). For each compact $K \subseteqq S$ there is $\mu$ in $P$ such that for each $\nu$ supported on $K$ $[\nu * \mu](T) = 1$.

(LL) *Left lumpy* (Day [4]). For each compact $K$ in $S$ there is $v$ in $S$ such that $Kv \subseteqq T$.

REMARK. We shall consider in later sections some conditions stronger than any of these: (LI) in § 4, B and § 5.4; $[P; j, k, l]$ in § 4, B. Wong [10] defined and used another condition (*) formally between (TLT) and (TLL); this paper began when I observed that (*) is equivalent to (TLL) in all locally compact semitopological semigroups.

These conditions between the extremes (LT) and (LL) are examples of the pattern described below:

$$[\text{For each } \varepsilon > 0][\text{for each } E \text{ in } H_i \text{ uniformly for } \nu \text{ in } E]$$
$$[\text{there is a } \mu \text{ in } Q_j \cap R_k \cap O_l]\ .$$

We list here five reasonable input classes $H_i$ of sets $E$ of probability measures on $S$:

| $i$ | $H_i$ |
|---|---|
| $F$ | $\{E \mid F$ is a finite subset of $S$ and $E = \{\nu \mid \nu(F) = 1\}\}$. |
| $F_c$ | $\{E \mid E = \{\nu\}$, where $\nu$ has finite support$\}$. |
| $P_c$ | $\{E \mid E = \{\nu\}$ where $\nu \in P_c$, that is, $\nu$ has compact support$\}$. |
| $U_\delta$ | $\{E \mid K$ is a compact subset of $S$ and $E = \{\nu \mid \nu = \delta_s$ and $s \in K\}\}$. |
| $U_c$ | $\{E \mid K$ is a compact subset of $S$ and $E = \{\nu \mid \nu$ supported on $K\}\}$. |

The restrictions on $\mu$ alone fall into two kinds, those on the nature of $\mu$,

| $j$ | $Q_j$ |
|---|---|
| $G$(eneral) | $P$ |
| $P_c$ | $P_c$ |
| $\delta$ | $\delta(S)$ |

and those on its relation to $T$,

| $k$ | $R_k$ |
|---|---|
| $G$(eneral) | $P$ |
| $A$(pproximate) | $P \cap \{\mu \mid \mu(T) > 1 - \varepsilon\}$ |
| $E$(xact) | $P \cap \{\mu \mid \mu(T) = 1\}$ |

There are two outcomes relating $\mu$ to $\nu$ and $T$:

| $l$ | $O_l$ |
|---|---|
| $A$(pproximate) | $P \cap \{\mu \mid [\nu * \mu](T) > 1 - \varepsilon\}$ |
| $E$(xact) | $P \cap \{\mu \mid [\nu * \mu](T) = 1\}$. |

We have now 90 formally distinct conditions indexed by $[i; j, k, l]$. It should be noted that in § 4, A we shall reformulate the approximate conditions in terms of limits of nets and in § 4, B will add other conditions not between (LT) and (LL).

The diagram below shows these conditions and notes the position of those defined earlier with the labels used in Wong [10]. In that same paper Wong showed that some of these are equivalent: (TLS), which is $[U_c; G, G, E]$, is equivalent to $[U_c; G, E, E]$. (TLT), which

| | | $j$ / $k$ | $G$ / $G$ | $G$ / $A$ | $G$ / $E$ | $P_c$ / $G$ | $P_c$ / $A$ | $P_c$ / $E$ | $\delta$ / $G$ | $\delta$ / $A$ | $\delta$ / $E$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $i$ | $l$ | | | | | | | | | | |
| $F$ | $A$ | | | | | | | | | | |
| $F_c$ | $A$ | | | | | | | | | | |
| $P_c$ | $A$ | | | | | (TLL$_1$) | (*) | | (TLL) | | |
| $U_\delta$ | $A$ | | | | | | | | | | |
| $U_c$ | $A$ | | (TLT) | | | | (TLT′) | | | | |
| $F$ | $E$ | | | | | | | | | | |
| $F_c$ | $E$ | | | | | | | | | | |
| $P_c$ | $E$ | | | | | | | | | | |
| $U_\delta$ | $E$ | | | | | | | | | | |
| $U_c$ | $E$ | | (TLS) | | | | | (TLS′) | | | |

is $[U_c; G, G, A]$, is equivalent to $[U_c; P_c, A, A]$, which is (TLT′). ($\text{TLL}_1$), which is $[P_c; P_c, G, A]$ is equivalent to $[P_c; \delta, G, A]$, which is (TLL). Wong [10] defined (*) which we shall see in the next section is also equivalent to (TLL).

**3. Equivalence classes of "left-thickness" conditions.** This section shows that there are not more than six equivalence classes of these conditions, even in general semitopological semigroups.

(3.1) For each $j, k, l$
(i) $[U_c; j, k, l] \longleftrightarrow [U_\delta; j, k, l]$,
(ii) $[F; j, k, l] \longleftrightarrow [F_c; j, k, l]$.

*Proof.* $\longrightarrow$. (i) Each $\delta_s$ is in $P_c$. (ii) $\operatorname{supp}\nu = F$, a finite set. $\longleftarrow$. (i) If $\nu$ is on $K$, then

$$[\nu * \mu](T) = \int_S [\delta_v * \mu](T)d\nu(v) = \int_K [\delta_v * \mu](T)d\nu(v)$$
$$= \int_K 1d\nu = 1 \quad \text{if} \quad l = E\,,$$
$$\geqq \int_K (1-\varepsilon)d\nu = 1-\varepsilon \quad \text{if} \quad l = A\,.$$

(ii) If $F$ is finite, let $\varphi = [\sum_{s\in F}\delta_s)/|F|$, and find $\mu$ such that $[\varphi * \mu](T) = 1$ if $l = E$ ($> 1 - \varepsilon/|F|$ if $l = A$). Then $[\delta_s * \mu](T) = 1$ ($> 1 - \varepsilon$) for all $s$ in $F$. Hence $[\nu * \mu](T) = 1$ ($> 1 - \varepsilon$) for all $\nu$ on $F$.

By these elementary calculations our collection of 10 by 9 formally distinct conditions has been reduced to at most 6 by 9 equivalence classes, but much more remains to be done.

(3.2) For each $i, k, l$, $[i; G, k, l) \longleftrightarrow [i; P_c, k, l]$.

*Proof.* $\longleftarrow$. Each $\mu$ in $P_c$ is in $P$.

$\longrightarrow$. If $k$ or $l$ is $A$ use $\varepsilon/3$ instead of $\varepsilon$ in $[i; G, k, l]$ and take $\varphi$ in $P$ such that $[\nu * \varphi]$ and $\varphi$ satisfy $[i; G, k, l]$ for $\varepsilon/3$. Because $\varphi$ is a regular Borel measure, there is a compact set $C$ in the support of $\varphi$ such that $\varphi(C) > 1 - \varepsilon/3$. Let $\mu = \varphi|_C/\varphi(C)$, so $\mu \in P_c$, $\|\varphi - \mu\| < 2\varepsilon/3$, and $\mu$ is absolutely continuous with respect to $\varphi$.

If $l = E$, then for all $\nu$ considered in Case $i$,

$$[\nu * \mu](T) = 1 = \int_S [\nu * \delta_u](T)d\mu(u)\,.$$

If $B = \{u \mid [\nu * \delta_u](T) < 1\}$, then $\varphi(B) = 0$, so $\mu(B) = 0$, and $[\nu * \mu](T) = \int_S [\nu * \mu](T)d\mu(u) = 1$.

If $l = A$, then $[\nu * \mu](T) > [\nu * \varphi](T) - 2\varepsilon/3 > 1 - \varepsilon$.
If $k = G$, nothing new is asked of $\mu$.
If $k = E$, then $\varphi(T) = 1$, so $\varphi(S\backslash T) = 0$, so $\mu(S\backslash T) = 0$ and $\mu(T) = 1$.
If $k = A$, then $\mu(T) > \varphi(T) - 2\varepsilon/3 > 1 - \varepsilon$.

(3.3) For each $l$ the nine conditions $[P_c; j, k, l]$ are equivalent to each other.

It suffices to prove that $[P_c; G, G, l]$ implies $[P_c; \delta, G, l]$ implies $[P_c; \delta, E, l]$; we give here the case $l = A$; the other is similar.

If $\nu$ is in $P_c$ and $\mu$ satisfies $[P_c; G, G, A]$, then

$$1 - \varepsilon < [\nu * \mu](T) = \int_S [\nu * \delta_v](T) d\mu(v) .$$

Then $[\nu * \delta_v](T) > 1 - \varepsilon$ for some $v$ in the support of $\mu$; this is $[P_c; \delta, G, A]$.

If $[P_c; \delta, G, A]$ is true, take any $u$ in $S$ and let $\theta = (\nu * \delta_u + \delta_u)/2$. Then $\theta$ also has compact support (=(support of $\nu)u \cup \{u\}$) and $\varepsilon/2 > 0$ so there is $v$ in $S$ such that $[\theta * \delta_v](T) > 1 - \varepsilon/2$. Let $t = uv$; then

$$[\nu * \delta_t](T) = [\nu * \delta_u * \delta_v](T) > 1 - \varepsilon$$

and

$$\delta_t(T) = [\delta_u * \delta_v](T) > 1 - \varepsilon > 0 ,$$

so $t$ belongs to $T$. (Note that $t$ can be found in each right ideal $uS$ of $S$.)

(3.4) The rows $[P_c; A]$ and $[P_c; E]$ of these conditions are in different equivalence classes.

See Example (3.10).

(3.5) The two (originally four) rows of conditions $[F; j, k, l]$ lie in one equivalence class; all of these eighteen (originally thirty six) conditions are equivalent to left thickness of $T$.

We need only show that $[F; G, G, A]$ implies left thick implies $[F; \delta, E, E]$.

If $F$ is a finite set in $S$ and $\mu$ in $P$ is chosen so that $[\delta_s * \mu](T) > 1 - \varepsilon/2|F|$ for all $s$ in $F$, then for each $s$ in $F$ $\mu(s^{-1}T) = [\delta_s * \mu](T) > 1 - \varepsilon/2|F|$. Also, because $\mu$ is a regular Borel measure, there is a compact $C$ in the support of $\mu$ such that $\mu(C) > 1 - \varepsilon/2$. Then

$$u(C \cap \bigcap_{s \in F} (s^{-1}T)) > 1 - \varepsilon/2 - |F|\varepsilon/2 \quad |F| = 1 - \varepsilon ,$$

so there must be some point $c$ in $C \cap \bigcap_{s \in F} (s^{-1}T)$. Then $sc \in T$ for each $s$ in $F$; that is, $Fc \subseteq T$; that is, $t$ is left thick.

If $F$ is finite and $T$ is left thick, take an arbitrary $u$ in $S$ and let $F_1 = Fu \cup \{u\}$. Then there is a $v$ in $S$ such that

$$Fuv \cup \{uv\} = F_1 v \subseteqq T \ .$$

Letting $t = uv$, we have $Ft \subseteqq T$ and $t \in T$. Hence $[\nu * \delta_t](T) = 1$ for each $\nu$ on $F$ and $\delta_t(T) = 1$; that is, $T$ is $[F; \delta, E, E]$.

Note again that $t$ can be taken in any right ideal $uS$ of $S$.

(3.6) The six (originally twelve) $[U; j, k, E]$ with $j = G$ or $P_c$ (but not $\delta$) are equivalent.

It will suffice to show that $[U_c; G, G, E]$ implies $[U_c; G, E, E]$ implies $[U_c; P_c, E, E]$.

Wong [10] shows the first of these. To repeat that proof, let $K$ be compact in $S$ and let $u$ be any element of $S$. For any $\nu$ on $K$ let $\theta = (\nu * \delta_u + \delta_u)/2$. Then $\theta$ is supported on the compact set $Ku \cup \{u\}$, so there is $\varphi$ in $P$ such that

$$[\nu * \delta_u * \varphi + \delta_u * \varphi](T) = [2\theta * \varphi](T) = 2 \ .$$

Set $\mu = \delta_u * \varphi$ to get

$$[\nu * \mu](T) = 1 = \mu(T) \ ; \quad \text{that is} \quad [U; G, E, E] \ .$$

The second implication is part of (3.2).

(3.7) The six (originally twelve) conditions $[U; j, k, A]$ with $j = G$ or $P_c$ (but not $\delta$) are equivalent.

Here (3.2) shows that $[U_c; G, G, A]$ implies $[U_c; P_c, G, A]$ so we need only show that $[U_c; P_c, G, A]$ implies $[U_c; P_c, A, A]$ implies $[U_c; P_c, E, A]$.

Take $K$ and $\nu$ on $K$ and let $t$ be arbitrary in $S$ and let $\theta = (\nu * \delta_t + \delta_t)/2$. $[U_c; P_c, G, A]$ shows that there is a $\varphi$ in $P_c$ with

$$[(\nu * \delta_t * \varphi + \delta_t * \varphi)/2](T) = [\theta * \varphi](T) > 1 - \varepsilon/6 \ ,$$

so setting $\psi = \delta_t * \varphi$, we have

$$[\nu * \psi](T) = [\nu * \delta_t * \varphi](T) > 1 - \varepsilon/3$$

and

$$\psi(T) = [\delta_t * \varphi](T) > 1 - \varepsilon/3 \ .$$

This $\psi$ satisfies $[U_c; P_c, A, A]$.

$\psi$ is regular so there is a compact $C \subseteqq T$ such that $\psi(C) > 1 - \varepsilon/3$. Let $\mu = \psi|_C/\psi(C)$. Then $\mu$ is absolutely continuous with respect to $\psi$ and $\|\mu - \psi\| < 2\varepsilon/3$. Hence $[\nu * \mu](T) > 1 - \varepsilon$, and $\mu(T) = \mu(C) = 1$. This $\mu$ satisfies $[U_c; P_c, E, A]$ so (3.7) is proved.

(3.8) The six (originally twelve) conditions $[U; \delta, k, l]$ are all equivalent to each other and to left lumpiness of $T$.

We need only prove $[U_\delta; \delta, G, A]$ implies left lumpy implies $[U_\delta; \delta, E, E]$. Take $\varepsilon < 1$ and compact $K \subseteqq S$ and $u$ in $K$; by hypothesis there is $s$ in $S$ with $\delta_{us}(T) = [\delta_u * \delta_s](T) > 1 - \varepsilon > 0$ for all $u$ in $K$, so $us \in T$ for all $u$ in $K$; that is, $Ks \subseteqq T$ and $T$ is left lumpy.

Next take a compact $K$ in $S$ and take $v$ in $S$; let $K_1 = Kv \cup \{v\}$ by (LL) there is a $w$ in $S$ such that $Kvw \cup \{vw\} = K_1 w \subseteqq T$. Let $t = vw$ to get $Kt \subseteqq T$ and $t \in T$. Then for each $u$ in $K$ $[\delta_u * \delta_t](T) = \delta_{ut}(T) = 1$ and $\delta_t(T) = 1$; that is $[U_\delta; \delta, E, E]$.

(3.9) Example. In $R$, $+$ the set $I$ of all irrational numbers is not left lumpy, but is topologically left substantial, so has all the other left thickness properties.

If $K$ is any closed interval of positive length, no translation of $K$ is contained in $I$ so $I$ is not left lumpy. If $\mu$ is Lebesgue measure on $[0, 1]$, and $\mu$ is 0 elsewhere, then for every $\nu$ in $P$,

$$[\nu * \mu](T) = \int_S \mu(Tu^{-1})\,d\nu(u) = \int_S 1 d\nu = 1\ .$$

(3.10) Example. To separate $l = E$ from $l = A$ wherever possible. Let $T = \bigcup_n [n + 1/n, n + 1]$. Then $T$ is not topologically left substantial (that is, $[U; P_c, E, E]$) nor is it $[P_c; P_c, E, E]$, but $T$ is topologically left thick and topologically left lumpy.

Take $\nu$ to be Lebsgue measure on $[0, 1]$; then $[\nu * \delta_u](T) < 1$ for every $u$ in $R$. Hence $[\nu * \mu](T) < 1$ for every $\mu$ in $P$.

However, if $\varepsilon > 0$ is given let $n$ be greater than $1/\varepsilon$ and let $\varphi =$ (Lebesgue measure on $[0, n])/n$. Then for every compact $K$ choose a $u$ so large that $K + u +$ (support of $\varphi$) is all beyond $n$, and let $\mu = \delta_u * \varphi$. Then for all $v$ in $K$ $[\delta_v * \mu](T) > 1 - \varepsilon$, so $[\nu * \mu](T) > 1 - \varepsilon$ for all $\nu$ on $K$; that is, $T$ is (TLT), and therefore, (TLL).

(3.11) Example. A left thick set in $R$, $+$ which is not topologically left lumpy.

The construction of $T$ is by induction. For $n = 1$ let $t_1 = 1$. Then there are $b_1 = \binom{2}{1}^1 = 2$ choices of an interval of the form $[(i - 1)/2, i/2]$ in $[0, 1]$. Copy these patterns in the halves of $[0, 2]$ to get $A_1$ and let $T_1 = t_1 + A_1 = \{t_1 + a \mid a \in A_1\}$. Let $t_n = 1 +$ largest integer in $T_{n-1}$. Then there are $b_n = \binom{2^n}{2^{n-1}}^n$ arrangements of subintervals $[(i - 1)/2^n, i/2^n]$ in the interval $[0, n]$ in such a way that for each $j$ with $1 \leqq j \leqq n$ exactly half of the interval $[j - 1, j]$ is covered.

Line up one copy of each such arrangement to get $A_n$ in the interval $[0, nb_n]$, and let $T_n = t_n + A_n$.

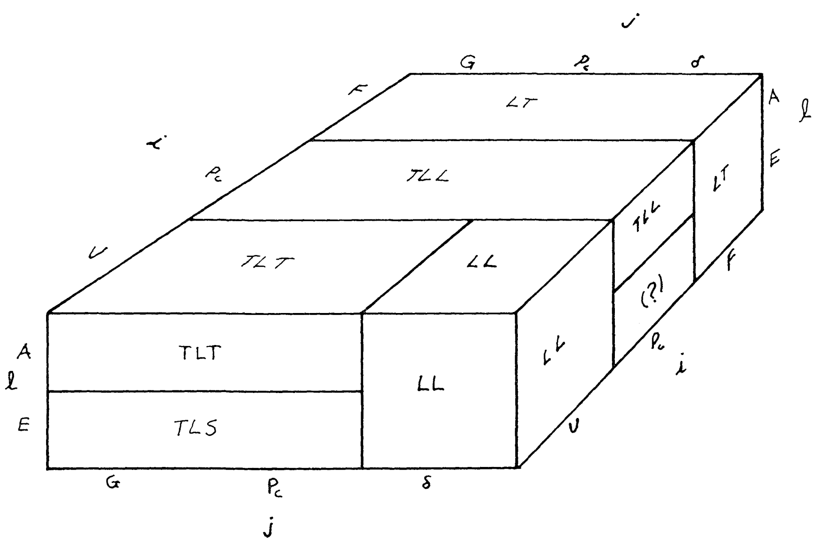
Define $T = \bigcup_n T_n$.

Then $T$ is left thick. Take a finite set $F$ in $R$, and move it until its smallest element is 0. Let $n$ be so large that $F \subseteqq [0, n]$ and $1/2^{n-1}$ is less than the smallest distance between distinct elements of $F$. Then $F$ can be covered by a union $E$ of distinct intervals $[(i-1)/2^n, i/2^n]$ and, since $|F \cap [j-1, j]| < 2^{n-1}$ for each $j$ with $1 \leqq j \leqq n$, $E$ is a subset of one of the arrangements of intervals used in defining $A_n$. Hence a translate of $F$ is contained in $T_n$.

$T$ is not topologically left lumpy. Let $\nu$ be the uniform probability density on some interval $[0, k]$ where $k \geqq 2$. Then $[\nu * \delta_s](T) < 1/2 + 1/2k < 3/4$ for all $s$ in $R$. This says that $T$ does not satisfy $[P_c; \delta, E, A]$; that is, $T$ is not (TLL).

Note that a more complicated version of this example is used in (4.7) to discuss the relation between thickness and invariant means.

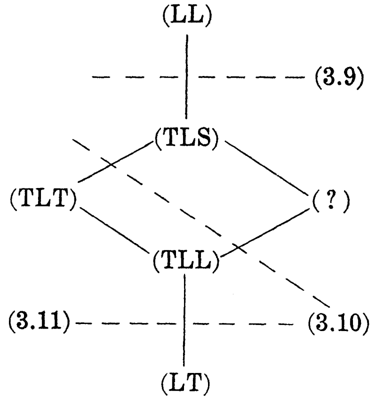
These results show that there are not more than six equivalence classes. Changing $k$ alone never moves a condition out of its class, so we can draw a block for $i$, $j$, $l$ and get a diagram in which up, left, or back gives weaker conditions.




The lattice diagram for these equivalence is: (see next page.)

Example (3.9) says that even in $R$, $+$ (LL) is stronger than the others. Example (3.11) says that (LT) is weaker than all the rest (even in $R$, $+$). Example (3.10) separates $E$ from $A$ conditions when (i) $i$ is $P_c$ and (ii) when $j$ is not $\delta$ and $i$ is $U$.

We have as yet no example to tell (TLS) from (?) or (TLT) from




(TLL). Frequently they are equivalent, as can be seen from Theorem 5.6 and the remark following the theorem.

## 4. Variations on these conditions.

A. *The "approximate" conditions in terms of nets.* The conditions (TLT) $= [U; P_c, E, A]$ and (TLL) $= [P_c; \delta, E, A]$ can be characterized simply in terms of convergence of nets of elements of $P$; that is, of functions defined from directed systems to $P$. A natural directed system for the $U$ conditions is the set of all $(K, \varepsilon)$ with $K$ a compact subset of $S$ and $\varepsilon$ a positive number; $(K, \varepsilon) > (K', \varepsilon')$ is defined to mean that $K \supseteqq K'$ and $\varepsilon \leqq \varepsilon'$. Then $[U_c; P_c, E, A]$ asserts that for each $K$ and $\varepsilon$ there is a $\mu = \mu(K, \varepsilon)$ such that $1 \geqq [\nu * \mu](T) > 1 - \varepsilon$ for all $\nu$ in $P_c$ supported on $K$. Therefore (TLT) implies (and is easily seen to be implied by) the following condition.

($\mathrm{TLT}_\lambda$) There is a net $(\mu_n)$ when $n$ runs over some directed system $\Delta$, such that for each compact $K \subseteqq S$, $\lim_{n \in \Delta} [\nu * \mu_n](T) = 1$ uniformly for $\nu$ in $P$ supported on $K$.

A similar reformulation of $[U_\delta; P_c, E, A]$ gives another equivalent condition.

($\mathrm{TLT}_{\lambda\delta}$) There exists a net $(\mu_n)$, such that for each compact $K \subseteqq S$, $[\delta_k * \mu_n](T) \to 1$ uniformly for $k$ in $K$.

In a similar way, using for $\Delta$ the set of finite subsets $\Phi = \{\nu_1, \cdots, \nu_p\}$ of $P_c$ and applying $[P_c; \delta, E, A]$ to $(\nu_1 + \cdots + \nu_p)/p$, gives a characterization of (TLL) in terms of convergence.

($\mathrm{TLL}_\lambda$) There is a net $(t_n) \subseteqq T$, where $n$ runs over a directed system $\Delta$ such that, $\lim_{n \in \Delta} [\nu * \delta_{t_n}](T) = 1$ for each $\nu$ in $P_c$.

This will be applied in (6.8) to show that in some kinds of semigroups (TLT) = (TLL).

(F) conditions are also expressible in terms of convergence; a convenient form is:

($F_\lambda$) There exists a net $(t_n) \subseteqq T$ such that for each $s$ in $S$ $\lim_{n \in \Delta} [\delta_s * \delta_{t_n}](T) \to 1$, that is, ultimately $st_n \in T$.

B. *Some conditions stronger than left-lumpiness.*

DEFINITIONS. For each $j$, $k$, and $l$ as in § 2 say that a Borel subset $T$ of $S$ satisfies $[P; j, k, l]$ when for each $\nu$ in $P$ there is $\mu$ in $Q_j \cap R_k \cap O_l$. Say that $T$ satisfies (LI) if $T$ contains left ideal of $S$.

Clearly each $[P; j, k, l]$ implies the corresponding $[P_c; j, k, l]$; the converse holds if $S$ is compact. (Wong [10] uses the label (LI) to mean that $T$ *is* a left ideal.)

(4.1) For each $j$ and $k$, $[P_c; j, k, A]$ is equivalent to $[P; j, k, A]$.
$\longleftarrow$ is obvious because $P_c \subset P$.
$\longrightarrow$ For $\nu$ in $P$ and $\varepsilon > 0$, take a compact set $K$ of the support of $\nu$ such that $\nu(S\backslash K) < \varepsilon/4$. Let $\varphi = \nu|_K/\nu(K)$, so that $\varphi \in P_c$, $\varphi$ is absolutely continuous with respect to $\nu$, and $\|\nu - \varphi\| < \varepsilon/2$. Take $\mu$ so that $[\varphi * \mu](T) > 1 - \varepsilon/2$; then $[\nu * \mu](T) > 1 - \varepsilon$.

(4.2) EXAMPLE. The set $I$ of irrationals in $R$, $+$ is an example of a set which satisfies $[P; P_c, E, E]$ but is not left-lumpy. $I \cap E$ is (TLS) if $E$ is any left-lumpy subset of $R$, $+$; if, in particular, $E = \bigcup_{n \in w} [2^{2n}, 2^{2n+1}]$, then $I \cap E$ is (TLS) and $[P_c; P_c, E, E]$ but is not $[P; P_c, E, E]$. Hence $[P; P_c, E, E]$ is none of (LL), (TLS), or $[P_c; P_c, E, E]$. $[P; P_c, E, E]$ is, of course, stronger than (TLT) in $R$, $+$ and is equivalent to (TLT) in compact semigroups.

(4.3) LEMMA. *If $T$ is* (LI), *then $T$ is $[P; \delta, E, E]$ and therefore is $[P; j, k, l]$ for all $j$, $k$, and $l$.*

*Proof.* Take $u$ in the left ideal $T'$ which is a subset of $T$ and let $\nu$ be any element of $P$. Then $[\nu * \delta_n](T) = \nu(Tu^{-1}) \geqq \nu(T'u^{-1}) = \nu(S) = 1$.

(4.4) Example, continued. If $I =$ irrationals in $R^+$, $+$ and if $J$ is an ideal, $t + R^+$, then $T = I \cap J$ is $[P; P_c, E, E]$ but is not (LI) or even (LL).

C. *Locally compact groups.* If $G$ is a locally compact group with left Haar measure $H$, there is an unnamed condition of "left-thickness" used in Day [3], Theorem 7.8:

($LL_A$) For each compact $K \subseteqq G$ and each $\varepsilon > 0$ there is $s$ in $G$ such that $H(K \cap Ts) \geqq (1 - \varepsilon)H(K)$.

*Note.* J. C. S. Wong [8] noted that this condition was misstated in the original article. $K\backslash Ts$ was printed instead of the correct form $K \cap Ts$, all through Theorem 7.8 and its proof.

(4.5) LEMMA. *$T$ satisfies* ($LL_A$) *if and only if $T$ satisfies*

($TLL_a$) *For each $\varepsilon > 0$ and each $\varphi$ in $P_c$ which is absolutely continuous with respect to Haar mearure, there is $s$ in $G$ such that* $[\varphi * \delta_s](T) > 1 - \varepsilon$.

REMARK. If $M_a$ is the set of all elements of $M$ which are absolutely contiuous with respect to $H$, then ($TLL_a$) is just (TLL) with the choice of $\nu$ restricted to $M_a \cap P_c$.

*Proof.* By absolute continuity of $\varphi$ there is $\eta = \eta(\varepsilon) > 0$ such that if $H(A) < \eta H(K)$, then $\varphi(A) < \varepsilon$. Take $K =$ support of $\varphi$ and apply ($LL_A$) to $K$ and $\eta$ to find an $s^{-1}$ for which $H(K \cap Ts^{-1}) > (1 - \eta)H(K)$, so that $H(K\backslash Ts^{-1}) < \eta H(K)$. But then $\varphi(K\backslash Ts^{-1}) < \varepsilon$, so $[\varphi * \delta_s](T) = \varphi(Ts^{-1}) = \varphi(K \cap Ts^{-1}) > 1 - \varepsilon$.

For the converse we need only consider $K$ with $H(K) > 0$. Let $\varphi = H|_K/H(K)$; then $\varphi$ is absolutely continuous with respect to $H$ and $\varphi(K \cap Ts)/H(K) = [\varphi * \delta_{s^{-1}}](T)$. By ($TLL_a$) this can be made $>1 - \varepsilon$ by proper choice of $s$; the same $s$ then fits ($LL_A$).

(4.6) COROLLARY. *If $G$ is a locally compact group, then for each Borel subset $T$ of $G$ the following conditions are equivalent*: (TLL), ($LL_A$), ($TLL_a$), (TLT).

(4.5) shows that ($TLL_a$) is equivalent to ($LL_A$). The other equivalences are in (5.8).

(4.7) EXAMPLE. Let $G = O_n \times R$, where $O_n$ is the orthogonal group in $n$ variables. We are interested in the cases $n \geqq 3$ so that $O_n$ treated as a discrete group has free subgroups and, therefore, is not amenable. For each $k$ choose $2k$ closed subsets of $O_n$ of equal measure $1/2k$ in such a way that $O_n$ is the union of the sets and that the intersection of any two of the sets is of measure zero with respect to $h$, the Haar measure in $O_n$. Then there are $C(2k, k)$ sets $E_{kl}$ where $E_{kl}$ is a union of $k$ chosen from these $2k$ pieces of $O_n$. Enumerate the sets $(E_{kl})$, $k = 1, 2, \cdots$, $l = 1, 2, \cdots, C(2k, k)$ in a sequence $B_k$, and let $C_k = B_k \times T_k$, where $T_k$ is the subset of $R$ defined in Example (3.11). Then $C$, the union of the $C_k$, is (as in (3.11)) (LT) but not (TLL).

If $(\mu_n)$ is supported on $T$ and if $(\mu_n)$ is ($LS\pi$) (see (6.3)), let

$\mu_n = c_n + d_n$, where $d_n$ is a discrete and $c_n$ is a continuous measure. Then $\|\delta_s * \mu_n - \mu_n\| = \|\delta_s * d_n - d_n\| + \|\delta_s * c_n - c_n\|$. If there were a subnet $(d_{n_i})$ for which $\|d_{n_i}\| \geqq \eta > 0$, then $\psi_{n_i} = d_{n_i}/\|d_{n_i}\|$ would be (LS$\pi$). But that would say that $G$ was left-amenable as a discrete group. Since $O_n$ is not, $G$ is not, so $\|d_n\| \to 0$.

5. **Special sets or semigroups.** We know from (3.9) and (3.10) that even in very good semigroups (such as $R$, $+$), LL $\neq$ TLS $\neq$ TLT $\neq$ LT.

We begin by showing an effect of closure on these conditions.

(5.1) THEOREM. *If $T$ is a topologically substantial Borel set in a locally compact semitopological semigroup, then the closure of $T$ is left-lumpy.*

*Proof.* By $[U_\delta; P_c, E, E]$ we have for each compact $K$ in $S$ a $\mu$ in $P_c$ such that for all $k$ in $K$, $\mu(k^{-1}T) = [\delta_k * \mu](T) = 1$ and $\mu(T) = 1$. But $\mu$ is supported on some compact $C$ in $S$, so $\mu(C) = 1 = \mu(C \cap k^{-1}T)$. Now $\mathrm{cl}(k^{-1}T)$ is a closed set with $1 \geqq \mu(\mathrm{cl}(k^{-1}T)) \geqq \mu(k^{-1}T) = 1$, so $\mathrm{cl}(k^{-1}T) \supseteqq C$ for each $k$ in $K$. Hence $kC \subseteqq k\,\mathrm{cl}(k^{-1}T) \subseteqq kk^{-1}\,\mathrm{cl}(T) \subseteqq \mathrm{cl}(T)$ for each $k$ in $K$; that is, $KC \subseteqq \mathrm{cl}(T)$. Choose $c$ in $C$; then $Kc \subseteqq \mathrm{cl}(T)$, and $\mathrm{cl}(T)$ is left-lumpy.

(5.2) COROLLARY. *For each locally compact semitopological semigroup $S$ and each closed subset $T$ of $S$, $T$ is topologically left substantial if and only if $T$ is left-lumpy.*

*Recall that the Examples* (3.10) *and* (3.11) *are of closed sets $T$, so the distinction between* (TLS) *and* (TLT) *and well as that between* (TLL) *and* (?) *is maintained even for closed $T$ in metric abelian grougs.*

In (6.4) closure of $T$ will again be useful, this time for showing that (LSU) for $T$ can sometimes be extended to all of $S$.

(5.3) EXAMPLE. Pointing out the difference between (5.1) and (5.2). Consider the semigroup $(R, \cdot)$. In this semigroup $T$ is (LL) or (LT) or anything in between if and only if $O \in T$. Hence the set $O^{-1}T$ is empty and $O^{-1}\,\mathrm{cl}(T) = S$ precisely in case $O \in \mathrm{cl}(T)\backslash T$, so the intersection of all $k\,\mathrm{cl}(k^{-1}T)$ may be much smaller than $\mathrm{cl}(T)$.

(5.4) THEOREM. *If $T$ is a left-thick subset of a compact semitopological semigroup, then the closure of $T$ is* (LI), *that is,* $\mathrm{cl}(T)$ *contains some left ideal of $S$.*

*Proof.* Let $\Delta$ be the directed system of all finite subsets of $S$

ordered by inclusion. If $T$ is left thick, there is for each finite subset $F$ an element $t_F$ of $T$ such that $Ft_F \subseteqq T$. Use compactness to choose a subset $(F_n)$ of $\Delta$ such that $t_{F_n}$ converges to some $t$ in $T$. For each $s$ in $S$, there is $n_s$ such that $s$ is in $F_n$ if $n > n_s$. For such an $n$, $st_{F_n}$ is in $T$, so $\lim_n st_{F_n} = st$ is in $\mathrm{cl}(T)$. This says that for all $s$ in $S$, $st$ is in $\mathrm{cl}(T)$; that is, $\mathrm{cl}(T)$ contains the principal left ideal $St$.

(5.5) COROLLARY. *If $T$ is left thick subset of a compact topological group, then $T$ is dense in $G$.*

*Proof.* $G$ is the only left ideal in $G$.

We turn next to a condition *on* $S$ which makes some of these conditions equivalent.

(5.6) THEOREM. *Assume that in $P_c$ there is a $\varphi$ such that the function $s \to \delta_s * \varphi$ is continuous from $S$ (a locally compact semi-topological semigroup) into $M$ (with its norm topology). Then for each Borel set $T$ in $S$,* (TLL) *is equivalent to* (TLT).

*Proof.* $\longleftarrow$ is known for all $S$ so we prove $\longrightarrow$.

Take a function $\varphi$ satisfying the hypothesis of this theorem. Take $\varepsilon > 0$ and $K$ compact. Then for each $k$ in $K$ there is an open set $E_k$ such that $\|\delta_k * \varphi - \delta_s * \varphi\| < \varepsilon/2$ if $s$ in $E_k$.

Use ($\mathrm{TLL}_c$) from § 4 to choose a net $(t_n)$ in $T$. Take a finite covering $E_{k_1}, \cdots, E_{k_p}$ of $K$. Then each $\delta_k * \varphi$ has compact support $k \operatorname{supp} \varphi$, so by ($\mathrm{TLL}_\lambda$) there is $m_\varepsilon$ in $\Delta$ such that for all $i \leqq p$, $[\delta_{k_i} * \varphi * \delta_{t_m}](T) > 1 - \varepsilon/2$ if $m > m_\varepsilon$. Hence for all $s$ in $E_{k_i}$ and $m > m_\varepsilon$,

$$[\delta_s * \varphi * \delta_{t_m}](T) > 1 - \varepsilon/2 - \|\delta_s * \varphi - \delta_{k_i} * \varphi\| > 1 - \varepsilon .$$

Let $\mu_m = \varphi * \delta_{t_m}$. Then $(\mu_m)$ is a net of elements of $P_c$ such that $[\delta_s * \mu_m](T) \to 1$ uniformly for $s$ in $K$. This is ($\mathrm{TLT}_{\lambda\delta}$) which, by § 4, A, is equivalent to (TLT).

REMARK. The need to go from $(\delta_{t_m})$ to $(\varphi * \delta_{t_m})$ in this proof again emphasizes the division of the family $[U; j, k, A]$ by the cases $j = \delta$ or $j \neq \delta$.

(5.7) COROLLARY. *If $S$ is a locally compact group or a discrete semigroup, then* (TLL) $=$ (TLT).

In a locally compact group $\varphi$ can be the indefinite integral with respect to Haar measure of a continuous nonnegative function with compact support and with integral over $S$ equal to 1.

REMARK. Wong [10] and Day [4] show that if $S$ supports a net $(\mu_n)$ satisfying (LSU) of § 6, and if $S$ is a locally compact semitopological semigroup, then (TLL) = (TLT). It would be interesting to know whether this very strong form of left amenability of such a semigroup implies the existence of a $\varphi$ with the continuity properties of this theorem or if instead, the amenability condition and its proof are really as different from this result as the know proofs suggest.

If $G$ is a locally compact group, we know that (TLL) and (TLT) are equivalent and that we have another condition ($\mathrm{LL_A}$) from § 4 which, according to Day [3], Theorem 7.8, characterizes those Haar measurable subsets of $G$ which can support an invariant mean on $L_\infty(G)$.

(5.8) THEOREM. *The following conditions on a Borel subset $T$ of a locally compact group are equivalent*:

$$(\mathrm{LL_A}),\ (\mathrm{TLL_a}) = (\mathrm{TLL})|_{M_a},\ (\mathrm{TLL}),\ (\mathrm{TLT})\ .$$

*Proof.* The equivalence of the first two conditions is (4.3) and that of the last two is (5.7). ($\mathrm{TLL_a}$) is a restriction on the $\nu$ used in (TLL), so (TLL) implies ($\mathrm{TLL_a}$).

If ($\mathrm{TLL_a}$) holds and $\nu \in P_c$, take $\varphi$ in $P_c \cap M_a$. $M_a$ is an ideal in $M$, so $\nu * \varphi$ is in $P_c \cap M_a$. By ($\mathrm{TLL_a}$) for each $\varepsilon > 0$ there is $s$ in $G$ such that $[(\nu * \varphi) * \delta_s](T) > 1 - \varepsilon$. Let $\varphi * \delta_s = \mu$ to get $[\nu * \mu](T) > 1 - \varepsilon$; that is, $[P_c; P_c, G, A]$, which is equivalent to (TLL).

## PART II. THE FLAVORS OF LEFT-AMENABILITY

**6. Reduction to strong left-amenability.** In a locally compact semi topological semigroup $S$ left-amenability can be characterized in terms of nets of elements in $P$ or in $P_c$, just as in the original paper (Day, [1]) where strong amenability was defined for discrete semigroups. The point of this shift of attention from elements of $P^{**}$ to net in $P$ is to keep all calculations down in $M$ and $S$, rather than up in $P^{**}$, when properties of $T$ are to be compared. (6.4) to (6.6) state the properties we need later.

If in a locally compact semitopological semigroup $S$ we choose $C_0$, the space of continuous functions vanishing at infinity, as our basic function space, then $M$, the space of bounded Borel measures, is like $C_0^*$, and with $M^*$ and $M^{**}$ this tower of four spaces is the generalization of the tower $c_0$, $l_1$, $m$, and $m^*$ which is used in discrete semigroups, and $P^{**}$, the positive face of the unit sphere in $M^{**}$, is the natural set in which to seek invariant means (see Wong, [8, 9]).

(6.1) DEFINITION. Say that an element $\Gamma$ of $P^{**}$ is

$P^{**}$-left-invariant if $\Xi * \Gamma = \Gamma$ for each $\Xi$ in $P^{**}$,
$P$-left-invariant if $Q\nu * \Gamma = \Gamma$ for each $\nu$ in $P$,
$P_c$-left-invariant if $Q\nu * \Gamma = \Gamma$ for each $\nu$ in $P_c$, and
$S$-left-invariant if $Q\delta_s * \Gamma = \Gamma$ for each $s$ in $S$.

If such a $\Gamma$ exists, $S$ or $M^*$ is called $P^{**}-$ or $P-$ or $P_c-$ or $S$-left-amenable, respectively.

$M^{**}$ is a Banach algebra if the Arens product (Arens 1951) is defined from the product in $M$. Because this product is $w^*$-continuous in its first variable, and because $QP$ and $QP_c$ are $w^*$-dense in $P^{**}$, we have

(6.2) *$\Gamma$ in $P^{**}$ is $P^{**}$-left-invariant if and only if $\Gamma$ is P-left-invariant and if and only if $\Gamma$ is $P_c$-left-invariant, but S-left-invariance need not imply the others.*

Retracing a proof from the discrete case, $w^*$-density of $QP_c$ shows that a $\Gamma$ in $P^{**}$ is $P-$ or $P_c-$ (or $S-$) left-invariant if and only if there is a net $(\mu_n)$ of elements of $P_c$ such that $Q\mu_n \to \Gamma$ and $Q\nu * Q\mu_n \to Q\nu * \Gamma = \Gamma$ in the $w^*$-topology of $P^{**}$, so $w^*$-$\lim_n Q(\nu * \mu_n - \mu_n) = 0$ if $\nu$ is in $P$ (or if $\nu$ is in $P_c$ or in $\delta(S)$). Then $\nu * \mu_n - \mu_n \to 0$ weakly in $P$ and (as in Day [4]) $(\mu_n)$ can be replaced by a net of averages $(\varphi_m)$ far out in $(\mu_n)$ so that $\|\nu * \varphi_m - \varphi_m\| \to$ in $P$ if $\nu$ is in $P$ (or if $\nu$ is in $P_c$ or in $\delta(S)$).

Conversely a net $(\varphi_m)$ in $P_c$ with this strong property has a subnet $(\mu_n)$ for which $(Q\mu_n)$ is $w^*$-convergent to some $\Gamma$ in $P^{**}$. It is easily seen that the norm convergence of $(\mu_n)$ to $P-$ (or $P_c-$ or $S-$) left-invariance forces $P-$ (or $P_c-$ or $S-$) left-invariance of $\Gamma$.

Let us list and label these properties along with a stronger property used by Reiter [7] in one form.

(6.3) DEFINITION. A net $(\mu_n)$ in $P$, or in $P_c$, may have one of the following flavors of strong convergence to left-invariance:

(LS$\pi$) (*Strong convergence to S-left-invariance*). For each $s$ in $S$, $\|\delta_s * \mu_n - \mu_n\| \to 0$.

(LSP) (*Strong convergence to P-left-invariance*). For each $\nu$ in $P$, or in $P_c$, $\|\delta_s * \mu_n - \mu_n\| \to 0$.

(LSU$_\delta$) (*Uniform-on-compacts strong convergence to S-left-invariance*) For each compact set $K$ in $S$, $\|\delta_s * \mu_n - \mu_n\| \to 0$ uniformly over $s$ in $K$, or else.

($LSU_c$) (*Uniform-on-compacts strong convergence to $P_c$-left-invariance*). For each compact set $K$ in $S$, $\|\nu * \mu_n - \mu_n\| \to 0$ uniformly for $\nu$ supported on $K$.

REMARK. The last two conditions are easily seen to be equivalent so we shall usually use (LSU) to refer to whichever is most convenient at the moment.

(6.4) THEOREM. *An element $\Gamma$ of $P^{**}$ is $P-$ (or $S-$) left-invariant if and only if there is a net $(\mu_n)$ of elements of $P$ or of $P_c$ such that $w^*$-$\lim_n Q\mu_n = \Gamma$ and $(\mu_n)$ is* (LSP) (*is* (LS$\pi$)).

In a *discrete* group or semigroup, all these conditions on a net $(\mu_n)$ are equivalent, but in a locally compact semigroup or group the first is weaker than the others. However, (see Day [2]), in a group $G$ if $\varphi$ in $M$ is defined by $\varphi(E) = \int_E f dH$, where $H$ is Haar measure on $G$ and $f$ is continuous, bounded nonnegative, with compact support, and $\int_S f dH = 1$, then for each net $(\mu_n)$ which satisfies (LS$\pi$) the net $(\varphi * \mu_n)$ is equicontinuous and satisfies (LSU) and so has more left-invariance than $(\mu_n)$ was assumed to have. In general locally compact semitopological semigroups, it is clear that for any net $(\mu_n)$, (LSU) implies (LSP) implies (LS$\pi$), but exact conditions under which existence of a net $(\mu_n)$ with one of the weaker properties implies existence of a net $(\varphi_n)$ with a stronger property are not known.

Recall that left thickness was designed to locate sets which could carry left-invariant means. We make the definitions necessary for this case. If $T$ is a Borel subset of $S$, $\chi_T$ is the characteristic function of $T$; $\chi_T(s) = 1$ if $s \in T$, $= 0$ if not. Let $q$ be the natural map of $BB$, the bounded Borel functions on $S$, into $M^*$ defined by: For all $\mu$ in $M$, $[qx](\mu) = \int_S x d\mu$. Let $\xi_T = q\chi_T$. Then

(6.5) $\Gamma$ in $P^{**}$ is supported on $T$ if and only if each net $(\mu_n)$ in $P$ which has $w^*$-$\lim Q\mu_n = \Gamma$ also has $\lim_n \mu_n(T) = 1$.

REMARK. By chopping the edges off such a net $(\mu_n)$; that is, replacing $(\mu_n)$ by $(\varphi_n) = \mu_n|_{C_n}/\mu_n(C_n)$, where $C_n$ is a compact subset of $T$ for which $\mu_n(C_n) \to 1$, it is possible to have $\|\mu_n - \varphi_n\| \to 0$ and $\varphi_n$ supported on $T$. The converse is trivial, so:

(6.6) A element $\Gamma$ of $P^{**}$ is supported on $T$ if and only if there is a net $(\mu_n)$ supported on $T$ such that $w^*$-$\lim_n Q\mu_n = \Gamma$.

**7. Left-thickness and strong left-amenability.** We quote Mitchell's theorems relating left thickness to the support of invariant means in discrete semigroups (Mitchell [6]).

THEOREM A. *If $S$ is a discrete left-amenable semigroup and if $T$ is a subset of $S$, then $T$ is left thick if and only if there is a left-invariant mean on $S$ which is supported on $T$.*

THEOREM B. *If $T$ is a left-thick subsemigroup of $S$, then $S$ is left amenable if and only if $T$ is left-amenable.*

Rephrasing Mitchell's theorems to our purposes merely replaces left-invariant *elements* of $m^*$ by strongly left invariant *nets* of elements of $l_1$ as in § 6.

In a semitopological semigroup we have six textures of left thickness and three flavors of strong left amenability which are not known to be equivalent. Some of these have been investigated by Wong [9, 10] and Day [4], but let us look at the full pattern and see what is known or can be shown here. Splitting "if" from "only if", we have four patterns to investigate; here $i$ and $k$ run over (LS$\pi$), (LSP), and (LSU) while $j$ runs over the six kinds of left thickness:

*For subsets*:

(A1) If $S$ has an $i$-net, and if the Borel subset $T$ of $S$ is $j$-thick, then $T$ supports a $k$-net (for $S$).

(A2) If $S$ has an $i$-net, and if $T$ supports a $k$-net (for $S$), then $T$ is $j$-thick.

*For subsemigroups*:

(B1) If $T$ is a $j$-thick Borel subsemigroup of $S$ and if $S$ has an $i$-net, then $T$ has a $k$-net (for $T$).

(B2) If $T$ is a $j$-thick Borel subsemigroup of $S$ and if $T$ has a $k$-net for $T$, then $S$ has an $i$-net for $S$.

In (A1), (A2), (B1), and (B2) we wish to know $k$, $j$, $k$, and $i$, respectively, in terms of the inputs.

Recall (Day [1, 2]) what can safely be done to a net $(\mu_n)$ to move it about without spoiling whatever left-invariance it may have.

(7.1) LEMMA. (a) *$(\mu_n)$ can be multiplied on the right by an element $\varphi$ of $P$ or by an net $(\varphi_m)$ of elements of $P$.*

(b) *It is safe to multiply all $\mu_n$ on the left by a single $\delta_u$ and for* (LSP) *by any $\psi$ in $P_c$.*

*Proof.* (a) If $\theta_{nm} = \mu_n * \varphi_m$ and if $(n, m) \geqq (n', m')$ means that

$n \geqq n'$ and $m \geqq m'$, then $\|\nu * \theta_{nm} - \theta_{nm}\| = \|(\nu * \mu_n - \mu_n) * \varphi_m\| \leqq \|\nu * \mu_n - \mu_n\|$ so $(\theta_{nm})$ tends strongly to any kind of left-invariance that $(\mu_n)$ had.

(b) If $\|\delta_s * \mu_n - \mu_n\| \to 0$ for all $s$ in $S$, then $\|\delta_s * \delta_u * \mu_n - \delta_u * \mu_n\| \leqq \|\delta_s * \delta_u * \mu_n - \mu_n\| + \|\mu_n - \delta_u * \mu_n\| \to 0$ also. In a similar way, if $\|\delta_s * \mu_n - \mu_n\| \to 0$ uniformly for all $s$ in each compact $K$ and if $u \in S$, then $Ku$ is compact because multiplication by $u$ is continuous in $S$ and $\delta_{su} = \delta_s * \delta_u$ has its support in $Ku$ as $s$ runs over $K$, so $\|\delta_s * \delta_u * \mu_n - \delta_u * \mu_n\|$ tends to zero uniformly for $S$ in $K$.

If $\|\varphi * \mu_n - \mu_n\| \to 0$ for all $\varphi$ in $P$, then for $\psi$ in $P$ let $\theta_n = \psi * \mu_n$. Then $\|\varphi * \theta_n - \theta_n\| = \|\varphi * \psi * \mu_n - \psi * \mu_n\| \leqq \|\varphi * \psi * \mu_n - \mu_n\| + \|\mu_n - \psi * \mu_n\| \to 0$.

DEFINITION. The locally compact semigroup $S$ is called (KK) if $\mathrm{cl}(K_1K_2)$, the closure of $K_1K_2$, is compact when the $K_i$ are compact subsets of $S$.

REMARK. $S$ is (KK) if (a) $S$ is discrete, or (b) $S$ is compact, or (c) multiplication in $S$ is jointly continuous. Of course, (a) is a special case of (c). The semigroup (under composition of operators) of all operators of norm $\leqq 1$ in Hilbert space, using the weak operator topology, is the standard example of (b) but not (c). $\boldsymbol{R}$ or any noncompact topological group is an example of (c) without (b).

(7.2) LEMMA. ( i ) *If* $(\mu_n)$ *is* (LS$\pi$), *so is* $(\delta_u * \mu_n * \varphi_m)$ *for each* $u$ *in* $S$ *and each net* $(\varphi_m)$ *in* $P$.

( ii ) *If* $(\mu_n)$ *is* (LSP), *so is* $(\psi * \mu_n * \varphi_m)$ *for each* $\psi$ *in* $P$ *and each net* $(\varphi_m)$ *in* $P$.

(iii) *If* $(\mu_n)$ *is* (LSU$_\delta$), *so is* $(\delta_u * \mu_n * \varphi_m)$ *for each* $u$ *in* $S$ *and each net* $(\varphi_m)$ *in* $P$.

(iv) *If* $S$ *is a* (KK) *semigroup and if* $(\mu_n)$ *is* (LSU$_c$) *then so is* $\psi * \mu_n * \varphi_m$ *for each* $\psi$ *in* $P_c$ *and each* $(\varphi_m)$ *in* $P$.

The proofs are applications of (7.1).

This can be applied to Pattern (A1). Clearly $T$ cannot support a net with more left-invariance than $S$ can support.

(7.3) LEMMA. *If* $T$ *is a* (TLL) *Borel subset of* $S$, *then* $T$ *supports a net with the same kind of strong left invariance as any net which* $S$ *supports.*

*Proof.* Take a net $(\mu_n)$ with some strong left-invariance, $\pi$, $P$, or $U$. By (4.1), $T$ has $[P; \delta, E, A]$, so for each index $n$ and each positive integer $k$ there is an element $t_{nk}$ of $T$ such that $[\mu_n * \delta_{t_{nk}}](T) > 1 - 1/k$. If $\theta_{nk} = \mu_n * \delta_{t_{nk}}$, then $\|\nu * \theta_{nk} - \theta_{nk}\| \leqq \|\nu * \mu_n * \mu_n\|$ so $(\theta_{nk})$ converges to the same strong left-invariance as $(\mu_n)$. But $\theta_{nk}(T) \to 1$ so we can replace $\theta_{nk}$ by $\varphi_{nk} = \theta_{nk}|_{C_{nk}}/\theta(C_{nk})$, where $\theta_{nk}(C_{nk}) \to 1$ and $C_{nk}$ is a compact subset of $T$. Then $(\theta_{nk})$ is supported on $T$ and $\|\nu * \varphi_{nk} - \varphi_{nk}\| \leqq \|\nu * \theta_{nk} - \theta_{nk}\| + \|\nu * (\varphi_{nk} - \theta_{nk})\| + \|\theta_{nk} - \varphi_{nk}\| \leqq \|\nu * \theta_{nk} - \theta_{nk}\| + 2\|\theta_{nk} - \varphi_{nk}\|$. The last term goes to zero; the first does too, uniformly on $K$ if $(\mu_n)$ did.

REMARK. It is probable that there are semigroups in which an (LT) subset of $S$ need not support an (LS$\pi$) net for $S$, but example (3.11) is not such a semigroup. $R, +$ is abelian and therefore is amenable when regarded as a discrete group. If $(\varphi_n)$ is a net of finite means existing because $R, +$ is amenable-as-discrete, there is a net $(t_n)$ determined by the left thickness of $T$ such that each $\varphi_n * t_n$ has it finite support in $T$, and this net is also (LS$\pi$).

For pattern (A2) we also have some direct results.

(7.4) LEMMA. *If $T$ is a Borel subset of $S$ which supports a $k$-net $(\mu_n)$ for $S$, then $T$ is $j$-thick where $k$ and $j$ are related by the following table:*

| | | | |
|---|---|---|---|
| $k$ | (LS$\pi$) | (LSP) | (LSU) |
| $j$ | (LT) | (TLL) | (TLT) . |

*Proof.* We have assumed that $\mu_n(T) = 1$ for each $n$.

(LS$\pi$) $\to$ (LT). For each finite set $F$ there is an $n$ such that $\mu_n(s^{-1}T) = [\delta_s * \mu_n](T) > 1 - \|\delta_s * \mu_n - \mu_n\| > 1 - 1/|F|$ for all $s$ in $F$. Then $\mu_n \bigcap_{s \in F} s^{-1}(T) > 1 - |F|/|F| = 0$, so there is a $v$ in $\bigcap_{s \in F} s^{-1}(T)$. Then $sv \in T$ for each $s$ in $F$, that is, $Fv \subseteqq T$. This is (LT).

(LSP) $\to$ (TLL). For each $\nu$ in $P_c$, $[\nu * \mu_n](T) > 1 - \|\nu * \mu_n - \mu_n\|$. $\to 1$ by condition (LSP). This is condition (TLL$_\lambda$) which was shown in § 4 to be equivalent to (TLL).

(LSU) $\to$ (TLT). $[\delta_s * \mu_n](T) > 1 - \|\delta_s * \mu_n - \mu_n\|$. For $K$ compact (LSU) asserts that the norm tends to zero uniformly in $K$, so $[\delta_s * \mu_n](T) \to 1$ uniformly in $K$; this is (TLT$_{\lambda\delta}$) of § 4, a condition equivalent to (TLT).

This gives another proof of a theorem of Day [4] and Wong [10].

(7.5) THEOREM. *If $S$ carries an* (LSU) *net $(\mu_n)$, that is, if $(\mu_n)$ is strongly convergent to left-invariance uniformly on compact subsets of $S$, then for each Borel subset $T$ of $S$, $T$ is* (TLT) *if and only if $T$ is* (TLL).

*Proof.* Always (TLT) implies (TLL). If $T$ is (TLL), (7.3) asserts that $T$ supports a net $(\varphi_n)$ which is $(\mathrm{LSU}_\delta)$. Then (7.4) asserts that $T$ is (TLT).

For subsemigroups more confusions can arise; a Borel subsemigroup of $S$ need not be locally compact, but since a Borel subset $E$ of a Borel subset $T$ of $S$ is a Borel subset of $S$, some of the difficulties are postponed. When it is necessary for $T$ to be locally compact we shall have to assume that $T$ is closed or open.

Again (TLL) is the most useful property for $T$ to have. Consider Pattern B1.

(7.6) LEMMA. *Let $T$ be a Borel subsemigroup of $S$ and assume that $T$ is* (TLL) *in $S$. Then $T$ supports a net with the same flavor of strong left invariance (either relative to $S$ or relative to $T$) that $S$ supports.*

*Proof.* It is an immediate consequence of (7.3) that if $(\mu_n)$ is a net in $P_c = P_c(S)$ which is (LSi), then a net $(t_n)$ in $T$ exists with $(\mu_n * \delta_{t_n})$ in $T$ and with strong left-invariance that $(\mu_n)$ had. Restricting the allowable $\nu$ to be supported on $T$ gives a net with the same strong left-invariance relative to $T$, even though $T$ may not be a locally compact semigroup.

For Pattern B2 something is also known.

(7.7) THEOREM. *If $T$ is a Borel subsemigroup of the locally compact semitopological semigroup $S$, and if $T$ supports a net $(\mu_n)$ which is* (LSk) *for $T$ and if $T$ satisfies a matching left thickness condition from the list below, then $S$ (and also $T$) support a net $(\varphi_n)$ with the same flavor,* (LSi) = LSk), *of left invariance for all of $S$.*

| $(\mu_n)$ | $T$ |
|---|---|
| $(\mathrm{LS}\pi)$ | (LT) |
| (LSP) | (TLL) |
| (LSU) | (a) (LL), *or* |
| | (B) (TLT) *and $T$ is closed and $S$ is* (KK) . |

*Proof.* It has been assumed that always $\mu_n(T) = 1$.

$(\mathrm{LS}\pi)$ Suppose that $\|\delta_t * \mu_n - \mu_n\| \to 0$ for all $t$ in $T$. Choose $u$ in $S$. Then for each $s$ in $S$ there is $v$ in $S$ such that $\{suv, uv\} = \{su, u\}v \subseteq T$; let $t = uv$. Then

$$\begin{aligned}\|\delta_s * \mu_n - \mu_n\| &\leqq \|\delta_s * \mu_n - \delta_s * \delta_t * \mu_n\| + \|\delta_s * \delta_t * \mu_n - \mu_n\| \\ &\leqq \|\mu_n - \delta_t * \mu_n\| + \|\delta_{st} * \mu_n - \mu_n\| .\end{aligned}$$

Both terms go to 0 because $t$ and $st$ are in $T$. Hence $(\mu_n)$ is (LS$\pi$) for $S$.

(LSP) Suppose that $\|\theta * \mu_n - \mu_n\| \to 0$ for all $\theta$ in $P_c(T)$. Take $\varepsilon > 0$ and $\nu$ in $P_c(S)$. Because $T$ is (TLL) there is $t$ in $T$ such that $[\nu * \delta_t](T) > 1 - \varepsilon/2$ so there is compact $C \subseteq T$ for which $[\nu * \delta_t](C) > 1 - \varepsilon/2$. Let $\varphi = [\nu * \delta_t]|_C/[\nu * \delta_t](C)$, so that $\|\varphi - \nu * \delta_t\| < \varepsilon$. Then

$$\begin{aligned}
&\|\nu * \mu_n - \mu_n\| \\
&\quad \leqq \|\nu * \mu_n - \nu * \delta_t * \mu_n\| + \|\nu * \delta_t * \mu_n - \varphi * \mu_n\| + \|\varphi * \mu_n - \mu_n\| \\
&\quad \leqq \|\mu_n - \delta_t * \mu_n\| + \varepsilon + \|\varphi * \mu_n - \mu_n\| \,.
\end{aligned}$$

The first and third terms go to zero with increasing $n$ because $\delta_t$ and $\varphi$ are in $P_c(T)$. Hence $\|\nu * \mu_n - \mu_n\| < 3\varepsilon$ if $n$ is sufficiently large, that is, $(\mu_n)$ is (LSP) for $S$ as well as for $T$.

(LSU) (a) (Wong [10]). If $T$ is left-lumpy, let $K$ be a compact subset of $S$ and let $t$ be an element of $T$ such that $K' = Kt \subseteq T$. Then

$$\begin{aligned}
\|\delta_s * \mu_n - \mu_n\| &\leqq \|\delta_s * \mu_n - \delta_s * \delta_t * \mu_n\| + \|\delta_{st} * \mu_n - \mu_n\| \\
&\leqq \|\mu_n - \delta_t * \mu_n\| + \|\delta_{st} * \mu_n - \mu_n\| \,.
\end{aligned}$$

The first term goes to zero since $\{t\}$ is a compact subset of $T$, and the second goes to zero uniformly for $s$ in $K$ because $K'$ is a compact subset of $T$, hence $(\mu_n)$ satisfies (LSU) for $S$.

(LSU) (b) If $T$ is a closed subsemigroup of $S$, then $T$ is also a locally compact semigroup. Take compact $K$ in $S$ and $\varepsilon > 0$. Because $T$ is (TLT) there is $\theta$ in $P_c(T)$ such that $[\delta_s * \theta](T) > 1 - \varepsilon/2$ for all $s$ in $K$. Let $\varphi_s = [\delta_s * \theta]|_T/[\delta_s * \theta](T)$ so $\|\delta_s * \theta - \varphi_s\| < \varepsilon$ and $\varphi_s$ is in $P_c(T)$. Since $K_1 =$ support of $\varphi$ is compact, $\mathrm{cl}\,(KK_1)$ is also compact so each $\varphi_s$ has its support in $sK_1 \subseteq K' = \mathrm{cl}\,(KK_1) \cap T$.

If $(\mu_n)$ is a net with $\|\varphi * \mu_n - \mu_n\| \to 0$ uniformly for $\varphi$ supported on a compact subset of $T$, we have for each $s$ in $K$ that

$$\begin{aligned}
&\|\delta_s * \mu_n - \mu_n\| \\
&\quad \leqq \|\delta_s * \mu_n - \delta_s * \theta * \mu_n\| + \|\delta_s * \theta * \mu_n - \varphi_s * \mu_n\| + \|\varphi_s * \mu_n - \mu_n\| \\
&\quad \leqq \|\mu_n - \theta * \mu_n\| + \varepsilon + \|\varphi_s * \mu_n - \mu_n\| \,.
\end{aligned}$$

Hence for $n$ large $\|\delta_s * \mu_n - \mu_n\| < 3\varepsilon$ uniformly for $s$ in $K$, because all the $\varphi_s$ are supported on $K' \subseteq T$. This says that $(\mu_n)$ is (LSU$_\delta$) for $S$.

(7.8) COROLLARY. *If $T$ is a* (TLT) *closed subsemigroup of a* (KK) *locally compact semitopological semigroup $S$, then $T$ supports an* (LSU) *net for $T$ if and only if $S$ supports an* (LSU) *net for $S$.*

*Proof.* The last part of (7.3) shows that (LSU) for $S$ implies

(LSU) for $T$. Part (iv) of (7.7) shows that if $(\mu_n)$ in $T$ is (LSU) for $T$, then $(\mu_n)$ is also (LSU) for $S$.

REMARK. Recall that $S$ is (KK) under any one of three common conditions: $S$ compact, or $S$ discrete, or multiplication jointly continuous in $S$. Hence (7.8) is stronger than Theorem 4.2 of Wong [10], which requires joint continuity of multiplication.

## 8. Conclusions and confusions.

It has been shown that the many generalizations of left thickness are really rather few in number and that the (TLL) family is the right one for the widest variety of purposes.

Examples are still needed to show when (TLL) is not (TLT) and when (?) is not (TLS).

A new set of problems has been opened up in the course of § 7. Some of the results deal with Borel subsemigroups of locally compact semitopological semigroups. These cannot be expected to be locally compact, but they share some of the properties of their containing groups in regards to thickness and invariant means. More of this should be learned.

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