PACIFIC JOURNAL OF MATHEMATICS
Vol. 101, No. 1, 1982

TAUBERIAN THEOREMS BETWEEN THE LOGARITHMIC AND ABEL-TYPE SUMMABILITY METHODS

DAVID BORWEIN AND BRUCE WATSON

The object of this paper is to show that if a series is summable by the logarithmic method L, then the series is also summable by the Abel method A_λ, provided a tauberian condition of the "slowly decreasing" type is satisfied.

1. Introduction. Suppose throughout that $\{s_n\}$ is a sequence of numbers, λ real is real, $\varepsilon_0^\lambda = 1$, $\varepsilon_n^\lambda = \binom{n+\lambda}{n}$ for $n = 1, 2, 3, \cdots$, and

$$v_n^\lambda = \frac{\varepsilon_n^\lambda \Gamma(\lambda+1)}{(n+1)^\lambda} \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

We are concerned with the methods of summability A_λ introduced and studied by Borwein [1] and the logarithmic method L. They are defined as follows. Let

$$(1) \qquad \sigma_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{y}{1+y}\right)^n , \quad \text{and}$$

$$(2) \qquad L(y) = \frac{1}{\log(1+y)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} \left(\frac{y}{1+y}\right)^{n+1} .$$

If $\sigma_\lambda(y)$ converges for $y > 0$ and tends to s as $y \to \infty$, then we say that the sequence $\{s_n\}$ is A_λ-convergent to s and write $s_n \to s(A_\lambda)$. The method A_0 is the ordinary Abel method.

If $L(y)$ converges for $y > 0$ and tends to s as $y \to \infty$, then we say that $\{s_n\}$ is L-convergent to s and write $s_n \to s(L)$.

Evidently, $s_n \to s(L)$ if and only if

$$-\frac{1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1}$$

converges for $0 < x < 1$ and tends to s as $x \to 1^-$.

LEMMA 1. *A_λ is regular for $\lambda > -1$. [That is, $s_n \to s$ implies $s_n \to s(A_\lambda)$].*

LEMMA 2. *L is regular.*

LEMMA 3. *$A_{\lambda+\varepsilon} \subset A_\lambda$ for $\lambda > -1$, and $\varepsilon > 0$. [That is, $s_n \to s(A_{\lambda+\varepsilon})$ implies $s_n \to s(A_\lambda)$ and there exists a sequence $\{s_n\}$, depending on λ and ε, such that $\{s_n\}$ is A_λ-convergent but not $A_{\lambda+\varepsilon}$-convergent.]*

LEMMA 4. *$A_\lambda \subset L$ for $\lambda > -1$.*

Lemmas 1 and 3 were established by Borwein in [1]. Lemma 4 was proved by Borwein in [2] as a particular case of a more general inclusion theorem on methods of summability based on power series. Lemma 2 is a standard result found, for example, in [4].

2. The main theorem. Suppose that Φ is a nonnegative, continuous, strictly increasing function on $[a, \infty)$, for some a, such that $\Phi(t) \to \infty$ as $t \to \infty$.

The real-valued function f is said to be *slowly decreasing with respect to* Φ if $\liminf \{f(y) - f(x)\} \geqq 0$ whenever $y \geqq x \to \infty$ and $\Phi(y) - \Phi(x) \to 0$.

THEOREM 1. *For $\lambda > -1$, if $s_n \to s(L)$ and $\sigma_\lambda(t)$ is slowly decreasing with respect to* $\log\log t$, *then $s_n \to s(A_\lambda)$.*

In connection with the methods A_λ, we proved the following lemma in [3].

LEMMA 5. *For $\lambda > -1$ and $\varepsilon > 0$, if $s_n \to s(A_\lambda)$ and $\sigma_{\lambda+\varepsilon}(t)$ is slowly decreasing with respect to* $\log t$, *then $s_n \to s(A_{\lambda+\varepsilon})$.*

3. Methods of summability based on power series. Suppose that $p_n \geqq 0$, $q_n \geqq 0$, $\sum_{v=n}^{\infty} p_v > 0$, and $\sum_{v=n}^{\infty} q_v > 0$ for $n = 0, 1, 2, \cdots$. Set

$$p(x) = \sum_{n=0}^{\infty} p_n x^n , \quad \text{and}$$

$$q(x) = \sum_{n=0}^{\infty} q_n x^n .$$

Let ρ_p and ρ_q denote their respective radii of convergence. We also write

$$p_s(x) = \frac{1}{p(x)} \sum_{n=0}^{\infty} p_n s_n x^n$$

$$q_s(x) = \frac{1}{q(x)} \sum_{n=0}^{\infty} q_n s_n x^n .$$

The power series method P is defined as follows. If $\rho_p > 0$, $\sum_{n=0}^{\infty} p_n s_n x^n$ converges for $0 < x < \rho_p$ and $\lim_{x \to \rho_p -} p_s(x) = s$, then we write $s_n \to s(P)$.

The method Q is defined similarly.

Borwein has proved [2] the following lemma.

LEMMA 6. (i) *If* $0 < \rho_p < \infty$, *then a necessary and sufficient condition for* P *to be regular is that* $\sum_{n=0}^{\infty} p_n(\rho_p)^n = \infty$.

(ii) *If* $\rho_p = \infty$ *then* P *is regular.*

Suppose that $\chi(t)$ is a function of bounded variation on [0, 1], and $\chi^*(t)$ is its associated normalized function. That is,

$$\chi^*(t) = \begin{cases} 0 & t = 0 \\ \dfrac{1}{2}\{\chi(t+) + \chi(t-)\} - \chi(0) & 0 < t < 1 \\ \chi(1) - \chi(0) & t = 1 . \end{cases}$$

A sequence $\{\mu_n\}$ is called an *m-sequence* if, for some χ,

$$\mu_n = \int_0^1 t^n d\chi(t) \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

If, in addition,

$$\mu_n \geqq \delta\int_0^1 t^n |d\chi^*(t)| \quad \text{for} \quad 0 < \delta \leqq 1 \quad \text{and}$$

$n = N, N + 1, \cdots$, then $\{\mu_n\}$ is called an *$\bar{m}$-sequence.*

LEMMA 7. *If* $p_n = \mu_n q_n (n = N, N + 1, \cdots)$, $\{\mu_n\}$ *is an $\bar{m}$-sequence,* $\rho_p = \rho_q > 0$, *and* P *is regular, then* $Q \subseteqq P$. *(That is,* $s_n \to s(Q)$ *implies* $s_n \to s(P)$.)

This result is due to Borwein (see [2], Theorem A′).

We require the following two lemmas.

LEMMA 8. *An m-sequence which converges to a positive limit is an $\bar{m}$-sequence.*

LEMMA 9. *The sequences* $\{v_n^\lambda\}$ *and* $\{1/v_n^\lambda\}$ *are $\bar{m}$-sequences for* $\lambda > -1$.

The proof of Lemma 8 is straightforward and Lemma 9 was established in [4], Theorem 211.

The next result is used in the proof of Theorem 1.

THEOREM 2. *Let* Q *be a regular power series method and suppose that* $\{\mu_n\}$ *is an $\bar{m}$-sequence such that* $\mu_n \to a > 0$. *Then* $\mu_n s_n \to as(Q)$

whenever $s_n \to s(Q)$.

Proof. Suppose that $s_n \to s(Q)$. Set $p_n = \mu_n q_n$ for $n = 0, 1, 2, \cdots$. Since $\mu_n \geqq 0$ and $\mu_n \to a$ it is easy to verify that $\rho_p = \rho_q$. If $\rho_p = \infty$, then P is regular by Lemma 6(ii). Otherwise, since $p_n \sim aq_n$, P is regular by Lemma 6(i).

Therefore, by Lemma 7, $s_n \to s(P)$. That is,

$$\frac{1}{p(x)}\sum_{n=0}^{\infty} s_n \mu_n q_n x^n \longrightarrow s \quad \text{as} \quad x \longrightarrow \rho_P^- . \tag{3}$$

In addition, since Q is regular,

$$\frac{p(x)}{q(x)} = \frac{1}{q(x)}\sum_{n=0}^{\infty} \mu_n q_n x^n \longrightarrow a \quad \text{as} \quad x \longrightarrow \rho_q^- . \tag{4}$$

Application of Q to $\{\mu_n s_n\}$ *yields*

$$\begin{aligned}\frac{1}{q(x)}&\sum_{n=0}^{\infty} \mu_n s_n q_n x^n \\ &= \frac{p(x)}{q(x)}\,\frac{1}{p(x)}\sum_{n=0}^{\infty} s_n \mu_n q_n x^n \\ &\longrightarrow as \quad \text{as} \quad x \longrightarrow \rho_q^- = \rho_p^- \text{ by (3) and (4)} .\end{aligned}$$

This completes the proof.

COROLLARY TO THEOREM 2. $s_n \to s(L)$ *if and only if* $v_n^\lambda s_n \to s(L)$.

This is immediate in view of Lemmas 8 and 9, and the fact that $v_n^\lambda \to 1$ as $n \to \infty$.

4. An integral transformation. The integral transformation $J_\lambda(w)$ of the function $f(t)$, for $\lambda > -1$ and $w > 0$, is defined as follows.

$$J_\lambda(w) = \frac{1}{\log(1+w)}\int_0^w (1+t)^{\lambda-1}\left(\log\frac{w(1+t)}{t(1+w)}\right)^\lambda f(t)dt . \tag{5}$$

THEOREM 3. *If* $\lambda > -1$ *and* $f(t) = \sigma_\lambda(t)$ *is convergent for all* $t > 0$, *then* $J_\lambda(w) \to s$ *as* $w \to \infty$ *if and only if* $s_n \to s(L)$.

Proof. Setting $u = (t(1+w))/(w(1+t))$ in $J_\lambda(w)$ gives

$$\begin{aligned}J_\lambda(w)&\\ &= \frac{1}{\log(1+w)}\int_0^w (1+t)^{\lambda-1}\left(\log\frac{w(1+t)}{t(1+w)}\right)^\lambda (1+t)^{-\lambda-1}\sum_{n=0}^{\infty}\varepsilon_n^\lambda s_n\left(\frac{t}{1+t}\right)^n dt\end{aligned}$$

$$= \frac{1}{\log(1+w)} \int_0^1 \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{w}{1+w}\right)^{n+1} u^n \left(\log \frac{1}{u}\right)^\lambda du$$
$$= \frac{1}{\log(1+w)} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{w}{1+w}\right)^{n+1} \int_0^1 u^n \left(\log \frac{1}{u}\right)^\lambda du$$
$$= \frac{\Gamma(\lambda+1)}{\log(1+w)} \sum_{n=0}^{\infty} \frac{\varepsilon_n^\lambda}{(n+1)^{\lambda+1}} s_n \left(\frac{w}{1+w}\right)^{n+1}$$
$$= \frac{1}{\log(1+w)} \sum_{n=0}^{\infty} \frac{v_n^\lambda s_n}{n+1} \left(\frac{w}{1+w}\right)^{n+1} .$$

The convergence, for $t > 0$, of the series defining $\sigma_\lambda(t)$ implies its absolute convergence. This justifies the integration term by term and, in view of the corollary to Theorem 2, the proof is complete.

5. Additional lemmas.

LEMMA 10. *For* $\lambda > -1$, $\sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n x^n$ *is absolutely convergent for* $|x| < 1$ *if and only if* $\sum_{n=0}^{\infty} (s_n/(n+1))x^n$ *is absolutely convergent for* $|x| < 1$.

We omit the simple proof.

LEMMA 11. *For* $0 < t < w$,

$$\log \frac{w(1+t)}{t(1+w)} > \frac{w-t}{w(1+t)} .$$

Proof. For $x > 1$,

$$\log x = \log x - \log 1 = \frac{x-1}{\theta} > \frac{x-1}{x}$$

where $1 < \theta < x$. The result follows by observing that, for $0 < t < w$, $x = (w(1+t))/(t(1+w)) > 1$.

LEMMA 12. *For fixed* $\gamma > 1$ *and* $\lambda > -1$,

$$I(x) = \int_0^x (1+t)^{\lambda-1} \left(\left(\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^\lambda - \left(\log \frac{x(1+t)}{t(1+x)}\right)^\lambda \right) dt$$
$$= O(1) .$$

Proof. Suppose $\lambda \geqq 1$. Then, for $x \geqq 1$,

$$|I(x)| = I(x)$$
$$\leqq \lambda \log \frac{x^\gamma(1+x)}{x(1+x^\gamma)} \int_0^x (1+t)^{\lambda-1} \left(\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^{\lambda-1} dt$$

$$\leqq \lambda \log \frac{x^\gamma(1+x)}{x(1+x^\gamma)}\left(\int_0^1 + \int_1^x\right)(1+t)^{\lambda-1}\left(\log\frac{1+t}{t}\right)^{\lambda-1}dt$$
$$= I_1(x) + I_2(x) .$$

Now,

$$\int_0^1 (1+t)^{\lambda-1}\left(\log\frac{1+t}{t}\right)^{\lambda-1}dt < \infty .$$

Hence,

$$I_1(x) = O(1) .$$

Also,

$$I_2(x) = O(1)\log\frac{x^\gamma(1+x)}{x(1+x^\gamma)}\int_1^x dt$$
$$= O(1)x\log\frac{1+x}{x} = O(1) .$$

Suppose $0 < \lambda < 1$. By Lemma 11 we have,

$$|I(x)| = I(x)$$
$$\leqq \lambda \log\frac{x^\gamma(1+x)}{x(1+x^\gamma)}\int_0^x (1+t)^{\lambda-1}\left(\log\frac{x(1+t)}{t(1+x)}\right)^{\lambda-1}dt$$
$$< \lambda\frac{M}{x}\int_0^x (1+t)^{\lambda-1}\left(\frac{x-t}{x(1+t)}\right)^{\lambda-1}dt$$

since $x\log(x^\gamma(1+x))/(x(1+x^\gamma)) \leqq M$.

Therefore

$$I(x) \leqq \lambda\frac{M}{x^\lambda}\int_0^x (x-t)^{\lambda-1}dt = M .$$

Suppose $-1 < \lambda < 0$. Then

$$|I(x)| = -I(x)$$
$$= \left(\int_0^{x/2} + \int_{x/2}^x\right)(1+t)^{\lambda-1}\left(\left(\log\frac{x(1+t)}{t(1+x)}\right)^\lambda - \left(\log\frac{x^\gamma(1+t)}{t(1+x^\lambda)}\right)^\lambda\right)dt$$
$$= I_1(x) + I_2(x) .$$

Using Lemma 11 and the fact that

$$\left|x\log\frac{x(1+x^\gamma)}{(1+x)x^\gamma}\right| \leqq M$$

we have

$$
\begin{aligned}
0 \leqq I_1(x) &\leqq \lambda\Big(\log \frac{x(1+x^\gamma)}{x^\gamma(1+x)}\Big)\int_0^{x/2}(1+t)^{\lambda-1}\Big(\log\frac{x(1+t)}{t(1+x)}\Big)^{\lambda-1}dt \\
&\leqq -\frac{\lambda M}{x}\int_0^{x/2}(1+t)^{\lambda-1}\Big(\frac{x-t}{x(1+t)}\Big)^{\lambda-1}dt \\
&= M((1/2)^\lambda - 1)\ .
\end{aligned}
$$

For $I_2(x)$, since $1 + t > x/2$,

$$
\begin{aligned}
0 \leqq I_2(x) &\leqq \int_{x/2}^{x}(1+t)^{\lambda-1}\Big(\log\frac{x(1+t)}{t(1+x)}\Big)^{\lambda}dt \\
&\leqq \int_{x/2}^{x}(1+t)^{\lambda-1}\Big(\frac{x-t}{x(1+t)}\Big)^{\lambda}dt \\
&= \frac{1}{x^\lambda}\int_{x/2}^{x}(x-t)^\lambda\frac{dt}{1+t} \\
&\leqq \frac{2}{x^{\lambda+1}}\int_{x/2}^{x}(x-t)^\lambda dt \\
&= \frac{1}{(\lambda+1)2^\lambda}\ .
\end{aligned}
$$

Hence, $I(x) = O(1)$ in this case.

Finally, since the case $\lambda = 0$ is trivial, the lemma is established.

LEMMA 13. *For* $\gamma > 1$, *and* $\lambda > -1$,

$$
\begin{aligned}
&\int_x^{x^\lambda}(1+t)^{\lambda-1}\Big(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\Big)^{\lambda}dt \\
&\qquad = (\gamma-1)\log(1+x) + o(\log(1+x))\ .
\end{aligned}
$$

Proof. Set $\{s_n\} = \{1\}$. Then $\sigma_\lambda(t) = 1$ and, by Theorem 3, putting $f(t) = \sigma_\lambda(t)$ in (5) gives

$$
J_\lambda(x) = 1 + o(1) \quad \text{as} \quad x \longrightarrow \infty\ .
$$

Now by Lemma 12,

$$
\begin{aligned}
&\int_x^{x^\lambda}(1+t)^{\lambda-1}\Big(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\Big)^{\lambda}dt \\
&\quad = \Big(\int_0^{x^\lambda} - \int_0^x\Big)(1+t)^{\lambda-1}\Big(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\Big)^{\lambda}dt \\
&\quad = \log(1+x^\gamma) + o(\log(1+x^\gamma)) - \log(1+x) + o(\log(1+x)) \\
&\qquad + o(1) \\
&\quad = (\gamma-1)\log(1+x) + o(\log(1+x))\ .
\end{aligned}
$$

This establishes the lemma.

6. A general tauberian result.

THEOREM 4. *Suppose that the following conditions hold:*

(6) *$K(w, t)$ is defined, real-valued, and nonnegative for $w > 0, t \geqq 0$; moreover, $\int_0^\infty K(w, t)dt$ exists in the sense of Lebesgue for each $w > 0$,*

(7) $$\int_0^\infty K(w, t)dt \longrightarrow 1 \quad as \quad w \longrightarrow \infty ,$$

(8) *f is real-valued and continuous on $(0, \infty)$,*

(9) *$F(w) = \int_0^\infty K(w, t)f(t)dt$ exists in the Cauchy-Lebesgue sense for each $w > 0$,*

(10) *$\liminf \{f(y) - f(x)\} \geqq -\mu$ for some fixed finite nonnegative μ, whenever $y \geqq x \to \infty$ and $\Phi(y) - \Phi(x) \to 0$,*

(11) $$\Phi(x) - \Phi(x-1) \longrightarrow 0 \quad as \quad x \longrightarrow \infty ,$$

(12) $$\int_0^x K(w, t)dt \longrightarrow 0 \quad whenever \quad w > x \longrightarrow \infty \quad and \quad \Phi(w) - \Phi(x) \longrightarrow \infty ,$$

(13) $$\int_x^\infty K(w, t)(\Phi(t) - \Phi(x))dt \longrightarrow 0 \quad whenever \quad x > w \longrightarrow \infty \quad and \quad \Phi(x) - \Phi(w) \longrightarrow \infty , \quad and$$

(14) $$F(w) = O(1) \quad for \quad w > 0 .$$

Then $f(t) = O(1)$ for $t > 0$.

This result was established in [5]. A version of this theorem with (10) replaced by the stronger condition that f be slowly decreasing with respect to Φ can be found in [3]. The proofs are very similar.

7. A theorem on boundedness. In this section we deduce a weakened form of Theorem 1 from the general tauberian result of § 6.

THEOREM 5. *If $\lambda > -1$, $\infty > \mu \geqq 0$, $s_n \to s(L)$, and $\liminf \{\sigma_\lambda(y) - \sigma_\lambda(x)\} \geqq -\mu$ whenever $y \geqq x \to \infty$ and $\Phi(y) - \Phi(x) \to 0$, then $\sigma_\lambda(t) = O(1)$.*

Proof. Set

$$K(w, t) = \begin{cases} \dfrac{1}{\log(1+w)}(1+t)^{\lambda-1}\left(\log\dfrac{w(1+t)}{t(1+w)}\right)^{\lambda} & o < t < w \\ 0 & \text{otherwise}\,, \end{cases}$$

$$\Phi(t) = \begin{cases} t/e^e & 0 \leqq t < e^e \\ \log\log t & e^e \leqq t\,, \end{cases}$$

and

$$f(t) = \sigma_\lambda(t)\,.$$

First, note that if $\{s_n\} = \{1\}$, then $s_n \to 1(L)$ and $\sigma_\lambda(t) = 1$. Hence, by Theorem 3 with $f(t) = \sigma_\lambda(t) = 1$ in (5), we have

$$\begin{aligned}\int_0^\infty & K(w, t)dt \\ &= \frac{1}{\log(1+w)}\int_0^w (1+t)^{\lambda-1}\left(\log\frac{w(1+t)}{t(1+w)}\right)^{\lambda} dt \\ &= J_\lambda(w) \longrightarrow 1 \quad \text{as} \quad w \longrightarrow \infty\,.\end{aligned}$$

This establishes (6) and (7).

Conditions (8), (9), (10) and (14) hold by hypotheses, and (11) clearly holds.

Furthermore, condition (13) is immediate since $K(w, t) = 0$ whenever $t \geqq w$. It remains to show (12). Suppose $-1 < \lambda < 0$. Then, by Lemma 11, we have

$$\begin{aligned}\int_0^x & K(w, t)dt \\ &= \frac{1}{\log(1+w)}\int_0^x (1+t)^{\lambda-1}\left(\log\frac{w(1+t)}{t(1+w)}\right)^{\lambda} dt \\ &\leqq \frac{1}{\log(1+w)}\int_0^x (1+t)^{\lambda-1}\left(\frac{w-t}{w(1+t)}\right)^{\lambda} dt \\ &= \frac{1}{\log(1+w)}\int_0^x (1-t/w)^{\lambda}\frac{dt}{1+t} \\ &\leqq \frac{(1-x/w)^{\lambda}}{\log(1+w)}\int_0^x \frac{dt}{1+t} \\ &= (1-x/w)^{\lambda}\frac{\log(1+x)}{\log(1+w)} = o(1)\end{aligned}$$

as $w > x \to \infty$ and $\log\log w - \log\log x \to \infty$, since the latter implies $\log x/\log w \to 0$ and $x/w \to 0$.

Suppose $\lambda \geqq 0$ and $x > 1$. Then

$$\log(1+w)\int_0^x K(w,t)dt = \int_0^x (1+t)^{\lambda-1}\left(\log\frac{w(1+t)}{t(1+w)}\right)^\lambda dt$$
$$\leqq \left(\int_0^1 + \int_1^x\right)(1+t)^{\lambda-1}\left(\log\frac{1+t}{t}\right)^\lambda dt$$
$$= I_1 + I_2 .$$

Setting $u = 1/t$ in I_1 gives

$$I_1 = \int_1^\infty (1+1/u)^{\lambda-1}(\log(1+u))^\lambda \frac{du}{u^2}$$
$$= O(1) .$$

Furthermore,

$$I_2 = O(1)\int_1^x (1+t)^{-1}dt$$
$$= O(1)\log(1+x) - O(1) .$$

Therefore,

$$\int_0^x K(w,t)dt$$
$$= \frac{1}{\log(1+w)}\{I_1 + I_2\}$$
$$= o(1) + O(1)\frac{\log(1+x)}{\log(1+w)} = o(1)$$

as $w > x \to \infty$ and $\log\log w - \log\log x \to \infty$.

This completes the proof.

8. **Proof of Theorem 1.** Assign $\varepsilon > 0$. Since $\sigma_\lambda(t)$ is slowly decreasing with respect to $\Phi(t) = \log\log t$, there exist positive numbers X and δ such that $\sigma_\lambda(y) - \sigma_\lambda(x) > -\varepsilon$ whenever $y > x > X$ and $\mathrm{loglog}\, y - \mathrm{loglog}\, x < \delta$; or equivalently, writing $\delta = \log\gamma$

(15) $$\sigma_\lambda(x) - \varepsilon < \sigma_\lambda(y) \quad \text{whenever} \quad X < x < y < x^\gamma .$$

Suppose, without loss of generality, that $s = 0$. Then $J_\lambda(w) \to 0$ as $w \to \infty$.

Relation (15) implies, for $x > X$, that

$$I_1 = \int_x^{x^\lambda} (1+t)^{\lambda-1}\left(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^\lambda (\sigma_\lambda(x) - \varepsilon)dt$$
$$\leqq \int_x^{x^\gamma} (1+t)^{\lambda-1}\left(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^\lambda \sigma_\lambda(t)dt$$
$$= I_2 .$$

Now, by Theorem 5 and Lemma 12,

$$\begin{aligned} I_2 &= \left(\int_0^{x^\gamma} - \int_0^x\right)(1+t)^{\lambda-1}\left(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^\lambda \sigma_\lambda(t)dt \\ &= \log(1+x^\gamma)J_\lambda(x^\gamma) - \log(1+x)J_\lambda(x) + O(1) \\ &= o(\log(1+x^\gamma)) + o(\log(1+x)) \\ &= o(\log(1+x)) \,. \end{aligned}$$

By Lemma 13,

$$\begin{aligned} I_1 &= (\sigma_\lambda(x) - \varepsilon)\int_x^{x^\gamma}(1+t)^{\lambda-1}\left(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^\lambda dt \\ &= (\sigma_\lambda(x) - \varepsilon)((\gamma-1)\log(1+x) + o(\log(1+x))) \,. \end{aligned}$$

But $I_1 \leqq I_2$ implies

$$\sigma_\lambda(x) - \varepsilon \leqq \frac{o(1)}{(\gamma-1) + o(1)} \,.$$

Therefore,

$$\limsup_{x\to\infty} \sigma_\lambda(x) \leqq \varepsilon \,. \tag{16}$$

In a similar fashion, we can show that

$$-\varepsilon \leqq \liminf_{x\to\infty} \sigma_\lambda(x) \,. \tag{17}$$

Combining (16) and (17) completes the proof of theorem.

9. A counterexample. In this section we give an example which shows that Theorem 1 would be false if $\log\log t$ were replaced by $\log t$. That is, a more delicate tauberian condition on $\sigma_\lambda(t)$ is required than what is obtained by using the standard definition of slowly decreasing.

LEMMA 14. *If $f(x)$ is absolutely continuous on $[0, T]$ for each $T > 0$ and $f'(x) > -M/x$ for all $x > 0$, then $f(x)$ is slowly decreasing with respect to* $\log x$.

Proof. Assign $\varepsilon > 0$. Then if $y > x > 0$

$$\begin{aligned} f(y) - f(x) &= \int_x^y f'(t)dt \\ &> -M\int_x^y \frac{1}{t}dt \\ &= -M(\log y - \log x) > -\varepsilon \end{aligned}$$

whenever $\log y - \log x < \varepsilon/M$. This completes the proof.

THEOREM 6. *There exists a sequence* $\{s_n\}$ *such that* $s_n \to s(L)$ *and, for every* $\lambda > -1$, $\sigma_\lambda(t)$ *is slowly decreasing with respect to* $\log t$, *but* $\{s_n\}$ *is not* A_λ*-convergent.*

Proof. Let $\{s_n\}$ be the real part of the sequence $\{\varepsilon_n^i\}$. For any $\lambda > -1$, $\sigma_\lambda(t)$ exists for $t > 0$, and we have

$$\varepsilon_n^i = \frac{\Gamma(\lambda + i + 1)}{\Gamma(\lambda + 1)\Gamma(i + 1)} \frac{\varepsilon_n^{\lambda+1}}{\varepsilon_n^\lambda} + o(1) .$$

Therefore, $\sigma_\lambda(t)$ is the real part of

$$(1 + t)^{-\lambda-1} \sum_{n=0}^\infty \frac{\Gamma(\lambda + i + 1)}{\Gamma(\lambda + 1)\Gamma(i + 1)} \varepsilon_n^{\lambda+i}\Big(\frac{t}{1 + t}\Big)^n + (1 + t)^{-\lambda-1} \sum_{n=0}^\infty \varepsilon_n^\lambda o(1)\Big(\frac{t}{1 + t}\Big)^n$$
$$= \frac{\Gamma(\lambda + i + 1)}{\Gamma(\lambda + 1)\Gamma(i + 1)}(1 + t)^i + o(1) .$$

The first term above has a derivative which is $O(1/t)$ and, hence, the real part of the first term has a derivative which is $O(1/t)$. The second term is $o(1)$ since A_λ is regular. Hence, the real part of this term is slowly decreasing with respect to any Φ. Therefore, by Lemma 14, $\sigma_\lambda(t)$ is slowly decreasing with respect to $\log t$.

Next, it is clear that $\{s_n\}$ is not A_λ-convergent.

However,

$$\begin{aligned} J_0(w) &= \frac{1}{\log(1 + w)} \int_0^w (1 + t)^{-1}\sigma_0(t)dt \\ &= \frac{1}{\log(1 + w)}\int_0^w \frac{\cos \log(1 + t)}{1 + t} dt \\ &= \frac{\sin \log(1 + w)}{\log(1 + w)} \longrightarrow 0 \quad \text{as} \quad w \longrightarrow \infty . \end{aligned}$$

Hence, by Theorem 3, $s_n \to O(L)$. This completes the proof.

REFERENCES

1. D. Borwein, *On a scale of Abel-type summability methods*, Proc. Cambridge Phil. Soc., **53** (1957), 318-322.

2. ———, *On methods of summability based on power series*, Proc. Royal Soc. Edinburgh, **64** (1957), 342-349.

3. D. Borwein and B. Watson, *Tauberian theorems on a scale of Abel-type summability methods*, Journal Fur Die Reine Und Angewandte Mathematik, **298** (1978), 1-7.

4. G. H. Hardy, *Divergent Series*, Oxford, 1949.

5. B. Watson, *Tauberian theorems on a scale of Abel-type summability methods*, Ph. D. Thesis, The University of Western Ontario, 1974.

Received July 30, 1979 and in revised form June 2, 1981. Supported in part by the Natural Sciences and Engineering Research Council of Canada, Grants A-2983 and A-4646.

THE UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO, CANADA N6A 5B7
AND
MEMORIAL UNIVERSITY OF NEWFOUNDLAND
ST. JOHN'S, NEWFOUNDLAND, CANADA A1B 3X7