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A SPECTRAL CONTAINMENT THEOREM ANALOGOUS TO THE SEMIGROUP THEORY RESULT $e^{t\sigma(A)} \subseteqq \sigma(e^{tA})$

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It is known that if A generates a (C_0) semigroup (e^{tA}), then $e^{t\sigma(A)} \subseteqq \sigma(e^{tA})$, where σ denotes "spectrum." This result is generalized to the context of solution operators of certain nth order linear differential equations.

1. Introduction. Let (T_t) be a (C_0) semigroup on a Banach space X with generator A. It is known [2, p. 457] that

$$e^{t\sigma(A)} \subseteqq \sigma(T_t) . \tag{1.1}$$

Since $T_t f$ solves the differential equation

$$\begin{cases} x' = Ax \\ x(0) = f , \end{cases} \tag{1.2}$$

$T_t f$ can be formally written as e^{tA}, so that (1.1) can be written as

$$e^{t\sigma(A)} \subseteqq \sigma(e^{tA}) . \tag{1.3}$$

In this paper it will be shown that if one replaces the first derivative in (1.2) by an nth order linear differential expression L producing the equation

$$\begin{cases} Lx = Ax \\ x(0) = f \\ x^{(j)}(0) = 0 \quad \text{for} \quad j = 2, \cdots, n-1 \end{cases}$$

solved by $S_t f$ for some linear operators S_t and replaces e^{tz}, the solution in C of

$$\begin{cases} x' = zx \\ x(0) = 1 , \end{cases}$$

by $\psi(t, z)$, the solution of

$$\begin{cases} Lx = zx \\ x(0) = 1 \\ x^{(j)}(0) = 0 \quad \text{for} \quad j = 2, \cdots, n-1 \end{cases}$$

(so that formally $S_t = \psi(t, A)$), then the analog

$$\psi(t, \sigma(A)) \subseteqq \sigma(S_t) ;$$
$$\psi(t, \sigma(A)) \subseteqq \sigma(\psi(t, A))$$

of (1.1) and (1.3) holds. The theorem will be stated precisely and be proved in §3.

2. Definitions and notation.

A. NOTATION. Let ϕ be C or X valued in $C^{(n)}$. Let $\hat{\phi} = (\phi, \phi', \cdots, \phi^{(n-1)})$, where n, the dimension of the vector, is the same as the order of differential expression appearing in context. If f is a constant, let $\hat{f} = (f, 0, 0, \cdots, 0)$.

B. DEFINITION. Let X be a Banach space over C. Let L be a linear nth order differential expression of the form

$$L = \sum_{j=0}^{n} a_j(t) \frac{d^j}{dt^j} ,$$

where the a_j are continuous complex valued functions on $[0, \infty)$ and $a_n \equiv 1$. Let A be a closed densely defined linear operator on X. The initial value problem

$$(P)\begin{cases} u \in C^{(n)}[0, \infty) \\ u(t) \in D_A \quad \text{for} \quad t \in [0, \infty) \\ Lu = Au \\ \hat{u}(0) = \hat{f} \end{cases}$$

is *well posed* if there are bounded linear operators $S_t \in \mathscr{B}(X)$ for $0 \leqq t < \infty$ and a vector subspace Y of X so that

(1) $Y \subseteqq D_A$ and $Y_0 = \{f: f \in Y \text{ and } Af \in Y\}$ is dense in D_A in the D_A norm. ($\|x\|_{D_A} = \|x\| + \|Ax\|$, the graph norm.)

(2) If $g \in Y$ and $u(t) = S_t g$, then $u(t)$ solves (P) with $f = g$.

(3) For each $f \in X$, $S_t f$ is continuous in t.

(4) If u solves

$$\begin{cases} u \in C^n[0, \infty) \\ u(t) \in D_A \quad \text{for} \quad t \in [0, \infty) \\ Lu = Au \quad \text{on} \quad [0, \infty) \\ \hat{u}(0) = \hat{0} , \end{cases}$$

then $u \equiv 0$ on $[0, \infty)$.

This definition is formulated so as to correspond with that in [3]. As will be shown in 5.B, one can take the subspace Y to be all of D_A, in which case $Y_0 = D_{A^2}$. In this case (1) is automatically satisfied whenever $\rho(A) \neq \varnothing$, because then D_{A^2} is dense in X. Thus the reader may recast the definition in less complicated terms without affecting the validity of the main theorem or its proof.

C. DEFINITION. The operators S_t described above such that $S_t f$ solves (P) for $f \in Y$ are called the *solution operators*.

D. DEFINITION. Let (P) be as in Definition B, the definition of "well posed." The corresponding *scalar equation* is

$$(p)\begin{cases} u \in C^n[0, \infty) \\ Lu = zu \\ \hat{u}(0) = \hat{1}\,. \end{cases}$$

(u is complex valued.)

For each $z \in \boldsymbol{C}$, this equation has a unique solution which will be denoted $\psi(t, z)$, so that formally $S_t = \psi(t, A)$.

3. Statement and proof of main theorem.

A. THEOREM. *Let X be a Banach space, A be a closed but not necessarily bounded linear operator on X, and*

$$(P)\begin{cases} u \in C^{(n)}[0, \infty) \\ u(t) \in D_A \quad for \quad 0 \leqq t < \infty \\ Lu(t) = Au(t) \quad for \quad 0 \leqq t < \infty \\ \hat{u}(0) = \hat{f} \end{cases}$$

be a well posed problem in X with solution operators S_t for $0 \leqq t < \infty$. Let

$$(p)\begin{cases} u \in C^{(n)}[0, \infty) \\ Lu(t) = zu(t) \quad for \quad 0 \leqq t < \infty \\ \hat{u}(0) = \hat{1} \end{cases}$$

be the corresponding scalar equation with solution $\psi(t, z)$. Then

$$\psi(t, \sigma(A)) \subseteqq \sigma(S_t)\,.$$

Note that denoting S_t by the suggestive notation $\psi(t, A)$, this statement become $\psi(t, \sigma(A)) \subseteqq \sigma(\psi(t, A))$.

Proof. The reverse containment of the complements will be shown, i.e., that if $\psi(t, \lambda) \in \rho(S_t)$, then $\lambda \in \rho(A)$. Assume λ is such that $\psi(t, \lambda) \in \rho(S_t)$.

By the variation of parameters formula there is a scalar valued kernel $K(t, s)$ which is $C^{(n)}$ in t and $C^{(1)}$ in s so that if $\tilde{K}\phi$ is defined by $(\tilde{K}\phi)(t) = \int_0^t K(t, s)\phi(s)ds$ for any continuous scalar valued or X valued function ϕ, then $u = \tilde{K}\phi$ is the unique scalar or X valued solution of

$$\begin{cases}(\lambda - L)u = \phi \\ \hat{u}(0) = \hat{0}\ .\end{cases}$$

Consider the problem

$$\begin{cases}(\lambda - L)u = S_t f \\ \hat{u}(0) = \hat{0}\ .\end{cases} \tag{3.1}$$

For any fixed $f \in X$, $S_t f$ is continuous in t. Hence, given $f \in X$, (3.1) has a unique $C^{(n)}$ solution $\tilde{K}S_t f$. Let $N_t f = -\tilde{K}S_t f$. Since $S_s f$ is continuous in s for each fixed f, the $\|S_s\|$ are uniformly bounded on compact intervals by the uniform boundedness principle. Since in addition K is continuous in s, N_t is seen to be $\in \mathscr{B}(X)$ for each fixed t.

Assume, as we may by 5.B, that $Y = D_A$ so that $S_t f$ solves $Lu = Au$, $\hat{u}(0) = 0$ for any $f \in D_A$.

If $f \in D_A$, then $LS_t f = AS_t f$, and since $LS_t f$ is continuous in t, so is $AS_t f$. Thus, $S_t f \in D_A$ and $S_t f$ is continuous in t with respect to the D_A norm. Hence, since $N_t f = -\tilde{K}S_t f = -\int_0^t K(t, s)S_s f ds$, $N_t f \in D_A$.

Furthermore

$$\begin{aligned} -(\lambda - A)N_t f &= \int_0^t K(t, s)(\lambda - A)S_s f ds \\ &= \int_0^t K(t, s)(\lambda - L)S_s f ds \\ &= \tilde{K}(\lambda - L)S_t f\ , \end{aligned} \tag{3.2}$$

since A is closed and $S_t f$ solves

$$\begin{cases}Lu = Au \\ \hat{u}(0) = \hat{f}\ .\end{cases}$$

Since $-(\lambda - A)N_t f = \tilde{K}(\lambda - L)S_t f$, by the variation of parameters formula $-(\lambda - A)N_t f$ must solve

$$\begin{cases}(\lambda - L)u(\lambda - L)S_t f \\ \hat{u}(0) = \hat{0}\ .\end{cases}$$

Clearly $S_t f$ solves

$$\begin{cases}(\lambda - L)u = (\lambda - L)S_t f \\ \hat{u}(0) = \hat{f}\ .\end{cases}$$

Hence $S_t f + (\lambda - A)N_t f$ solves

$$\begin{cases}(\lambda - L)u = 0 \\ \hat{u}(0) = \hat{f} .\end{cases} \tag{3.3}$$

But (3.3) is obviously solved by $\psi(t, \lambda)f$. By uniqueness in the initial value problem $S_t f + (\lambda - A)N_t f = \psi(t, \lambda)f$. Hence

$$(\lambda - A)N_t f = (\psi(t, \lambda) - S_t)f \quad \text{for all} \quad f \in D_A . \tag{3.4}$$

Since A and S_s commute by Lemma 5.A, $(\lambda - A)N_t f = (\lambda - A)\int_0^t -K(t, s)S_s f ds = \int_0^t -K(t, s)S_s(\lambda - A)f ds = N_t(\lambda - A)f$. Hence from (3.4)

$$N_t(\lambda - A)f = (\psi(t, \lambda) - S_t)f \quad \text{for all} \quad f \in D_A . \tag{3.5}$$

By using the density of D_A in X and closure and continuity properties, one concludes from (3.4) that

$$(\lambda - A)N_t f = (\psi(t, \lambda) - S_t)f , \qquad f \in X . \tag{3.6}$$

Since $\psi(t, \lambda) \in \rho(S_t)$, $(\psi(t, \lambda) - S_t)$ is a bijection. Thus, (3.6) shows that $(\lambda - A)$ is onto, and (3.5) shows that $(\lambda - A)$ is one to one. Thus $(\lambda - A)\colon D_A \to X$ is a bijection, and from the closed graph theorem, $(\lambda - A)^{-1}$ exists in $\mathscr{B}(X)$, i.e., $\lambda \in \rho(A)$. This completes the proof.

Note. For λ such that $\psi(t, \lambda) \in \rho(S_t)$, $(\lambda - A)^{-1} = N_t(\psi(t, \lambda) - S_t)^{-1}$. This fact follows from (3.6) and from the fact that $(\lambda - A)^{-1}$ exists as a bounded linear operator.

4. Remarks. There are a variety of well posed second order problems to which the theorem applies. Any problem solved by a cosine function [1, Ch. 2, §8] is well posed as are all problems considered by Stafney in [3].

Containment may be strict in the semigroup $e^{t\sigma(A)} \subsetneqq \sigma(e^{tA})$ case. A genuine second order example of strict containment in the main theorem also exists where $L = d^2/dt$. Hille and Phillips mention [2, §26.16] a (C_0) group (T_t) whose generator A has empty spectrum. Then

$$\begin{cases}x'' = A^2 x \\ x(0) = f \\ x'(0) = 0\end{cases}$$

is well posed and solved by the cosine function $S_t = 1/2(T_t + T_{-t})$. Yet, spectral equality fails in $\cosh(t\sqrt{\sigma(A^2)}) \subsetneqq \sigma(S_t)$, because

$\sigma(A^2) = \varnothing$, but $\sigma(S_t) \neq \varnothing$.

It appears that the initial conditions on (p_n) and (P_n) in the statement of the theorem can be altered to other conditions differing in only one coordinate from the zero initial conditions. We think our proof could be altered to reflect the change.

5. Commutativity results.

A. LEMMA. *Let*

$$\begin{cases} Lx = Ax \\ \hat{x}(0) = \hat{f} \end{cases}$$

be a well posed problem in a Banach space X. For all $t > 0$, if $f \in D_A$, then $S_t f \in D_A$, and $AS_t f = S_t A f$.

Proof. This lemma can be proved exactly as the special case in [3] is proved.

In case $\rho(A) \neq \varnothing$, a more direct proof can be constructed by noting that $(\lambda - A)^{-1} S_t (\lambda - A) f$ and $S_t f$ both solve the same initial value problem and so must be equal.

B. COROLLARY. *In the Definition* 2.B *of "well posed," Y can be chosen to be all D_A.*

Proof. Suppose that Y and Y_0 satisfy the conditions of 2.B. The only condition on Y that is not obviously met by D_A is that $u = S_t f$ satisfy

$$\begin{cases} u \in C^{(n)}[0, \infty) \\ Lu = Au \\ \hat{u}(0) = \hat{f} \end{cases} \tag{5.1}$$

for each $f \in D_A$.

Let Y and Y_0 be as originally given. Choose $f \in D_A$. Choose a sequence (f_n) from Y so that $f_n \to f$ in the D_A norm, that is $f_n \to f$ and $Af_n \to Af$ in the norm of X.

Then since the norms of S_t are bounded on finite t intervals, $S_t A f_n \to S_t A f$ uniformly on finite t intervals. Since S_t and A commute

$$AS_t f_n \to AS_t f \quad \text{uniformly on finite } t \text{ intervals.} \tag{5.2}$$

For any $\psi \in C([0, \infty), X)$ define $\tilde{K}\psi$ to the unique solution of

$$\begin{cases} Lu = \psi \\ \hat{u}(0) = \hat{0}\ . \end{cases}$$

In particular this solution u is in $C^{(n)}$. By the variation of parameters formula $\tilde{K}\psi$ is given by

$$(\tilde{K}\psi)(t) = \int_0^t K(t, s)\psi(s)ds$$

for some continuous kernel K.

Let ϕ be the unique (scalar) solution of

$$\begin{cases} Lu = 0 \\ \hat{u}(0) = \hat{1} . \end{cases}$$

Then ϕg is the unique X valued solution of

$$\begin{cases} Lu = 0 \\ \hat{u}(0) = \hat{g} . \end{cases}$$

Both $S_t f_n$ and $\tilde{K}AS_t f_n + \phi f_n$ solve

$$\begin{cases} Lu = Au \\ \hat{u}(0) = \hat{f}_n , \end{cases}$$

and so by uniqueness for the well posed problem,

$$S_t f_n = \tilde{K}AS_t f_n + \phi f_n . \tag{5.3}$$

Since $\tilde{K}$ is defined by an integral with continuous kernel and since by (5.1) $AS_t f_n \to AS_t f$ uniformly on finite t intervals,

$$\tilde{K}AS_t f_n \longrightarrow \tilde{K}AS_t f .$$

Clearly $\phi f_n \to \phi f$. Hence, from (5.3), taking limits of both sides,

$$S_t f = \tilde{K}AS_t f + \phi f . \tag{5.4}$$

$\tilde{K}AS_t f$ solves $Lu = AS_t f$, $AS_t f$ is continuous, and ϕf solves $Lu = 0$. Hence, both $\tilde{K}AS_t f$ and ϕf are in $C^{(n)}$, and their sum $S_t f$ is also in $C^{(n)}$. This establishes the first condition in (5.1). Applying L to both sides of (5.4),

$$LS_t f = AS_t f . \tag{5.5}$$

Since $(\tilde{K}AS_t f)\hat{}_{t=0} = \hat{0}$ and $(\phi(t)f)\hat{}_{t=0} = \hat{\phi}(0)f = \hat{1}f = \hat{f}$, $(S_t f)\hat{}_{t=0} = \hat{f}$. Thus $u(t) = S_t f$ satisfies the other conditions of (5.1).

C. Lemma. *Let X be a Banach space and let*

$$\begin{cases} Lu = Au \\ \hat{u}(0) = \hat{f} \end{cases}$$

be a well posed problem in X with solution operators S_t. Then the solution operators commute with each other (that is, $S_sS_t = S_tS_s$, $s, t \geqq 0$).

Proof. Consider $s \geqq 0$ as fixed and t as variable, so that L depends on t and takes derivatives in t. By Corollary B, take Y in the Definition 2.B of "well posed" to be all of D_A. For any $f \in D_A$, $S_sf \in D_A$ by Lemma A (commutativity result). Then by the definition of S_t,

$$(5.6)\qquad \begin{cases} LS_tS_sf = AS_tS_sf \\ (S_tS_sf)\hat{}_{t=0} = (S_sf)\hat{} \,. \end{cases}$$

But using the fact that S_s is linear and continuous, S_s commutes with differentiation. S_s also commutes with scalar multiplication. Hence S_s commutes with L. Thus

$$LS_sS_tf = S_sLS_tf = S_sAS_tf = AS_sS_tf \,,$$

and

$$(S_sS_tf)\hat{}_{t=0} = (S_sf)\hat{} \,.$$

Hence,

$$(5.7)\qquad \begin{cases} LS_sS_tf = AS_sS_tf \\ (S_sS_tf)_{t=0} = (S_sf)\hat{} \,. \end{cases}$$

Now (5.6) and (5.7) show that both S_tS_sf and S_sS_tf solve

$$\begin{cases} Lu = Au \\ \hat{u}(0) = (S_sf)\hat{} \,, \end{cases}$$

and so since $Lu = Au$ is well posed, the two solutions must be the same. Thus

$$S_sS_tf = S_tS_sf$$

for all $f \in D_A$. Since D_A is dense in X and S_t and S_s are continuous, $S_sS_f = S_tS_s$ on X.

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