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APPROXIMATING CELLULAR MAPS BETWEEN LOW DIMENSIONAL POLYHEDRA

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A compact subset X of a polyhedron P is cellular in P if there is a pseudoisotopy of P shrinking precisely X to a point. A proper surjection between polyhedra $f: P \to Q$ is cellular if each point inverse of f is cellular in P. It is shown that if $f: P \to Q$ is a cellular map with either (i) dim $P \leqq 3$, or (ii) dim $Q \leqq 3$, then f is approximable by homeomorphisms.

Introduction. As a generalization of the concept of cellularity in a manifold, J. W. Cannon proposed in [3] that a set X in a polyhedron P be called cellular if X is compact and there is a pseudoisotopy of P which shrinks precisely X. He then defined a cellular map between polyhedra P and Q to be a proper surjection $f: P \to Q$ such that for each $q \in Q$, $f^{-1}(q)$ is cellular in P. Cannon first asked if a cellular map f is approximable by homeomorphisms when either P or Q is an n-manifold, $n \neq 4$. He conjectured that an affirmative solution to that question would lead to a solution of the more general problem of approximating cellular maps between arbitrary polyhedra. It was shown in [6] that if P or Q is an n-manifold, $n \neq 4$, then $f: P \to Q$ is approximable by homeomorphisms. Here we prove that if $\dim P \leqq 3$ or $\dim Q \leqq 3$, then f is approximable by homeomorphisms. This, then, can be viewed as an extension of the approximation theorem of Armentrout [1].

While the proof of the approximation theorem given here relies in many cases on the techniques used by Handel [5], it should be pointed out that the type of map considered by Handel is more restrictive than those considered here and in [6].

The reader is encouraged to read at least §§ 1 and 2 of [6] to gain an understanding of the stratification and cellular sets being used here before reading this paper.

1. Definitions and background. A *polyhedron* P is a subset of some Euclidean space $\boldsymbol{R}^n$ such that each point $b \in P$ has a neighborhood $N = bL$, the join of b and a compact subset L of P. Throughout, P and Q will denote polyhedra. A homotopy $H_t: P \to P$ for which H_t, $0 \leqq t < 1$, is a homeomorphism is a *pseudoisotopy*. A compact subset X of P is *cellular* in P if there is a pseudoisotopy $H_t: P \to P$ such that X is the only nondegenerate point preimage of H_1. A proper surjection $f: P \to Q$ is a *cellular map* if for each

$y \in Q$, $f^{-1}(y)$ is cellular in P.

The *intrinsic dimension* of a point x in P, denoted $I(x, P)$, is given by $I(x, P) = \max\{n \in \boldsymbol{Z} \mid \text{there is an open embedding } h: \boldsymbol{R}^n \times cL \to P \text{ with } L \text{ a compact polyhedron and } h(\boldsymbol{R}^n \times cL) \text{ a neighborhood of } h(0 \times c) = x\}$, where cL is the open cone on L. The *intrinsic n-skeleton* of P is $P^{(n)} = \{x \in P \mid I(x, P) \leqq n\}$, and the *intrinsic n-stratum* of P is $P[n] = P^{(n)} - P^{(n-1)}$. It can easily be shown that given a triangulation T of P, there is a subcomplex K_n of T such that $|K_n| = P^{(n)}$. Also, $P[n]$ is always a topological n-manifold.

Three results from [6] will form the basis for the proof of the main result. We list them here.

THEOREM 1.1 ([6], Thm. 2.2). *The following are equivalent*:

(1) *X is cellular in P*

(2) *The projection $\pi: P \to P/X$ is approximable by homeomorphisms*

(3) *$X = \bigcap_{i=1}^{\infty} N_i$, where the N_i's are homeomorphic cellular neighborhoods with $\overline{N_{i+1}} \subset N_i$.*

A *cellular neighborhood* is an open set U which is homeomorphic to $\boldsymbol{R}^n \times cL$, the type of set used in determining the intrinsic dimension of a point in P.

THEOREM 1.2 ([6], Thm. 4.1). *Let $f: P \to Q$ be a cellular map with $Q[4] = \varnothing$. Then $P[i] \neq \varnothing$ if and only if $Q[i] \neq \varnothing$, and $f|P^{(i)} = f_i: P^{(i)} \to Q^{(i)}$ is a cellular map with $Q[i] = f_i(P[i] - f_i^{-1}(f_i(P^{(i-1)})))$.*

THEOREM 1.3 ([6], Thm. 4.2). *Let $f: P \to Q$ be a cellular map. If P or Q is an n-manifold, possibly with boundary, and $Q[4] = \varnothing$, then f is approximable by homeomorphisms.*

It should be noted that the statements of Theorem 1.2 and Theorem 1.3 given here differ from those of Theorems 4.1 and 4.2 of [6]. It has been pointed out that the proof of Theorem 4.1 of [6] depends only on Theorem 3.5 of [6]. Thus we need only restrict the possible dimensions of strata of Q and not P. Hence the hypothesis that $P[4] = \varnothing$ need not appear in Theorems 1.2 and 1.3.

The last theorem of this section is an application of the local contractability of the manifold homeomorphism group and the approximation theorem of Armentrout for $n = 3$ or that of Siebenmann [9] $n \neq 4, 5$.

THEOREM 1.4. *Suppose that $f: M^n \to N^n$ is a cellular map* *between n-manifolds with boundary,* $n \neq 4, 5$. *Then for each* $\varepsilon: M^n \to (0, \infty)$, *there is a* $\delta: \partial M^n \to (0, \infty)$ *such that if* $g: \partial M^n \to \partial N^n$ *is a homeomorphism which* δ*-approximates* $f \mid \partial M^n$, *then there is a homeomorphism* $h: M^n \to N^n$ *which* ε*-approximates* f *and* $h \mid \partial M^n = g$.

2. Decomposing cellular maps. The purpose of this section is to show how to consider a cellular map $f: P \to Q$ as a collection of cellular maps defined on closed subpolyhedra of P. The spirit of this idea is similar to that of Theorem 1.2. However, rather than restricting the map f to a particular intrinsic skeleton, we will want to consider the map f restricted to the closure $\bar{A}$ of a component A of a stratum $P[i]$ of P.

LEMMA 2.1. *Suppose that* U *is a cellular neighborhood in* P *homeomorphic to* $\boldsymbol{R}^n \times cL$, C *is a compact subset of* U, N *is a neighborhood of* C *in* U, *and* $\varepsilon > 0$. *Then there is a stratum preserving homotopy* $h_t: P \to P$ *such that*

(1) $h_0 = \mathrm{id}$

(2) $h_t(U) \subset U$

(3) h_t *is the identity off of* N *and on a neighborhood of* $P[n] \cap U$

(4) $h_1(C) \subset N_\varepsilon(P[n])$.

Proof. The proof is essentially that of Proposition 1.5 of [6], except that one takes a simplicial neighborhood N^* of C in N and use that neighborhood N^* to redefine the homotopy of Proposition 1.5 to be the identity off of N^*. This technique is described in the proof of Lemma 5.2 of [6]. It should be noted that h_t will not, in general, be an isotopy.

At this point, we want to consider a closed subset of $\bar{A} - A$, with A as above. Let D be a closed subset of $\bar{A} - A$ such that $D = D_1 \cup D_2 \cup \cdots \cup D_m$, where each D_j is a component of a stratum of P and $\dim D_j \leqq \dim D_{j+1}$. We note that $\bar{A} - A$ is such a closed set.

PROPOSITION 2.2. *With* D *as above, let* $U_j = D_j - f^{-1}(f(\bigcup_{i<j} D_i))$. *Then given* $\varepsilon > 0$ *and a neighborhood* V *of* $f^{-1}(f(U_j))$, *there is a neighborhood* N *of* $f^{-1}(f(U_j))$ *and a stratum preserving homotopy* $h_t: N \to V$ *such that* $h_1(N) \subset N_\varepsilon(D_j)$.

Proof. Cover $f^{-1}(f(U_j))$ with a locally finite collection of saturated open sets $\{U_\alpha^n\}$, where $n = \dim D_j$, such that for each U_α^n, there is a cellular neighborhood C_α^n of the form $\boldsymbol{R}^n \times cL$ such that $\overline{U_\alpha^n} \subset C_\alpha^n \subset V$. Let T_n be a triangulation of U_j such that for each simplex

$\sigma \in T_n$, $f^{-1}(f(\sigma))$ lies in some U_α^n. Then for each simplex $\tau \in T_n^{n-1}$, the $(n-1)$-skeleton of T_n, cover $f^{-1}(f(\tau))$ by a finite number of saturated open sets $\{U_\beta^{n-1}\}$ such that for each U_β^{n-1}, there is a cellular neighborhood $C_\beta^{n-1} \cong \boldsymbol{R}^n \times cL_\beta$ such that if $U_\alpha^n \supset f^{-1}(f(\sigma))$, then $\bar{U}_\beta^{n-1} \subset C_\beta^{n-1} \subset \bar{C}_\beta^{n-1} \subset U_\alpha^n$. Thus $\{U_\beta^{n-1}\}$ is a locally finite open cover of $f^{-1}(f(T_n^{n-1}))$. Let T_{n-1} be a sub-division of T_n^{n-1} such that for each simplex $\sigma \in T_{n-1}$, $f^{-1}(f(\sigma)) \subset U_\beta^{n-1}$ for some β. Similarly, we may inductively define T_{k-1}, $\{U_\beta^{k-1}\}$, and $\{C_\beta^{k-1}\}$ given T_k, $\{U_\alpha^k\}$, and $\{C_\alpha^k\}$. We also require that $\{C_\gamma^0\}$ be a collection of pairwise disjoint cellular neighborhoods.

We now want to identify the neighborhood N. Let $N_0 = \cup\{U_\gamma^0\}$. For each 1-simplex σ in T_1, let N_σ be a saturated open set containing $f^{-1}(f(\sigma - N_0))$ and lying in some U_α such that if σ and τ are different 1-simplices of T_1, $N_\sigma \cap N_\tau = \varnothing$. Define $N_1 = \cup\{N_\sigma | \sigma$ is a 1-simplex of $T_1\}$. Similarly construct N_k, $1 \leqq k \leqq m$. Let $N = \bigcup_{k=0}^m N_k$.

The desired homotopy will first pull N_0 close to D_j, then it will keep N_0 fixed and pull N_1 close to D_j, and so forth. We can apply Lemma 2.1 to each of the disjoint $\bar{U}_\gamma^0$'s using the cone structure on the disjoint C_γ^0's to find a homotopy $H_t^0: N \to V$ such that $H_1^0(N_0)$ lies close to D_j. At the following stages, we will not be trying to homotopically move the $\bar{U}_\alpha^k$'s close to D_j, but the subsets $H_1^{k-1} \cdots H_1^0(\bar{N}_\sigma) \subset U_\alpha^k$. We can, however, construct the homotopies moving the $H_1^{k-1} \cdots H_1^0(\bar{N}_\sigma)$ close to D_j to be in a sense the restriction to the set $H_1^{k-1} \cdots H_1^0(\bar{N}_\sigma)$ of homotopies which do move $\bar{U}_\alpha^k$ close to D_j. There is a neighborhood w_α^k of $D_j \cap C_\alpha^k$ on which such homotopies are the identity. If we have defined $H^{k-1}H^{k-2} \cdots H^0: N \to V$ so that for each k-simplex σ in T_k with $N_\sigma \subset U_\alpha^k$ we have $H_1^{k-1}H_1^{k-2} \cdots H_1^0(N_\tau) \subset w_\alpha^k$ for each $\tau \in T_i$, $i < k$, such that $\tau \subset \sigma$, then we can apply Lemma 2.1 to $H_1^{k-1} \cdots H_1^0(N_\sigma)$ and C_α^k to get a homotopy which fixes $H_1^{k-1} \cdots H_1^0(N_0 \cup \cdots \cup N_{k-1})$, and pulls $H_1^{k-1} \cdots H_1^0(N_\sigma)$ into w_β^{k+1} for each U_β^{k+1} containing C_α^k. Piecing these homotopies together, we can then define the product homotopy $H^kH^{k-1} \cdots H^0: N \to V$ so that $H_1^k \cdots H_1^0(N_0 U \cdots N_k)$ lies so close to D_j that subsequent homotopies will not move $H_1^k \cdots H_1^0(N_0 \cup \cdots \cup N_k)$. The product homotopy $H^mH^{m-1} \cdots H^0: N \to V$ is then the desired homotopy.

Proposition 2.3. *Let $f: P \to Q$ be a cellular map with $Q[4] = \varnothing$, and suppose that A is a component of a stratum of P. Then given a closed subset D of $\bar{A} - A$ as before, $A - f^{-1}(f(D))$ is a nonempty connected set.*

Proof. We first note that the fact that $A - f^{-1}(f(D))$ is nonempty follows from Theorem 1.2.

If we assume that the closed set $\bigcup_{i<j} D_i$ has the desired pro-

perty, we need only show that $f^{-1}(f(U_j))$ does not separate $A - f^{-1}(f(\bigcup_{i<j} D_i))$, where $U_j = D_j - f^{-1}(f(\bigcup_{i<j} D_i))$. Note that this will also cover the initial inductive case when $D = D_1 = D_j$.

Assume that x_0 and x_1 lie in different components of the subpolyhedron $A - f^{-1}(f(D))$ of A. There is an embedding $\alpha: [0, 1] \to A - f^{-1}(f(\bigcup_{i<j} D_i))$ with $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Let V be a neighborhood of $f^{-1}(f(U_j))$ in P such that $V \cap \{x_0, x_1\} = \varnothing$. There is a neighborhood N of $f^{-1}(f(U_j))$ in V and a stratum preserving homotopy $h_t: N \to V$ such that $h_1(N) \cap \alpha([0, 1]) = \varnothing$. Let N^* be a regular neighborhood of a simplicial neighborhood of $f^{-1}(f(U_j)) \cap A$ in $A - f^{-1}(f(\bigcup_{i<j} D_i))$ such that $N^* \subset N$. We may assume that h and α are in general position. It then follows that there is a component M of the boundary of N^* such that $\alpha([0, 1]) \cap M$ consist of an odd number of points, which we may assume to be one point. The component of $h^{-1}(\alpha([0, 1])) \cap (M \times I)$ containing the point $h^{-1}(\alpha([0, 1])) \cap (M \times \{0\})$ must be homeomorphic to $[0, 1)$, which is not possible. Therefore $A - f^{-1}(f(D))$ must be connected.

THEOREM 2.4. *Let $f: P \to Q$ be a cellular map with $Q[4] = \varnothing$, B a component of $Q[n]$, and A the component of $P[n]$ containing $f_n^{-1}(B)$. Then $f_A = f \mid \bar{A}: \bar{A} \to \bar{B}$ is a cellular map.*

Proof. Since each component of each stratum of P is an isotopy class of P (see Proposition 1.2 of [6]), the restriction of a pseudoisotopy of P to $\bar{A}$ will yield a pseudoisotopy of $\bar{A}$. Therefore, if $y \in Q$ and $f^{-1}(y) \cap \bar{A} \neq \varnothing$, $f^{-1}(y) \cap \bar{A}$ is cellular in A. Thus we need only show that $f(\bar{A}) = \bar{B}$.

If $\dim B = 0$, then by Theorem 1.2, $f(\bar{A}) = f(A) = B = \bar{B}$. We now assume that this theorem is true for components of strata of dimension less than n, and that $\dim A = n$.

Let $B_1, \cdots, B_m$ be the components of strata of Q such that $\bar{B} - B = \bigcup_{i=1}^{m} B_i$. Since $\dim B_i < n$, for each i, there is a component A_i of a stratum of P such that $f(\bar{A}_i) = \bar{B}_i$. It then follows that $f^{-1}(\bar{B}) \cap (\bar{A} - A) = D$ is a closed subset of $\bar{A} - A$ consisting of the union of components of strata of P. We now apply Proposition 2.3 to conclude that $A - f^{-1}(f(D))$ is a connected nonempty open subset of A. If $E = (\bar{A} - A) - D$ is nonempty, there is a path β in $\bar{A}$ from $f^{-1}(B)$ to $f^{-1}(f(E))$ which misses $f^{-1}(f(D))$. But this implies that $f(\beta) \cap B \neq \varnothing$, $f(\beta) \cap (Q^{(n)} - B) \neq \varnothing$, and $f(\beta) \cap \bar{B} - B = \varnothing$. Therefore $E = \varnothing$ and $f(\bar{A}) \subset \bar{B}$. Since $B \subset f(A)$, $\bar{B} \subset f(\bar{A})$ and the proof is complete.

3. **The approximation theorem.** We now present the main result.

THEOREM 3.1. *Let $f: P \to Q$ be a cellular map with* (i) $\dim P \leqq 3$, *or* (ii) $\dim Q \leqq 3$. *Then f is approximable by homeomorphisms.*

Proof. We first note that if $\dim P \leqq 3$, we must have $\dim Q \leqq 3$ since $\dim Q < \infty$ [4]. Now Theorem 1.2 gives us that $\dim P = \dim Q$. If $\dim Q \leqq 3$, the same theorem applies and again $\dim P = \dim Q$.

Let $\varepsilon: P \to (0, \infty)$ be given. We must find a homeomorphism $g: P \to Q$ such that $d(f(x), g(x)) < \varepsilon(x)$ for each $x \in P$.

Since $f(P[0]) = Q[0]$ and $Q[0]$ is a discrete set in Q, it follows from Theorem 1.1 that we may assume that f is $1-1$ over $Q[0]$. The problem is then reduced to approximating $f|P-P[0]: P-P[0] \to Q-Q[0]$ by a homomorphism which may be extended to agree with f on $P[0]$. Therefore, it may now be assumed that $P[0]=Q[0]=\emptyset$.

The following lemma contains the key to the proof.

LEMMA 3.2. *Let $f: P \to Q$ be a cellular map with $Q[0]=P[0]=\emptyset$ and $\dim P = \dim Q \leqq 3$. Then if w is a cellular neighborhood of a point in Q, there is a cellular map $h: P \to Q$ such that $h|f^{-1}(w)$ is a homeomorphism and $h|P - f^{-1}(w) = f|P - f^{-1}(w)$.*

We will complete the proof of the theorem and then return to the proof of the lemma.

Let $\delta: Q \to (0, \infty)$ be such that for each $x \in P$, $\delta(f(x)) < \varepsilon(x)$. Choose a locally finite open cover w of Q with $w = w_0 \cup w_1 \cup \cdots \cup w_n$, where $n = \dim Q$ and each w_i consist of open sets $w_{i1}, w_{i2}, \cdots$ such that (1) $w_{ij} \cap w_{ik} = \emptyset$ for $j \neq k$, (2) w_{ij} is a cellular neighborhood of a point in Q, and (3) $\operatorname{diam} st^{n+1}(w_{ij}, w) < \inf\{\delta(y) | y \in w_{ij}\}$.

We may apply Lemma 3.2 to all of the elements in w_0 and to f at the same time to obtain a cellular map $h_0: P \to Q$ which is a homeomorphism when restricted to $f^{-1}(\cup w_0)$ and agrees with f on $P - f^{-1}(\cup w_0)$. Now apply Lemma 3.2 to $\cup w_1$ and h_0. Proceeding in the same manner, we obtain $h_n: P \to Q$ which is a homeomorphism over $w_0 \cup w_1 \cup \cdots \cup w_n = Q$. The desired homeomorphism is thus h_n if we can show that $d(f(x), h_n(x)) < \varepsilon(x)$ for each x in P.

For $x \in P$, $\{f(x), h_n(x)\} \subset st^{n+1}(w_{ij}, w)$ for each w_{ij} containing $f(x)$. But $\operatorname{diam} st^{n+1}(w_{ij}, w) < \inf\{\delta(y) | y \in w_{ij}\} \leqq \delta(f(x)) < \varepsilon(x)$. Therefore $d(f(x), h_n(x)) < \varepsilon(x)$.

Proof of Lemma 3.2. In order to show that f can be approximated by a homeomorphism on $f^{-1}(w)$, we need to fully understand the structure of the cellular neighborhood w. We will consider the possible structures of w in the order of increasing dimension of w. Since $Q[0] = \emptyset$, we begin with cellular neighborhoods which are

1-dimensional.

Case I. $\dim w = 1$.

Again, since $Q[0] = \varnothing$, we must have $w \cong \boldsymbol{R}^1$.

Case II. $\dim w = 2$.

There are two possibilities for the cellular neighborhood. Either (a) $w \cong \boldsymbol{R}^2$, or (b) $w \cong \boldsymbol{R}^1 \times c(p_1, \cdots, p_n)$, $n \neq 2$, where $p_1, \cdots, p_n$ are distinct points. In the latter situation, w can be viewed as $\boldsymbol{R}^2_+ \bigcup_{R^1} \boldsymbol{R}^2_+ \cdots \bigcup_{R^1} \boldsymbol{R}^2_+$, the union of n-copies of $\boldsymbol{R}^2_+$ identified along the common $\boldsymbol{R}^1$ boundary.

Case III. $\dim w = 3$.

Here, either (a) $w \cong \boldsymbol{R}^3$, (b) $w \cong \boldsymbol{R}^2 \times c(p_1, \cdots, p_n)$, $n \neq 2$, or (c) $w \cong \boldsymbol{R}^1 \times cL$, where L is a compact, 1-dimensional polyhedron. The neighborhood of type (b) is seen to be $\boldsymbol{R}^3_+ \bigcup_{R^2} \cdots \bigcup_{R^2} \boldsymbol{R}^3_+$, the union of n-copies of $\boldsymbol{R}^3_+$ identified along the common $\boldsymbol{R}^2$.

The third possibility is the most interesting. In order to understand w, we need to look at the relationship between the stratification of L and the stratification of $\boldsymbol{R}^1 \times cL$. It follows from Prop. 1.4 of [6] that there is a subcomplex L_2 of L such that $(\boldsymbol{R}^1 \times cL)^{(2)} = \boldsymbol{R}^1 \times cL_2$.

Claim. $L_2 = L[0]$.

Proof of claim. If $z \in L[0]$, z has a neighborhood in L homeomorphic to zJ, where $J = \varnothing$ or $J = \{q_1, \cdots, q_n\}$, $n \neq 2$. Now $\boldsymbol{R}^1 \times c(zJ) - \boldsymbol{R}^1 \times c \cong \boldsymbol{R}^1 \times \boldsymbol{R}^1 \times zJ \cong \boldsymbol{R}^2 \times zJ$. Hence $\boldsymbol{R}^1 \times c(z) - \boldsymbol{R}^1 \times c$ is a subset of $(\boldsymbol{R}^1 \times cL)[2]$ and $z \in L_2$.

If $z \in L[1]$, then z has a neighborhood homeomorphic to zJ, where $J = \{q_1, q_2\}$. Hence $\boldsymbol{R}^1 \times c(zJ) - \boldsymbol{R}^1 \times \boldsymbol{R}^1 \times c \cong \boldsymbol{R}^1 \times \boldsymbol{R}^1$ and $z \notin L_2$. This completes the proof of the claim.

We now know that $(\boldsymbol{R}^1 \times cL)[3]$ consist of open sets of the form $(\boldsymbol{R}^1 \times cK) - (\boldsymbol{R}^1 \times c)$, where K is a component of $L[1]$. Each component K of $L[1]$ is homeomorphic to either $\boldsymbol{R}^1$ or S'.

If $K \cong S^1$, then $\boldsymbol{R}^1 \times cK \cong \boldsymbol{R}^3$. Note that in this case, we must have $K \neq L$ since w is a cellular neighborhood of a point in $Q[1]$.

If $K \cong \boldsymbol{R}^1$, it is not as important to consider $\boldsymbol{R}^1 \times cK$ as $\boldsymbol{R}^1 \times c\bar{K}$, where $\bar{K}$ is the closure of K in L. Note that $\bar{K} - K$ will consist of either one or two points. In either case, $\bar{K} - K \subset L[0]$. The set $\bar{K}$ is homeomorphic to either I^1 or S^1. Hence $\boldsymbol{R}^1 \times c\bar{K}$ is homeomorphic to either $\boldsymbol{R}^3_+$ or $\boldsymbol{R}^3$, respectively.

At this point, we want to reconstruct w. We already know that $(\boldsymbol{R}^1 \times cL)^{(2)} = \boldsymbol{R}^1 \times cL[0] \cong \boldsymbol{R}^2_+ \bigcup_{R^1} \cdots \bigcup_{R^1} \boldsymbol{R}^2_+$, with one copy of

$\boldsymbol{R}^2_+$ for each point of $L[0]$.

We now adjoin to $\boldsymbol{R}^2 \times cL[0]$ a copy of $\boldsymbol{R}^1 \times c\bar{K}$ for each component K of $L[1]$. If $K \cong S^1$, then we identify the $\boldsymbol{R}^1 \times c$ in $\boldsymbol{R}^1 \times cS^1$ to the $\boldsymbol{R}^1 \times c$ in $\boldsymbol{R}^1 \times cL[0]$.

If $K \cong \boldsymbol{R}^1$ and $\bar{K} \cong I^1$, we attach a copy of $\boldsymbol{R}^1 \times c(I^1)$ to $\boldsymbol{R}^1 \times cL[0]$ with $\boldsymbol{R}^1 \times c(bd\ I^1)$ being identified with the two copies of $\boldsymbol{R}^2_+$ corresponding to the two points of $\bar{K} - K$ in $L[0]$.

When $K \cong \boldsymbol{R}^1$ and $\bar{K} \cong S^1$, a copy of $\boldsymbol{R}^1 \times c(S^1)$ is attached to $\boldsymbol{R}^1 \times cL[0]$ with $\boldsymbol{R}^1 \times cz$ being identified with the copy of $\boldsymbol{R}^2_+$ in $\boldsymbol{R}^1 \times cL[0]$ corresponding to the point $\bar{K} - K = z \in L[0]$.

Now that the structure of w has been determined, the cellular map $h: P \rightarrow Q$ can be constructed. We proceed by working on neighborhoods w of increasing dimension.

If $\dim w = 1$, then it follows from Theorem 1.3 that there is a homeomorphism $h': f^{-1}(w) \rightarrow w$ which may be extended to agree with f on $P - f^{-1}(w)$.

There are two cases to be considered when $\dim w = 2$. If $w \cong \boldsymbol{R}^2$, then we may apply Theorem 1.3 as above.

Otherwise, $w \cong \boldsymbol{R}^2_+ \bigcup_{R^1} \cdots \bigcup_{R^1} \boldsymbol{R}^2_+$. The construction in this case provides a good example of the proof technique for the remaining cases. It follows from Theorem 2.4 that for each component B of $w \cap Q[2]$, there is a component A of $f^{-1}(w) \cap P[2]$ such that $f_A = f \mid \bar{A}: \bar{A} \rightarrow \bar{B}$ is a cellular map, where $\bar{A}$ and $\bar{B}$ are the closures of A and B in $f^{-1}(w)$ and w, respectively.

Given $\varepsilon: \mathrm{cl}\,(f^{-1}(w)) \rightarrow [0, \infty)$ such that $\mathrm{cl}\,(f^{-1}(w)) - f^{-1}(w) = \varepsilon^{-1}(0)$, this cellular map may be ε-approximated by a homeomorphism h_A by Theorem 1.3. Furthermore, there is a δ_A such that if $g_A: f^{-1}(w) \cap P[1] \rightarrow w \cap Q[1]$ is a homeomorphism within δ_A of $f \mid f^{-1}(w) \cap P[1]$, we may assume that $h_A \mid f^{-1}(w) \cap P[1] = g_A$ according to Theorem 1.4. Let $g: f^{-1}(w) \cap P[1] \rightarrow w \cap P[1]$ be a homeomorphism such that g is so close to f that each homeomorphism h_A may be chosen to agree with g on $P[1] \cap f^{-1}(w)$. The desired map $h: P \rightarrow Q$ can now be defined by

$$h(x) = \begin{cases} f(x),\ x \notin f^{-1}(w) \\ h_A(x),\ x \in \bar{A},\ A \text{ a component of } P[2] \cap f^{-1}(w)\ . \end{cases}$$

Suppose now that $\dim w = 3$. If $w \cong \boldsymbol{R}^3$, the construction of h is straightforward. When $w \cong \boldsymbol{R}^3_+ \bigcup_{R^2} \cdots \bigcup_{R^2} \boldsymbol{R}^3_+$, we proceed as in the similar case where $\dim w = 2$.

The interesting case is then $w \cong \boldsymbol{R}^1 \times cL$, where L is a 1-dimensional polyhedron. The first thing to be noted is that $w \cap Q^{(2)} \cong \boldsymbol{R}^1 \times c(L[0])$. If $L[0] = \varnothing$, then $w \cap Q^{(2)} \cong \boldsymbol{R}^1$. Otherwise, $w \cap Q^{(2)} \cong \boldsymbol{R}^1 \times c(p_1, \cdots, p_j)$ for some $j \geqq 1$. In either case, we can use the previous techniques to approximate $f_2|P^{(2)} \cap f^{-1}(w): P^{(2)} \cap f^{-1}(w) \to Q^{(2)} \cap w$ by a homeomorphism g is as close as desired to $f_2|P^{(2)} \cap f^{-1}(w)$.

If B is a component of $w \cap Q[3]$ such that $\bar{B}$, the closure of B in w, is homeomorphic to $\boldsymbol{R}^3_+$, then $f_A: \bar{A} \to \bar{B}$ is approximable by homeomorphisms h_A according to Theorem 1.3. Here $\bar{A}$ is the closure of the component of $f^{-1}(w) \cap P[3]$ given by Theorem 2.4 applied to B and $f|f^{-1}(w)$. Also, it follows from Theorem 1.4 that if $g_A: \bar{A} - A \to \bar{B} - B$ is a homeomorphism which closely approximates $f_A\bar{A}-A$, we can assume that $h_A|\bar{A} - A = g_A$.

The next possibility to be considered is a component B of $w \cap Q[3]$ such that both B and $\bar{B}$ are homeomorphic to $\boldsymbol{R}^3$. This corresponds to $\boldsymbol{R}^1 \times c(\bar{K})$, where K is a component of $L[1]$ such that $\bar{K} \cong S^1$ and $\bar{K} - K = z \in L[0]$. We know from Theorem 1.3 that $\bar{A} = f_A^{-1}(\bar{B})$ is homeomorphic to $\boldsymbol{R}^3$. Also, $P^2 \cap f_A^{-1}(Q^{(2)} \cap w)$ is a subcomplex of $f_A^{-1}(\bar{B}) = \bar{A}$ which is homeomorphic to $\boldsymbol{R}^2_+$. We must construct a homeomorphism $h_A: \bar{A} \to \bar{B}$ such that $h_A(\bar{A} - A) = \bar{B} - B$ and h_A approximates f_A.

Let B the space obtained from $\bar{B}$ by removing $\bar{B} \cap Q[2]$ and replacing that copy of $\boldsymbol{R}^2$ with two copies of $\boldsymbol{R}^2$ in the natural fashion. There is a natural projection $\pi_B: \tilde{B} \to \bar{B}$ which is $1-1$ over $\bar{B} - (\bar{B} \cap Q[2])$ and $2-1$ over $\bar{B} \cap Q[2]$.

Similarly, we split $\bar{A}$ along $\bar{A} \cap P[2]$, the subcomplex of $\bar{A}$ homeomorphic to $\boldsymbol{R}^2$, and then attach two copies of $\bar{A} \cap P[2]$ to obtain a 3-manifold with boundary $\tilde{A}$. Again, there is the projection $\pi_A: \tilde{A} \to \bar{A}$ which is $1-1$ over $\bar{A} - (\bar{A} \cap P[2])$ and $2-1$ over $\bar{A} \cap P[2]$.

Each nondegenerate point inverse $f_A^{-1}(y)$ has a defining sequence of neighborhoods of the form $\boldsymbol{R}^3$, $\boldsymbol{R}^2 \times c\{p_1, p_2\}$, or $\boldsymbol{R}^1 \times c(S^1)$, where the lowest dimensional stratum of $\bar{A}$ that $f^{-1}(y)$ intersects is 3, 2, or 1, respectively. The splitting of $\bar{A}$ will then leave point preimages of the first type unchanged. Those of the second type, with neighborhoods homeomorphic to $\boldsymbol{R}^2 \times c\{p_1, p_2\}$, will be split into two pieces, each having a defining sequence of neighborhoods homeomorphic to $\boldsymbol{R}^3_+$. The last type of nondegenerate point preimage will be split along $\bar{A} \cap P[2]$, but will still be connected. This split cellular set will have a defining sequence of neighborhoods homeomorphic to $\boldsymbol{R}^1 \times c(I^1)$, and hence be cellular in $\tilde{A}$ according to Theorem 1.1. Thus the induced map $\tilde{f}: \tilde{A} \to \tilde{B}$ is a cellular map between polyhedra, each of which is a 3-manifold with boundary.

Given $\tilde{\varepsilon}_A: \tilde{A} \to (0, \infty)$, there is a $\tilde{\delta}_A: \partial\tilde{A} \to (0, \infty)$ such that a homeomorphism that $\tilde{\delta}_A$-approximates $\tilde{f}|\partial\tilde{A}$ can be extended to an $\tilde{\varepsilon}_A$-approximation of $\tilde{f}$. We can find a homeomorphism $g_A: \bar{A} - A \to \bar{B} - B$ which induces a $\tilde{\delta}_A$-approximation $\tilde{g}_A: \partial\tilde{A} \to \partial\tilde{B}$ such that if

$\pi_A(x_1) = \pi_A(x_2)$, then $\pi_B \tilde{g}_A(x_1) = \pi_B \tilde{g}_A(x_2)$. Let $\tilde{h}_A$ be the $\tilde{\varepsilon}_A$-approximation extending $\tilde{g}_A$. The desired approximation to f_A is the homeomorphism $h_A: \bar{A} \to \bar{B}$ given by $h_A(x) = \pi_B \tilde{h}_A \pi_A^{-1}(x)$.

The last case to be considered is the one where $\bar{B} \cong \boldsymbol{R}^1 \times c(S^1)$, with $\boldsymbol{R}^1 \times c = \bar{B} \cap Q[1]$. Again, $\bar{B}$ and $\bar{A}$ are both homeomorphic to $\boldsymbol{R}^3$. We must find an approximating homeomorphism which takes $\bar{A} \cap P[1]$ onto $\bar{B} \cap Q[1]$.

Given a point $y \in \bar{B} - B$, we will show how to approximate $f_A: \bar{A} \to \bar{B}$ by a cellular map g_y which is a homeomorphism over a neighborhood of y in $\bar{B}$, takes $\bar{A} - A$ onto $\bar{B} - B$, and equals f_A outside of that neighborhood. The approximating homeomorphism can then be constructed in the same way that the lemma is used to construct the general approximation theorem.

Since $f_A(\bar{A} - A) = \bar{B} - B$, we can assume that f_A is a homeomorphism over B.

There is a cellular neighborhood $N \cong \boldsymbol{R}^1 \times c(S^1)$ of $f_A^{-1}(y)$ in $\bar{A}$. Choose y_1 and y_2 to be points in $\bar{B} - B$ such that the arc α in $\bar{B} - B$ with endpoints y_1 and y_2 containts y in its interior and $f_A^{-1}(\alpha) \subset N$. Since each of $f_A^{-1}(y_1)$ and $f_A^{-1}(y_2)$ is cellular, we can assume that $f_A^{-1}(y_1)$ and $f_A^{-1}(y_2)$ are points in $N \cap P[1]$. Let S be a tame 2-sphere in $\bar{B}$ bounding the 3-cell D in $f_A(N)$ such that (D, α) is homeomorphic to the pair (B^3, B^1). Since f_A is a homeomorphism over S, $f_A^{-1}(S)$ is a 2-sphere in N which bounds a 3-cell E in N. Also, $f_A^{-1}(S) \cap (\bar{A} - A)$ consist of the two points $f_A^{-1}(y_1)$ and $f_A^{-1}(y_2)$. Both $(\bar{A} - A) \cap N$ and $f_A^{-1}(S)$ are tame subsets of N, and we would like to conclude that $[(\bar{A} - A) \cap N] \cup f_A^{-1}(S)$ is tame in N. Let β_1 and β_2 be disjoint subarcs of $\bar{A} - A$ which lie in N such that $\beta_i \cap f^{-1}(S) = f_A^{-1}(y_i)$. It then follows from [7] that $f_A^{-1}(S) \cup \beta_1 \cup \beta_2$ is tame in N, and so is $[(\bar{A} - A) \cap N] \cup f_A^{-1}(S)$.

Since $(\bar{A} - A) \cap N$ corresponds to $\boldsymbol{R}^1 \times c$ in $\boldsymbol{R}^1 \times c(S^1)$, we can find a simple closed curve γ in N such that $\gamma \cap (\bar{A} - A) = \beta_1 \cup \beta_2 \cap (E \cap P[1])$ and γ bounds an embedded tame 2-cell in N. Therefore γ is unknotted in N and $(E, E \cap P[1])$ is homeomorphic to the standard pair (B^3, B^1). Since $f_A | \partial E$ is a homeomorphic taking $\partial E \cap P[1]$ onto $\partial D \cap \alpha$, there is a homeomorphism $g'_y: (E, E \cap P[1]) \to (D, \alpha)$ which agrees with f_A on ∂E. Hence we may define g_y to be g'_y on E and f_A on $\bar{A} - E$.

Since $P[1] \cap \bar{A}$ has a neighborhood U in $\bar{A}$ such that $(U, P[1] \cap \bar{A})$ is homeomorphic to $(\boldsymbol{R}^3, \boldsymbol{R}^1)$, given any $\varepsilon: \bar{A} \to (0, \infty)$, there is a $\delta: \bar{A} - A \to (0, \infty)$ such that if $g_A: \bar{A} - A \to \bar{B} - B$ is a homeomorphism which δ-approximates $f_A | \bar{A} - A$, then we can find h_A, an ε-approximation of f_A which extends g_A.

We have now shown that if we are given a component B of $w \cap Q[3]$ with the corresponding component A of $f^{-1}(w) \cap P[3]$ and

$\varepsilon_A: \bar{A} \to (0, \infty)$, there is a $\delta_A: \bar{A} - A \to (0, \infty)$ such that if $g_A: \bar{A} - A \to \bar{B} - B$ is a homeomorphism which δ-approximates $f_A | \bar{A} - A$, then we can find h_A, an ε-approximation of f_A which extends g_A.

To complete the proof, we let $\varepsilon: \mathrm{cl}\,(f^{-1}(w)) \to [0, \infty)$ be a continuous function such that $\varepsilon^{-1}(0) = \mathrm{cl}\,(f^{-1}(w)) - f^{-1}(w))$. This induces $\varepsilon_A: \bar{A} \to (0, \infty)$ for each component A of $P[3] \cap f^{-1}(w)$. Let δ_A be the function described above. We now choose $\delta: f^{-1}(w) \cap P^{(2)} \to (0, \infty)$ to be a positive function such that if $x \in \bar{A} - A$ for any component A of $P[3] \cap f^{-1}(w)$, then $\delta(x) < \delta_A(x)$. Otherwise, we require that for $x \in P^{(2)} \cap f^{-1}(w)$, $\delta(x) < \varepsilon(x)$. We now approximate $f | f^{-1}(w) \cap P^{(2)}: f^{-1}(w) \cap P^{(2)} \to w \cap Q^{(2)}$ by a δ-approximation g. For each component A of $P[3] \cap f^{-1}(w)$, we approximate $f_A: \bar{A} \to \bar{B}$ by a homeomorphism h_A such that $h_A | \bar{A} - A = g | \bar{A} - A$. Then $h^*: f^{-1}(w) \to w$ defined by

$$h^*(x) = \begin{cases} g(x), & x \in f^{-1}(w) \cap P^{(2)} \\ h_A(x), & x \in \bar{A},\ A \text{ a component of } P[3] \cap f^{-1}(w) \end{cases}$$

extends to $h: P \to Q$ by the map f on $P - f^{-1}(w)$.

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