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A HAUSDORFF-YOUNG INEQUALITY FOR B-CONVEX BANACH SPACES

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A vector valued analogue of the classical Hausdorff-Young inequalities for characters of groups is obtained.

Introduction. For Banach space notions and terminology not explained here, we refer the reader to [5], [6] and the several papers which are mentioned further on. Let us start by recalling the definition of type and cotype of normed spaces. We say that a normed space X, $\| \ \|$ has type p (resp. cotype q) if there is a constant $M < \infty$ (resp. $\delta > 0$) such that for every integer m and every choice of vectors $(x_i)_{1 \leqq i \leqq m}$ in X

$$\left\{\int \|\sum \varepsilon_i(t) x_i\|^2 dt\right\}^{1/2} \leqq M(\sum \|x_i\|^p)^{1/p} \tag{1}$$

respectively

$$\left\{\int \|\sum \varepsilon_i(t) x_i\|^2 dt\right\}^{1/2} \geqq \delta(\sum \|x_i\|^q)^{1/q} \tag{2}$$

holds, where (ε_i) denotes the sequence of Rademacker functions. Take further p_X the supremum of all types $1 \leqq p \leqq 2$ of X and q_X the infimum of all cotypes $2 \leqq q \leqq \infty$. The space X is said to have type (resp. cotype) provided $p_X > 1$ (resp. $q_X < \infty$).

The numbers p_X and q_X have a geometrical interpretation. As shown in [7], if X is an infinite dimensional Banach space, then ℓ^{p_X} and ℓ^{q_X} are both finitely representable in X (see also [8]). In particular, X has type (resp. cotype) if and only if ℓ^1 (resp. ℓ^∞) is not finitely representable in X. The first of those properties is also called B-convexity, a notion which was introduced in [4]. Very recently, see [12], it was proved that if X is B-convex, then p_X and q_{X^*} are conjugate exponents, i.e., $(p_X)^{-1} + (q_{X^*})^{-1} = 1$.

One may ask for an analogue of (1), (2) if the Rademacker functions (ε_i) are replaced by distinct Walsh functions $w_S = \prod_{i \in S} \varepsilon_i$ on the Cantor group $\{1, -1\}^N$ or, more generally, by characters of an arbitrary compact abelian group (integrating with respect to the Haar measure). In this spirit, we will prove

THEOREM 1. *If X is a B-convex Banach space, there exist some $\tilde{p} > 1$ and $\tilde{q} < \infty$ and constants $M < \infty$, $\delta > 0$, such that*

$$\left\{\int \|\sum x_\gamma \gamma(t)\|^2 dt\right\}^{1/2} \leqq M(\sum \|x_\gamma\|^{\tilde{p}})^{1/\tilde{p}} \tag{3}$$

and

$$(4) \qquad \left\{\int \| \sum x_\gamma \gamma(t) \|^2 dt\right\}^{1/2} \geqq \delta(\sum \| x_\gamma \|^{\tilde{q}})^{1/\tilde{q}}$$

whenever $\{x_\gamma\}_{\gamma \in \Gamma}$ *is a finitely supported sequence of elements of* X *and* Γ *the spectrum of a compact abelian group.*

Theorem 1 is a Hausdorff-Young theorem for B-convex spaces and gives a new characterization of B-convexity. In [1], Th. 1 was established for the circle group, under the strong hypothesis that X is super reflexive. Again, we may introduce $\tilde{p}_X$ as the spremum of the $\tilde{p}$ and $\tilde{q}_X$ as the infimum of the $\tilde{q}$. A standard duality argument shows then that $\tilde{p}_X$ and $\tilde{q}_{X^*}$ are conjugate. Obviously $p_X \geqq \tilde{p}_X$ and $q_X \leqq \tilde{q}_X$ and there is not necessarily equality. If we take indeed for X the space L^α with $2 \leqq \alpha < \infty$, then $p_X = 2$ and $\tilde{p}_X = \alpha/(\alpha - 1)$.

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Preliminary results. This section deals with certain facts which are needed for the proof of Theorem 1. The first result concerns an extremal representation of uniformly bounded orthogonal systems of real functions. It seems to be important in the proof of Theorem 1 except in the case of the group $\{1, -1\}^N$ for which it is not needed. Assume $(\Omega, \boldsymbol{P})$ a diffuse probability space and fix a positive integer n. Consider the Banach space $B = \bigoplus_n L^2(\boldsymbol{P})$ obtained by taking direct sum of n copies of the real $L^2(\boldsymbol{P})$-space. Let $\mathcal{O}$ be the subset of B consisting of the n-tuples $\xi = (\xi_1, \cdots, \xi_n)$, where the ξ_k are uniformly bounded by 1, of mean zero and mutually orthogonal. Obviously $\mathcal{O}$ is norm closed in B.

We agree to call extreme point of a set in a vector space any point of the set which is not midpoint of two different points of the set.

The next fact is elementary. We omit its proof since it is essentially contained in [2] (see Lemma 2.3).

PROPOSITION 1. *The set* $\mathcal{E}$ *of extreme points of* $\mathcal{O}$ *consists of the* $\xi = (\xi_1, \cdots, \xi_n)$ *in* $\mathcal{O}$ *such that each function* ξ_k *is* ± 1*-valued.*

Our aim is to obtain each member of $\mathcal{O}$ as barycenter of a measure supported by $\mathcal{E}$. Remark that since $\mathcal{O}$ is not weakly compact in B one can not use simply Choquet's integral representation theorem (cf. [9]). However, we can apply here a more general result due to G. Edgar (see [3]), in order to obtain the following

PROPOSITION 2. *For all $\xi = (\xi_1, \cdots, \xi_n)$ in $\mathcal{O}$ there exists a Borel probability measure μ on $\mathscr{E}$ for which*

$$\xi_k = \int \eta_k \mu(d\eta) \quad \textit{for} \quad k = 1, \cdots, n .$$

It is important here to notice that the measure P may be taken separable. The space B is then a separable Banach space with the so called Radon-Nikodym property in which frame Edger's theorem applies. Actually, the result as stated in [3] requires also the convexity of the norm-closed set but in fact this additional hypothesis is never used in the proof which is based on a martingale technique. Let us point out that the measure μ obtained by Edgar's argument also satisfies for $k = 1, \cdots, n$

$$\xi_k(t) = \int \eta_k(t) \mu(d\eta) \quad \text{for almost all} \quad t \in \Omega .$$

If A is a finite set, denote $\#A$ its cardinality. For positive integers d, let D_d be the set $\{1, -1\}^d$. The next fact which we need is the following probabilistic lemma.

PROPOSITION 3. *Let $\xi = (\xi_1, \cdots, \xi_n)$ be a fixed member of $\mathscr{E}$ and fix a positive integer d. For each $\varepsilon \in D_d$ and $t_1, \cdots, t_d$ in Ω, define*

$$A_\varepsilon(t_1, \cdots, t_d) = \{k = 1, \cdots, n; \xi_k(t_1) = \varepsilon_1, \cdots, \xi_k(t_d) = \varepsilon_d\} .$$

For $\kappa > 0$, consider the following subset of Ω^d

$$\Omega_{d,\kappa} = \{(t_1, \cdots, t_d) \in \Omega^d; |\#A_\varepsilon(t_1, \cdots, t_d) - 2^{-d}n| \leqq \kappa n \ \textit{for all} \ \varepsilon \in D_d\} .$$

Then the product-measure of $\Omega_{d,\kappa}$ in Ω^d is at least

$$(1 - (1 + \sqrt{2})/\kappa)(2^d/n)^{1/2} .$$

Proof. For fixed $\varepsilon \in D_d$, we estimate the measure of the set

$$C_\varepsilon = \{(t_1, \cdots, t_d) \in \Omega^d; |\#A_\varepsilon(t_1, \cdots, t_d) - 2^{-d}n| > \kappa n\} .$$

Define the following functions on Ω^d

$$f_0 = \text{constant function } n$$

and for $j = 1, \cdots, d$

$$f_j(t_1, \cdots, t_d) = \sum_k \prod_{i=1}^{j} [1 + \varepsilon_i \xi_k(t_i)] .$$

Clearly

$$f_d(t_1, \cdots, t_d) = 2^d \#A_\varepsilon(t_1, \cdots, t_d) .$$

Hence

$$2^d \int |\# A_\varepsilon(t_1, \cdots, t_d) - 2^{-d} n| dt_1 \cdots dt_d = \int |f_d - f_0| \leqq \sum_{j=0}^{d-1} \int |f_{j+1} - f_j| .$$

Now

$$f_{j+1} - f_j = \varepsilon_{j+1} \sum_k \prod_{i=1}^{j} [1 + \varepsilon_i \xi_k(t_i)] \xi_k(t_{j+1})$$

and thus, by orthogonality of $\xi_1, \cdots, \xi_n$,

$$\begin{aligned} \left\{\int |f_{j+1} - f_j|\right\}^2 &\leqq \int \left\{\int |f_{j+1} - f_j|^2 dt_{j+1}\right\} dt_1 \cdots dt_j \\ &= 2^j \int \left\{\sum_k \prod_{i=1}^{j} [1 + \varepsilon_i \xi_k(t_i)]\right\} dt_1 \cdots dt_j \\ &= 2^j n . \end{aligned}$$

Therefore,

$$\int |f_d - f_0| \leqq \sum_{j=0}^{d-1} \sqrt{2}^j \sqrt{n} \leqq \frac{\sqrt{2}^d}{\sqrt{2} - 1} \sqrt{n} ,$$

which shows that C_ε has measure less than $(1 + \sqrt{2})/\kappa(2^d n)^{1/2}$, by Chebychef's inequality. The statement of the proposition is now immediate.

Assume X a normed space. Let p be a type of X^* and denote C the type constant (cf. [1]).

PROPOSITION 4. *If d is a positive integer and $(f_\varepsilon)_{\varepsilon \in D_d}$ are functions in $L^2_X(\Omega, \boldsymbol{P})$, then*

$$\left\{\frac{1}{d} \sum_{i=1}^{d} \int \|\sum_\varepsilon \varepsilon_i f_\varepsilon(t)\|^2 dt\right\}^{1/2} \leqq C d^{-1/p'} 2^d \max_\varepsilon \left\{\int \|f_\varepsilon(t)\|^2 dt\right\}^{1/2}$$

where $p' = p/(p-1)$.

Proof. If $(x_\varepsilon)_{\varepsilon \in D_d}$ are vectors in X, then

$$\{\sum_i \|\sum_\varepsilon \varepsilon_i x_\varepsilon\|^{p'}\}^{1/p'} \leqq C \sqrt{2}^d \{\sum_\varepsilon \|x_\varepsilon\|^2\}^{1/2} .$$

This follows from a duality argument, seeing the ε_i as Rademacher functions. Hence, by Hölder's inequality,

$$\{\sum_i \|\sum_\varepsilon \varepsilon_i x_\varepsilon\|^2\}^{1/2} \leqq C d^{1/p - 1/2} \sqrt{2}^d \{\sum_\varepsilon \|x_\varepsilon\|^2\}^{1/2} .$$

Replacing the x_ε by functions f_ε, we find

$$\left\{\sum_i \int \|\sum_\varepsilon \varepsilon_i f_\varepsilon(t)\|^2 dt\right\}^{1/2} \leqq C d^{1/p - 1/2} \sqrt{2}^d \left\{\sum_\varepsilon \int \|f_\varepsilon(t)\|^2 dt\right\}^{1/2}$$

from which the required estimate follows.

Proof of the main result. It is clear that only part [3] of Th. 1 must be shown. Indeed, if X is B-convex, then X^* is also B-convex and [4] for X follows from a dualization of [3] for X^*. For a given (complex) Banach space X, we introduce the numbers

$$\varphi(n) = \sup \left\{ \int \| \sum_{\gamma \in \Lambda} x_\gamma \gamma(t) \|^2 dt \right\}^{1/2} .$$

The supremum is taken here over all subsets Λ of the spectrum $\Gamma \backslash \{1\}$ of a compact abelian group G and families $(x_\gamma)_{\gamma \in \Lambda}$ in the unit ball of X, assuming $\sharp \Lambda = n$ and $\gamma_1 \neq \bar{\gamma}_2$ for $\gamma_1 \neq \gamma_2$ in Λ.

Remark that the Haar measure of G can be taken diffuse, since G may always be replaced by the group $G \times \{1, -1\}^N$. The probability space $(\Omega, \boldsymbol{P})$ considered in the previous section will be the group G equipped with its Haar measure.

Our purpose is to establish a recursive estimate on the $\varphi(n)$. For $\gamma \in \Gamma$, denote $\operatorname{Re} \gamma$ (resp. $\operatorname{Im} \gamma$) the real (resp. imaginary) part of the character γ. By the assumptions on Λ, both sets

$$\{\operatorname{Re} \gamma; \gamma \in \Lambda\} \quad \text{and} \quad \{\operatorname{Im} \gamma; \gamma \in \Lambda\}$$

belong clearly to $\mathscr{O}$ introduced in the previous section. Application of Prop. 2 gives Borel probability measures μ and ν on $\mathscr{E}$ satisfying

$$\operatorname{Re} \gamma = \int \xi_\gamma \mu(d\xi) \quad \text{and} \quad \operatorname{Im} \gamma = \int \xi_\gamma \nu(d\xi)$$

for all $\gamma \in \Lambda$.

Using the fact that the γ are group characters and the translation invariance of the Haar measure, we get by substitution

$$\begin{aligned} &\left\{ \int \| \sum x_\gamma \gamma(t) \|^2 dt \right\}^{1/2} \\ &\quad = \left\{ \iint \| \sum x_\gamma \gamma(u) \gamma(t) \|^2 du dt \right\}^{1/2} \\ &\quad \leqq \left\{ \iint \| \sum \operatorname{Re} \gamma(t) x_\gamma \gamma(u) \|^2 du dt \right\}^{1/2} + \left\{ \iint \| \sum \operatorname{Im} \gamma(t) x_\gamma \gamma(u) \|^2 du dt \right\}^{1/2} \\ &\qquad + \left\{ \iiint \| \sum \xi_\gamma(t) x_\gamma \gamma(u) \|^2 du dt \mu(d\xi) \right\}^{1/2} \\ &\qquad + \left\{ \iiint \| \sum \xi_\gamma(t) x_\gamma \gamma(u) \|^2 du dt \nu(d\xi) \right\}^{1/2} . \end{aligned}$$

Our next purpose is to estimate an integral of the form

$$\left\{ \iint \| \sum \xi_\gamma(t) x_\gamma \gamma(u) \|^2 du dt \right\}^{1/2}$$

where $(\xi_\gamma)_{\gamma \in \Lambda}$ belongs to $\mathscr{E}$.

Let p be a type for X^* with type constant C. Then

PROPOSITION 5. *For any positive integer c, the following inequality holds*

$$\left\{\iint \|\sum \xi_\gamma(t)x_\gamma\gamma(u)\|^2 du dt\right\}^{1/2} \leqq Cd^{-1/p'}2^d\varphi(2^{-d+1}n) + 2\left(\frac{8^d}{n}\right)^{1/4}\varphi(n)\ .$$

Proof. Take in Prop. 3 $\kappa = 2^{-d}$ and let $\Omega_0 = \Omega_{d,\kappa}$, which has measure at least $1 - (1 + \sqrt{2})(8^d/n)^{1/2}$.

We have

$$\iint \|\sum \xi_\gamma(t)x_\gamma\gamma(u)\|^2 du dt = \int\left\{\frac{1}{d}\sum_{i=1}^{d}\int \|\sum \xi_\gamma(t_i)x_\gamma\gamma(u)\|^2 du\right\}dt_1 \cdots dt_d$$

$$\leqq \sup_{(t_1,\cdots,t_d)\in\Omega_0}\left\{\frac{1}{d}\sum_i\int \|\sum \xi_\gamma(t_i)x_\gamma\gamma(u)\|^2 du\right\} + (1 + \sqrt{2})\left(\frac{8^d}{n}\right)^{1/2}\varphi(n)^2\ .$$

For fixed $(t_1, \cdots, t_d)$ in Ω_0, we estimate

$$\left\{\frac{1}{d}\sum_i\int \|\sum \xi_\gamma(t_i)x_\gamma\gamma(u)\|^2 d\mu\right\}^{1/2}$$

which can be rewritten as

$$\left\{\frac{1}{d}\sum_i\int \|\sum_\varepsilon \varepsilon_i \sum_{\gamma\in A_\varepsilon} x_\gamma\gamma(u)\|^2 du\right\}^{1/2}$$

where ε ranges in D_d and $A_\varepsilon = A_\varepsilon(t_1, \cdots, t_d)$.

Taking $f_\varepsilon = \sum_{\gamma\in A_\varepsilon} x_\gamma\gamma$, application of Prop. 4 gives then the estimate

$$Cd^{-1/p'}2^d \max_\varepsilon\left\{\int \|\sum_{\gamma\in A_\varepsilon} x_\gamma\gamma(u)\|^2 du\right\}^{1/2} \leqq Cd^{-1/p'}2^d\varphi(\# A_\varepsilon)$$

$$\leqq Cd^{-1/p'}2^d\varphi(2^{-d+1}n)$$

by definition of the set Ω_0.

Combining inequalities, the required result is obtained. As a consequence of Prop. 5 and the estimate given above involving the extremal representation, we find

PROPOSITION 6. *For any positive integer d, the following holds*

$$\varphi(n) \leqq 2Cd^{-1/p'}2^d\varphi(2^{-d+1}n) + 4\left(\frac{8^d}{n}\right)^{1/4}\varphi(n)\ .$$

PROPOSITION 7. *Assume $p > 1$ a type for X^* and C the corresponding type constant. Then, there is a constant $K < \infty$ such that*

(*) $$\varphi(n) \leqq Kn^{1-\tau} \quad \textit{for all} \quad n$$

where $1/\tau = (17C)^{p'}$.

Proof. Fix a positive integer d satisfying $(16C)^{p'} < d < (17C)^{p'}$ and let K be such that

$$\varphi(n) \leqq Kn^{1-\tau} \quad \text{for} \quad n \leqq 8^{d+4} .$$

We now show that (*) also holds for $n > 8^{d+4}$ proceeding by induction. Application of Prop. 6 yields namely

$$\varphi(n) \leqq 2Cd^{-1/p'}2^d K(2^{-d+1}n)^{1-\tau} + \frac{1}{2}\varphi(n)$$

and thus

$$\varphi(n) \leqq 8Cd^{-1/p'}2^{d\tau}Kn^{1-\tau} \leqq Kn^{1-\tau}$$

by the choice of d.

If now Λ is a finite subset of Γ, one may write Λ as disjoint union $\Lambda = \Lambda' \cup \Lambda''$ of two sets Λ', Λ'' such that $\gamma_1 \neq \bar{\gamma}_2$ for $\gamma_1 \neq \gamma_2$ in Λ' or in Λ''. Hence

$$\left\{\int \|\sum_{\gamma \in \Lambda} x_\gamma \gamma(t)\|^2 dt\right\}^{1/2} \leqq 2\varphi(\#\Lambda) \leqq 2K(\#\Lambda)^{1-\tau} .$$

Using a well-known technique (see [10], Lemma 2), [3] of Th. 1 follows for any $\tilde{p} < 1/(1-\tau)$.

This concluded the proof of Th. 1.

Remarks and questions.

1. It is clear from the preceding that the numbers $\tilde{p}$, $\tilde{q}$, M, δ in Th. 1 only depend on the type and type constant of X and X^*.

2. Theorem 1 has an analogue if we replace the L_X^2-norm by the L_X^α-norm for $1 < \alpha < \infty$. If p is a type for X^* with type constant C, the same argument yields

$$\left\{\int \|\sum x_\gamma \gamma(t)\|^\alpha dt\right\}^{1/\alpha} \leqq M_{\alpha,\tilde{p}}(\sum \|x_\gamma\|^{\tilde{p}})^{1/\tilde{p}}$$

for some constant $M_{\alpha,\tilde{p}}$, provided

$$\tilde{p} < \frac{(17C)^{p'}}{(17C)^{p'} - 1} \quad \text{if} \quad 2 \leqq \alpha \leqq p'$$

and

$$\tilde{p} < \frac{(17C)^\alpha}{(17C)^\alpha - 1} \quad \text{if} \quad \alpha > p' .$$

3. One may ask the question whether or not Th. 1 remains valid for arbitrary ortonormal systems. Using results of [11], a positive solution should solve the following conjecture affirmatively.

Question. Does there exist for all $p > 1$ and $C < \infty$ some $\varepsilon > 0$ and $K < \infty$ such that

$$d(E, l^2(\dim E)) \leqq K(\dim E)^{1/2-\varepsilon}$$

holds if E is a finite dimensional normed space of type p with type constant C? (d is the Banach-Mazur distance).

The fact that the characters is heavily used in our argument. Using Prop. 3 and Prop. 4, one can show estimates for general ortonormal systems, but these are only of logarithmic order.

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