FIXED POINT INDEX AND CHAIN APPROXIMATIONS

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Localizing the Lefschetz number of certain chain approximations of upper semi continuous multivalued mappings a new approach to fixed point index is given. It turns out that this fixed point index satisfies the commutativity property as well as the mod-p property (known from the single-valued case). In particular, in the single-valued case the proof of the mod-p property is a natural consequence of a corresponding property of (global) Lefschetz number.

Introduction. The aim of the present note is to provide a unified approach to fixed point index theory for single-valued mappings as well as for certain classes of multivalued mappings (e.g., acyclic mappings) on compact polyhedra.

Fixed point index theory was initiated in 1926/1927 by S. Lefschetz [27] with his celebrated fixed point formula on manifolds (with triangulation), and it was generalized one year later by H. Hopf [23] to compact polyhedra. (For a good survey of fixed point index theory we recommend [14].) Later, after the classical paper of J. Leray and J. Schauder [30] (1934) the problem of generalizing and extending the Leray-Schauder theory resp. the fixed point index theory arose in a natural way. In the forties J. Leray [28], [29] succeeded in developing an index theory for the category of 'convexoid spaces' which are in some sense homological generalizations of simplicial complexes, see also [4]. The techniques introduced by Leray are very general; indeed, his cohomology theory is the precursor of modern sheaf cohomology.

In 1953 B. O'Neill [34] rederived—for the category of compact polyhedra—the principle results of Leray's theory (and Hopf's results), and, moreover, he proved that fixed point index is uniquely determined by certain axioms (in the category of compact polyhedra). In 1960 F. E. Browder [5] extended fixed point index theory to 'semi-complexes' by finding appropriate algebraic analogues of the properties of fixed point index for chain mappings.

All these results use definitions of fixed point index based on certain induced chain mappings. A purely homological approach to fixed point index (on manifolds) was discovered in 1965 by A. Dold [9], [10], see also [11] where Dold extends his technique to coincidence problems.

There are essentially two ways to handle multivalued fixed point problems. The first one is based on homological arguments and roots

in the early work of L. Vietoris [44] (and of E. Begle [1]) where the 'Vietoris-Begle mapping theorem' is proved, and where for acyclic maps induced homomorphisms in (Vietoris-) homology are defined. Using the Vietoris (−Begle) mapping theorem S. Eilenberg and D. Montgomery [12] in 1946 generalized the Lefschetz fixed point theorem to acyclic mappings (on compact metric ANR's). The other way to deal with multivalued fixed point problems is influenced by homotopic considerations, and detects fixed points of the multivalued map Φ by fixed points of single-valued 'approximations' of Φ. Early examples of this procedure were given by J. von Neumann [33] and by S. Kakutani [24]. For a survey of both methods we recommend [2].

The first approach to degree resp. fixed point index of multivalued (convex or acyclic) mappings goes back to A. Granas [19] and to A. Granas and J. W. Jaworowski [20] in 1959. Further approaches to degree of multivalued mappings based on single-valued approximations were introduced by A. Cellina and A. Lasota [8] and by Yu. G. Borisovič et al. [3].

In 1970 B. D. Calvert [7] gave a homological definition of fixed point index for acyclic maps which runs along lines similar to [9]: Using the Vietoris-Begle mapping theorem Calvert applies Dold's technique to the induced homomorphism of the acyclic map. In [21] and [26] there are similar homological approaches. We refer also to [15] where Calvert's index of 'repulsive' fixed points is computed.

A common lack of all these approaches to fixed point index of multivalued (e.g., acyclic) maps is that it is unclear whether the commutativity property holds, see [7], [15], [18]. Beside some special cases there is no proof for this property [7], [15]. Moreover, it is not known whether the mod-p property in the sense of Zabreĭko-Krasnosel'skiĭ-Steinlein, $i(X, f, U) \equiv i(X, f^p, U) \bmod p$, p prime, is satisfied, see [15], [18], [37].

In this paper we will give an affirmative answer to these two questions provided an alternative definition of fixed point index is used. We shall define a fixed point index for a multivalued (acyclic) mapping by associating to this map a chain map ("chain approximation"), and by computing the 'local' Lefschetz number (in the sense of O'Neill and Browder) of that chain map.

The paper is organized as follows:

In Chapter I we describe algebraic analogues of properties of the fixed point index for chain mappings (including commutativity and the mod-p property). In Chapter II chain approximations and corresponding fixed point indices are discussed, and in Chapters III and IV the results of I and II are applied to acyclic mappings on polyhedra. Finally, we remark that in the single-valued case proofs

(e.g., the proof of the mod-p property) become considerably easier provided classical simplicial approximation is used.

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I. **Index and chain mappings.** Since the fundamental papers of Leray [28], O'Neill [34], and, in particular, of Browder [5] one knows that 'local Lefschetz numbers' of chain mappings have algebraic properties which are in analogy to corresponding properties of the fixed point index (Homotopy invariance, Normalization, Additivity, Commutativity). In the following we will recapitulate this fact, and, moreover, we will find a mod-p property which is easy to prove, and which provides the corresponding mod-p property for the fixed point index.

In what follows $K = (K, \tau)$ and $L = (L, \mu)$ will always denote compact polyhedra with fixed triangulations τ resp. μ.

0. **Preliminaries and notations.** For $l \in N^1$ and a triangulation τ we denote by τ^l the lth *barycentric subdivision* of τ. A subset $U \subset K$ is called *polyhedral* provided there is an integer $l \in N$ such that τ^l induces a triangulation of the closure $\bar{U}$ of U. Let $B \subset K$ and $k \in N$. The kth *star of B in τ,*

$$st^k(B, \tau) ,$$

is defined recursively as follows:

$$st^1(B, \tau) := st(B, \tau) := \cup \{\sigma \in \tau \mid \sigma \cap B \neq \varnothing\}$$
$$st^k(B, \tau) := st(st^{k-1}(B, \tau), \tau) , \quad k > 1 .$$

(Simplices are always assumed to be closed.)

For $l \in N$ and a field F denote by $C_*(K, l)$ the oriented chain complex $C_*(K, \tau^l; F)$, see [39, p. 159].

If C and C' are two chain complexes (over F) denote by $\hom(C, C')$ the space of all chain mappings $\varphi: C \to C'$. Let $c \in C_*(K, \tau)$. The *carrier of c*, carr(c), is the smallest polyhedral subset $X \subset K$ such that $c \in C_*(X, \tau)$, see [10, p. 198].

DEFINITION. Let $\varphi: C_*(K, \tau) \to C_*(K, \tau)$ be a chain mapping, and let $\sigma, \sigma' \in C_n(K, \tau)$ be two simplices of dimension n.

σ' *is contained in* $\varphi(\sigma)$, $\sigma' \in \varphi(\sigma)$, provided σ' is contained in the irreducible representation of $\varphi(\sigma)$ (with respect to the basis of $C_n(K, \tau)$ which consists of all simplices of dimension n) with a nonzero coefficient.

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[1] N = natural numbers.

σ is called a *fixed simplex of* φ iff $\sigma \in \varphi(\sigma)$.

Let $l \in N$. Then $b: C_*(K, \tau) \to C_*(K, l)$ is the subdivision chain map which maps each chain of $C_*(K, \tau)$ into its lth barycentric subdivision [10, p. 40]. By $\chi: C_*(K, l) \to C_*(K, \tau)$ we denote any chain mapping which is induced by a simplicial approximation of Id: $(K, \tau^l) \to (K, \tau)$, [10, p. 119].

Let $V = \{V_q\}_{q \in N}$ be a graded F-vector space with $\dim V < \infty$, and let $\beta = \{\beta_q\}_{q \in N}$ be an endomorphism of V of degree zero. Then

$$\Lambda(\beta) := \sum_{q \in N} (-1)^q \operatorname{tr}(\beta_q) \quad (\operatorname{tr} = \text{trace})$$

is called the *Lefschetz number of* β, [10, p. 208].

1. Index systems.

DEFINITION 1.1. Let $U \subset K$ be open and polyhedral. A graded set

$$M(K, U) = \{M(K, U)_k\}_{k \in N}$$

where $M(K, U)_k \subset \hom(C_*(\bar{U}, k), C_*(K, k))$, is called *Index system* (*I-system*) *on* U provided the following properties are satisfied:

There is an integer $k_0 = k_0(M(K, U))$ such that for all $l, k \in N$, $l \geq k \geq k_0$, and for all $\varphi_1 \in M(K, U)_k$, $\varphi_2 \in M(K, U)_l$

(i) φ_1 and $\chi \varphi_2 b$ are *admissible homotopic*, i.e., the diagram

$$
\begin{array}{ccc}
C_*(\bar{U}, k) & \xrightarrow{\ \varphi_1\ } & C_*(K, k) \\
b \downarrow & & \uparrow \chi \\
C_*(\bar{U}, l) & \xrightarrow{\ \varphi_2\ } & C_*(K, l)
\end{array}
$$

is homotopy commutative with a chain homotopy D which satisfies the following *boundary condition*,
for any simplex $\sigma \in \tau^k$ with $st(\sigma, k) \cap \partial U \neq \varnothing$,

$$\operatorname{carr} D\sigma \cap st(\sigma, k) = \varnothing .$$

$(st(\cdot, \tau^k)$ is abbreviated by $st(\cdot, k)$ in the following.)

(ii) $\varphi_2 b \chi |_{\bar{U}}$ and φ_2 are admissible homotopic.

$$(\chi |_{\bar{U}} := \chi |_{C_*(\bar{U}, l)}) .$$

REMARK. In the definition above we did assume that U is already polyhedral in τ, otherwise we have a graded set $\{M(K, U)_k\}_{k \geq l_0}$ where U is polyhedral in τ^{l_0}

To any I-system $M(K, U)$ we associate a 'local' Lefschetz number which will be called the index of $M(K, U)$.

LEMMA AND DEFINITION 1.2. *Let $M(K, U)$ be an I-system, and let $l \geqq k \geqq k_0(M(K, U))$.*

Let $\varphi_1 \in M(K, U)_k$, $\varphi_2 \in M(K, U)_l$.
Then

$$\Lambda(\pi\varphi_1) = \Lambda(\pi\varphi_2) \,,$$

where $\pi = \pi_{\bar{U}} \colon C_*(K, k) \to C_*(U, k)$ resp. $C_*(K, l) \to C_*(\bar{U}, l)$ are the natural projections.

$$I(M(K, U)) := \Lambda(\pi\varphi) \in F \,, \quad \varphi \in M(K, U)_k \,, \quad k \geqq k_0(M(K, U)) \,,$$

*is called the **index of the I-system** $(M(K, U))$.*

Proof. Since φ_1 and $\chi\varphi_2 b$ are admissible homotopic we have $\Lambda(\pi\varphi_1) = \Lambda(\pi\chi\varphi_2 b)$, [5, p. 274]. Moreover, one computes $\Lambda(\pi\chi\varphi_2 b) = \Lambda(\chi\pi\varphi_2 b)$. Since the trace is commutative, and since $\varphi_2 b \chi_{|\bar{U}}$ and φ_2 are admissible homotopic, Browder's lemma applies again,

$$\Lambda(\chi\pi\varphi_2 b) = \Lambda(\pi\varphi_2) \,.$$

Hence, $\Lambda(\pi\varphi_1) = \Lambda(\pi\varphi_2)$.

In the following we list some properties of $I(M(K, U))$:

PROPOSITION 1.3 *(Additivity).* *Let $M = M(K, U)$ be an I-system, and let $U_1, U_2 \subset U$ be open, disjoint subsets which are polyhedral. Assume that*

$$M_i := M(K, U)|_{\bar{U}_i} := \{M(K, U)_k|_{\bar{U}_i}\}_{k \in N} \,,$$

where $M(K, U)_k|_{\bar{U}_i} := \{\varphi|_{\bar{U}_i} \mid \varphi \in M(K, U)_k\}$, are I-systems for $i = 1, 2$. Assume further that the following condition is satisfied:
There is an integer $k' \in N$ such that for all $k \geqq k'$ there is a map $\varphi \in M(K, U)_k$ with the following property,
for any simplex $\sigma \in \tau^k$ with $st(\sigma, k) \cap (\bar{U} \backslash (U_1 \cup U_2)) \neq \varnothing$,

$$\operatorname{carr} \varphi(\sigma) \cap st(\sigma, k) = \varnothing \,.$$

Then

$$I(M) = I(M_1) + I(M_2) \,.$$

DEFINITION 1.4. Let $M(K, U)$ and $M'(K, U)$ be I-systems. $M(K, U)$ and $M'(K, U)$ are called **homotopic**, provided there is an integer $k'' \in N$ such that for all $k \geqq k''$ there are maps $\varphi \in M(K, U)_k$ and $\varphi' \in M'(K, U)_k$ which are admissible homotopic.

PROPOSITION 1.5 *(Homotopy Invariance).* *Let $M(K, U)$ and $M'(K, U)$ be two I-systems which are homotopic. Then*

$$I(M(K,\ U)) = I(M'(K,\ U))\ .$$

2. Commutativity and mod-p property. If C, C', C'' are chain complexes (over F), and if $A \subset \hom(C,\ C')$, $B \subset \hom(C',\ C'')$, define

$$B \circ A: = \{\psi \circ \varphi \mid \psi \in B,\ \varphi \in A\} \subset \hom(C,\ C'')\ .$$

PROPOSITION 1.6 *(Commutativity).* *Let $U \subset K$, $V \subset L$ be open and polyhedral subsets of K resp. L, and let*

$$A(K,\ L) = \{A(K,\ L)_k\}_{k \in N}\ ,\quad A(L,\ K) = \{A(L,\ K)_k\}_{k \in N}$$

be graded sets where

$$A(K,\ L)_k \subset \hom(C_*(K,\ k),\ C_*(L,\ k))$$
$$A(L,\ K)_k \subset \hom(C_*(L,\ k),\ C_*(K,\ k))\ .$$

Assume

$$A(K,\ L) \circ A(L,\ K): = \{A(K,\ L)_k \circ A(L,\ K)_k\}_{k \in N}$$

and

$$A(L,\ K) \circ A(K,\ L): = \{A(L,\ K)_k \circ A(K,\ L)_k\}_{k \in N}$$

are I-systems on U resp. V, and denote them by $M(K,\ U)$ resp. $M(L,\ V)$.
Assume further that the following condition is satisfied,
there is an integer $\tilde{k} \in N$ such that for all $k \geq \tilde{k}$ |there are maps $\varphi \in A(K,\ L)_k$, $\psi \in A(L,\ K)_k$ with the following properties:
(i) *For any simplex $\sigma \in \mu^k$, $\sigma \subset \bar{V}$,*

$$\sigma_1 \in \psi(\sigma),\ \sigma_1 \not\subset \bar{U} \implies \sigma \notin \varphi(\sigma_1)$$

(ii) *For any simplex $\sigma \in \tau^k$, $\sigma \subset \bar{U}$,*

$$\sigma_1 \in \varphi(\sigma),\ \sigma_1 \not\subset \bar{V} \implies \sigma \notin \psi(\sigma_1)\ .$$

Then

$$I(M(K,\ U)) = I(M(L,\ V))\ .$$

Proof. Choose k sufficiently large such that

$$I(M(K,\ U)) = \varLambda(\pi_{\bar{U}} \circ \psi \circ \bar{\varphi})\ ,\quad I(M(L,\ V)) = \varLambda(\pi_{\bar{V}} \circ \varphi \circ \bar{\psi})\ ,$$

where $\varphi \in A(K,\ L)_k$, $\psi \in A(L,\ K)_k$, $\bar{\varphi}: = \varphi|_{\bar{v}}$, $\bar{\psi}: = \psi|_{\bar{v}}$, and where φ and ψ satisfy properties (i) and (ii). Choose bases of $C_i(K,\ k)$ and $C_i(L,\ k)$ consisting of simplices of dimension i, $0 \leq i \leq \dim K$, $\dim L$, and compute the matrix representations of

$$\pi_{\bar{U}} \circ \psi \circ \bar{\varphi}: C_i(\bar{U},\ k) \longrightarrow C_i(\bar{U},\ k)$$
$$\pi_{\bar{V}} \circ \varphi \circ \bar{\psi}: C_i(\bar{V},\ k) \longrightarrow C_i(\bar{V},\ k)$$

with respect to those bases.

It follows immediately that the traces of $\pi_{\bar{U}} \circ \psi \circ \bar{\varphi}$ and $\pi_{\bar{V}} \circ \varphi \circ \bar{\psi}$ coincide, and, hence, we obtain

$$I(M(K, U)) = I(M(L, V)) .$$

PROPOSITION 1.7 (Mod-p Property). Let $F = Z_p$, p prime. Let $U \subset K$ be open and polyhedral.

Let $A(K, K) = \{A(K, K)_k\}_{k \in N}$ be a graded set where

$$A(K, K)_k \subset \hom(C_*(K, k), C_*(K, k)) .$$

Assume that $A(K, K)$ and $A(K, K) \circ A(K, K) \circ \cdots \circ A(K, K)(p \text{ times})$ are I-systems on U, and denote them by $M(K, U)$ and $M^p(K, U)$. Assume further that the following condition is satisfied, there is an integer $\bar{k} \in N$ such that for all $k \geq \bar{k}$ there is a map $\varphi \in A(K, K)_k$ with the following property: For any simplex $\sigma \in \tau^k$, $\sigma \subset \bar{U}$, with $\sigma \in \varphi^p(\sigma)$,

$$\sigma_1 \in \varphi^{p-j}(\sigma), \sigma_1 \not\subset \bar{U}, \sigma_1 \in \tau^k \Longrightarrow \sigma \in \varphi^j(\sigma_1) \quad \text{for all} \quad j, 1 \leq j < p .$$

Then

$$I(M(K, U)) = I(M^p(K, U))(in \ Z_p) .$$

Proof. We give a proof for $p = 2$, for $p > 2$ the proof runs along the same lines.

Choose k sufficiently large such that

$$I(M(K, U)) = \varLambda(\pi\bar{\varphi}) , \quad I(M^2(K, U)) = \varLambda(\pi\varphi\bar{\varphi}) ,$$

where $\varphi \in A(K, K)_k$, $\bar{\varphi} = \varphi|_{\bar{U}}$, and where φ satisfies the assumption of the proposition. Using the lemma in [36, p. 441] ("Fermat's theorem") we have

$$\varLambda(\pi\bar{\varphi}\pi\bar{\varphi}) = \varLambda(\pi\bar{\varphi}) \quad (\text{in } Z_2) .$$

But the assumption of the proposition implies that

$$\text{tr } \pi\bar{\varphi}\pi\bar{\varphi} \mid_{C_i(\bar{U},k)} = \text{tr } \pi\varphi\bar{\varphi} \mid_{C_i(\bar{U},k)}$$

for i, $0 \leq i \leq \dim K$. Hence the mod-p property follows.

II. Approximation systems and fixed point indices. In this chapter we will study chain mappings and their approximation properties. It turns out that in order to define a fixed point index à la O'Neill [34] or Browder [5] one does not need a chain mapping which is induced by simplicial approximation but a chain map which has

certain approximation properties.   We will show how such "approxi-
mation systems" lead to fixed point index theories even for mul-
tivalued mappings in a natural way.   In the following (K, τ), (L, μ)
and (M, ν) will always denote compact polyhedra with fixed triangula-
tions τ, μ and ν.

Let $\Phi\colon K \to K$ be an upper semi continuous (u.s.c.) multivalued
map[1] i.e., the graph $\{(x, y) \in K \times K \mid y \in \Phi(x)\}$ is closed, and let $U \subset$
K be an open subset of K.   We will say that the triple (K, Φ, U)
is *admissible* iff $x \notin \Phi(x)$ for all $x \in \partial U$.

1. Approximation systems.

DEFINITION 2.1.   Let $\Phi\colon K \to L$ be u.s.c., and let $k, n \in N$.
A chain mapping

$$\varphi_1\colon C_*(K, l) \longrightarrow C_*(L, k) \quad (l \geqq k)$$

is called $n - k$-*approximation of* Φ, provided the following condition
is satisfied:
For any simplex $\sigma \in \tau^l$ there is a point $y(\sigma) \in K$ such that

$$\sigma \subset st^n(y(\sigma), k)$$
$$\operatorname{carr} \varphi_1\sigma \subset st^n(\Phi(y(\sigma)), k) \ .$$

EXAMPLE 2.2.   Let $f\colon K \to L$ be continuous and single-valued.
Then any chain map $f'\colon C_*(K, l) \to C_*(L, k)$ which is induced by a
simplicial approximation $\bar{f}$ of f, $\bar{f}\colon (K, \tau^l) \to (L, \mu^k)(l \geqq k)$, is a $2 - k$-
approximation of f.

DEFINITION 2.3.   Let $\Phi\colon K \to L$ be u.s.c.   A graded set

$$A(\Phi) = \{A(\Phi)_j\}_{j \in N} \ ,$$

where $A(\Phi)_j \subset \operatorname{hom}(C_*(K, j), C_*(L, j))$, is called *approximation system*
(*A-system*) *for* Φ, provided there is an integer $n = n(A(\Phi))$ such that
the following properties are satisfied:
  (1) $\varphi \in A(\Phi)_j \Rightarrow \varphi = \varphi_1 b$, where φ_1 is a

$n - j$-approximation of Φ $(b = \text{subdivision map})$.

  (2)   For all $j \in N$ there is an integer $j_1 = j_1(A(\Phi))$ such that for
all $l, m \geqq j_1(m \geqq l)$ and for all $\varphi = \varphi_1 b \in A(\Phi)_l$, $\psi = \psi_1 b \in A(\Phi)_m$

$\varphi_1\chi$ and $\chi\psi_1$ are *homotopic with a small homotopy* ,

i.e., the diagram

---

[1] We assume always $\phi(x) \neq \varnothing$ and closed for all $x \in K$.

$$
\begin{array}{ccc}
C_*(K,\, l_1) & \xrightarrow{\ \varphi_1\ } & C_*(L,\, l) \\
\chi\uparrow & & \uparrow\chi \qquad (m_1 \geqq l_1) \\
C_*(K,\, m_1) & \xrightarrow[\ \psi_1\ ]{} & C_*(L,\, m)
\end{array}
$$

is homotopy commutative with a chain homotopy D which satisfies the following *smallness condition*:

For any simplex $\sigma \in \tau^{m_1}$ there is a point $z(\sigma) \in K$ such that

$$\sigma \subset st^n(z(\sigma),\, j)$$
$$\operatorname{carr} D\sigma \subset st^n(\Phi(z(\sigma)),\, j)\ .$$

EXAMPLE 2.4.   Let $f\colon K \to L$ be continuous and single-valued. Then, an approximation system for f is obtained in a natural way by simplicial approximation:

$$A^*(f) = \{A^*(f)_j\}_{j \in N}\ ,\quad \text{with}$$
$$A^*(f)_j\colon = \{\varphi = f'b \mid f'\colon C_*(K,\, l) \longrightarrow C_*(L,\, j)\}$$

where f' is induced by simplicial approximation, see 2.2.
$A^*(f)$ is called the *approximation system induced by f*.

Proof.   Set $n = n(A^*(f))\colon = 2$, and for $j \in N$ set $j_1 = j_1(A^*(f))\colon = j$.

Then, it is well known [22, p. 36/37] that $f'\chi$ and $\chi f''$ are (contiguous) homotopic with a small homotopy, provided $f'b \in A^*(f)_l$, $f''b \in A^*(f)_m$, where $m \geqq l \geqq j$.

The following lemma detects an important relation between I-systems and A-systems.

LEMMA AND DEFINITION 2.5.   *Let $U \subset K$ be open and polyhedral, and let $A(\Phi)$ be an A-system for $\Phi\colon K \to K$.*

Then $A(\Phi)$ is an I-system on U provided $(K,\, \Phi,\, U)$ is admissible.

This I-system is called the I-system induced by $A(\Phi)$ and denoted by $M(K,\, U;\, A(\Phi))$.

Proof.   Without loss of generality we assume $n(A(\Phi)) = 1$; for $n(A(\Phi)) > 1$ the same arguments apply.   Choose $k \in N$ sufficiently large such that the following boundary condition is satisfied:

$$y \in K,\ st^3(y,\, k) \cap \partial U \neq \varnothing \implies st^3(y,\, k) \cap st(\Phi(y),\, k) = \varnothing\ .$$

Since $A(\Phi)$ is an A-system we obtain for $j = k$ an integer $j_1 = j_1(A(\Phi))$.

We claim that $A(\Phi)$ is an I-system on U with $k_0 = k_0(M(K,\, U;\, A(\Phi)))\colon = j_1$:

Therefore let $l, m \in N$, $m \geq l \geq k_0$, and let

$$\varphi = \varphi_1 b \in A(\Phi)_l \, , \quad \psi = \psi_1 b \in A(\Phi)_m \, .$$

We have to show that φ and $\chi \psi b$ are admissible homotopic, see 1.1.

$$
\begin{array}{ccccc}
C_*(\bar{U}, l) & \xrightarrow{\ b\ } & C_*(\bar{U}, l_1) & \xrightarrow{\ \varphi_1\ } & C_*(K, l) \\
\scriptstyle b \big\downarrow & \diagdown \scriptstyle (I) & \big\uparrow \scriptstyle \chi & \scriptstyle (II) & \big\uparrow \scriptstyle \chi \\
& \scriptstyle b \diagdown & & & \\
C_*(\bar{U}, m) & \xrightarrow[\ b\ ]{} & C_*(\bar{U}, m_1) & \xrightarrow[\ \psi_1\ ]{} & C_*(K, m)
\end{array}
$$

Diagram (I) is commutative, and diagram (II) is homotopy commutative with a small homotopy D. Thus, φ and $\chi \psi b$ are homotopic with the homotopy $D_1 := Db$.

We prove that D_1 satisfies the boundary condition of an I-system:

Assuming that D_1 does not satisfy the boundary condition we have that there exists a simplex $\sigma \in \tau^l$ such that

$$st(\sigma, l) \cap \partial U \neq \varnothing \quad \text{and}$$
$$st(\sigma, l) \cap \operatorname{carr} D_1 \sigma \neq \varnothing \, .$$

Hence, there is a simplex $\sigma_1 \in \tau^l$ such that

$$\sigma_1 \subset st(\sigma, l) \quad \text{and} \quad \sigma_1 \subset \operatorname{carr} D_1 \sigma \, .$$

Let σ_2 be a simplex in τ^{m_1} such that

$$\sigma_2 \in b\sigma \quad \text{and} \quad \sigma_1 \subset \operatorname{carr} D\sigma_2 \, .$$

Since the homotopy D is small, there is a point $z(\sigma_2) \in K$ such that

$$\sigma_2 \subset st(z(\sigma_2), k) \quad \text{and}$$
$$\operatorname{carr} D\sigma_2 \subset st(\Phi(z(\sigma_2)), k) \, .$$

Hence,

$$st(\sigma, l) \cap st(\Phi(z(\sigma_2)), k) \neq \varnothing \, .$$

Because of $\sigma_2 \subset \sigma$, and, thus, $st^2(z(\sigma_2), k) \supset \sigma$, we see that

$$st^3(z(\sigma_2), k) \cap st(\Phi(z(\sigma_2)), k) \neq \varnothing$$

and

$$st^3(z(\sigma_2), k) \cap \partial U \neq \varnothing$$

which is impossible because of the choice of k. Thus, D_1 is admissible.

That $\psi b \chi |_{\bar{U}}$ and ψ are admissible homotopic is proved in a similar way.

2. Fixed point index for multivalued mappings.

DEFINITION 2.6.   Let $U \subset K$ be open and polyhedral, and let $\Phi \colon K \to K$ be u.s.c.. Let (K, Φ, U) be admissible, and let $A = A(\Phi)$ be an A-system for Φ.   Then

$$I_A(K, \Phi, U) := I(M)(K, U; A(\Phi)) \in F$$

is called the **fixed point index of Φ in U** (with resp. to A).

In the following proposition we list some properties of the fixed point index.

LEMMA 2.7.   *Let* $\Phi_1 \colon K \to L$ *and* $\Phi_2 \colon L \to K$ *be u.s.c., and let* $A(\Phi_1), A(\Phi_2)$ *be A-systems for Φ_1 and Φ_2. Let $C \subset K$ be a closed subset of K such that for all $x \in C, x \notin \Phi_2\Phi_1(x)$.*
There is an integer $l_0 \in N$ such that for all $l \geq l_0$ and for all $\varphi \in A(\Phi_1)_l, \psi \in A(\Phi_2)_l$ the following property holds:
For any simplex $\sigma \in \tau^l$ with $\sigma \subset st(C, l_0)$ and for any simplex $\bar{\sigma} \in \varphi(\sigma)$

$$\sigma \notin \psi(\bar{\sigma}) \ .$$

Proof.   Without loss of generality we assume $n(A(\Phi_1)) = n(A(\Phi_2)) = 1$.
Since Φ_1 and Φ_2 are u.s.c., and since C is compact, there is $k \in N$ such that

$$st(\Phi_2(st^2(\Phi_1(st(\sigma, k)), k)), k) \cap st(\sigma, k) = \varnothing$$

for all simplices $\sigma \subset st(C, k)$.
For $j = k$ we find integers $j_1 = j_1(A(\Phi_1)), j_1' = j_1(A(\Phi_2))$.   Then the lemma follows with $l_0 := \max\{j_1, j_1'\}$.

PROPOSITION 2.8 (*Additivity*).   *Let $U \subset K$ be open and polyhedral, and let $\Phi \colon K \to K$ be u.s.c..*
Let $U_1, U_2 \subset U$ be open, disjoint subsets of U which are polyhedral.
Assume that the fixed point set of $\Phi|_{\bar{U}}$ is contained in $U_1 \cup U_2$,

$$\mathrm{Fix}(\Phi|_{\bar{U}}) \subset U_1 \cup U_2 \ .$$

Let $A = A(\Phi)$ be an A-system for Φ.

Then

$$I_A(K, \Phi, U) = I_A(K, \Phi, U_1) + I_A(K, \Phi, U_2) .$$

Proof. The proof follows immediately from 2.7 and 1.3.

Beside the ADDITIVITY we obtain (with the same arguments) an excision property which allows to define the fixed point index on arbitrary open sets.

PROPOSITION 2.9 (*Excision*). *Let* $U_1 \subset U \subset K$ *be open and polyhedral, and let* $\Phi: K \to K$ *be u.s.c.. Let* $A = A(\Phi)$ *be an A-system for* Φ. *If* $\mathrm{Fix}(\Phi \mid_{\bar{U}}) \subset U_1$, *then*

$$I_A(K, \Phi, U) = I_A(K, \Phi, U_1) .$$

DEFINITION 2.10. Let $V \subset K$ be open, and let $\Phi: K \to K$ be u.s.c..
Let (K, Φ, V) be admissible, and let $U \subset V$ be open and polyhedral such that $\mathrm{Fix}(\Phi \mid_{\bar{V}}) \subset U$. Let $A = A(\Phi)$ be an A-system for Φ.

$$I_A(K, \Phi, V) := I_A(K, \Phi, U) .$$

Observe that ADDITIVITY and EXCISION generalize immediately to this fixed point index on open sets.

EXAMPLE 2.11. Let $f: K \to K$ be continuous and single-valued, and let $A^*(f)$ be the A-system induced by f, see 2.4. Then Definition 2.10 of fixed point index coincides with the definition given by O'Neill in [34]:

$$I_{A^*}(K, f, V) = i(K, f, V_1) .$$

3. Homotopy of *A*-systems.

DEFINITION 2.12. Let $U \subset K$ be open and polyhedral, and let $\Phi_1, \Phi_2: K \to L$ be u.s.c.. Let $H: \bar{U} \times I \to L$ be an u.s.c. homotopy joining $\Phi_1 \mid_{\bar{U}}$ and $\Phi_2 \mid_{\bar{U}}$. $(I = [0, 1])$ Let $A(\Phi_1)$, $A(\Phi_2)$ be A-systems for Φ_1, Φ_2. $A(\Phi_1)$ and $A(\Phi_2)$ are called **H-homotopic** provided there is an integer $m \in N$ such that the following property is satisfied:
For all $j \in N$ there is an integer $j_2 \in N$ such that for all $l \geqq j_2$ there are maps

$$\varphi = \varphi_1 b \in A(\Phi_1)_l , \quad \psi = \psi_1 b \in A(\Phi_2)_l ,$$

where

$$\varphi_1, \psi_1: C_*(K, l_1) \longrightarrow C_*(K, l) \quad (l_1 \geqq l)$$

such that

$$\bar{\varphi}_1 := \varphi_1|_{C_*(\bar{U},l_1)} \quad \text{and} \quad \bar{\psi}_1 := \psi_1|_{C_*(\bar{U},l_1)}$$

are homotopic with an homotopy D satisfying the following condition, for any simplex $\sigma \in \tau^{l_1}$ there is a point $d(\sigma) \in K$ such that

$$\sigma \subset st^m(d(\sigma), j)$$
$$\text{carr } D\sigma \subset st^m(H(d(\sigma) \times I), j) .$$

EXAMPLE 2.13. Let $f_1, f_2: K \to L$ be continuous and single-valued, and let $A^*(f_1)$, $A^*(f_2)$ be the A-systems induced by f_1, f_2. Let $U \subset K$ be open and polyhedral, and let $H: \bar{U} \times I \to L$ be a homotopy joining $f_1|_{\bar{U}}$ and $f_2|_{\bar{U}}$. Then, $A^*(f_1)$ and $A^*(f_2)$ are H-homotopic.

Proof. Define

$$K_1 := K \times \{0\} \cup \bar{U} \times I \cup K \times \{1\}$$
$$\tau_1 := \tau \times \{0\} \cup (\tau \times I) \cup \tau \times \{1\} .$$

(By $\tau \times I$ we mean a product triangulation $\tau \times \rho$, where ρ is a triangulation of I.)
Let $H_1: K_1 \to L$ be the extension of H with

$$H_1|_{K \times \{0\}} = f_1 , \quad H_1|_{K \times \{1\}} = f_2 .$$

Let $j \in N$. Set $j_2 := j$.
Choose $H_1'b \in A^*(H_1)_l (l \geqq j_2)$, and set

$$\varphi := H_1'b|_{C_*(K \times \{0\})} , \quad \psi := H_1'b|_{C_*(K \times \{1\})} ,$$

and $D\sigma := H_1'b \ (\sigma \times I) \ (\sigma \times I$ chain in $(\tau \times I)^l)$. Thus, $A^*(f_1)$ and $A^*(f_2)$ are H-homotopic with $m = 2$.

LEMMA 2.14. *Let $U \subset K$ be open and polyhedral, and let Φ_1, Φ_2: $K \to K$ be u.s.c.. Let $A(\Phi_1)$, $A(\Phi_2)$ be A-systems for Φ_1, Φ_2 which are H-homotopic.*

If (K, H_t, U) is admissible for all $t \in I$, then the induced I-systems $M(K, U; A(\Phi_1))$ and $M(K, U; A(\Phi_2))$ are homotopic $(H_t(x) := H(x, t))$.

Proof. We assume without loss of generality $n(A(\Phi_1)) = n(A(\Phi_2)) = 1$. Choose $k \in N$ sufficiently large such that for all $(z, t) \in \bar{U} \times I$,

$$st^3(z, k) \cap \partial U \neq \varnothing \Longrightarrow st^3(z, k) \cap st(H(z, t), k) = \varnothing .$$

For $j = k$ we obtain from 2.12 an integer j_2 such that for all $l \geqq j_2$ there are maps

$$\varphi = \varphi_1 b \in A(\Phi_1)_l \ , \quad \psi = \psi_1 b \in A(\Phi_2)_l$$

such that the restrictions $\bar{\varphi}_1$ and $\bar{\psi}_1$ are homotopic with an homotopy D satisfying the properties described in 2.12.

Thus, the restrictions $\bar{\varphi}$ and $\bar{\psi}$ are homotopic with Db.

The proof that $\bar{\varphi}$ and $\bar{\psi}$ are admissible homotopic is similar to the proof of 2.5, and is therefore omitted.

PROPOSITION 2.15 (*Homotopy Invariance*). *Let* $V \subset K$ *be open, and let* $\Phi_1, \Phi_2 \colon K \to K$ *be u.s.c.. Let* $H \colon \bar{V} \times I \to K$ *be an u.s.c. homotopy joining* $\Phi_1|_{\bar{V}}$ *and* $\Phi_2|_{\bar{V}}$*, such that* (K, H_t, V) *is admissible for all* $t \in I$*. Let* $A_1 = A(\Phi_1)$*,* $A_2 = A(\Phi_2)$ *be A-systems for* Φ_1, Φ_2*, which are H-homotopic on* U*, where* $U \subset V$ *is an open, polyhedral subset such that* $\bar{V} \setminus U$ *contains no fixed points of* H_t*,* $t \in I$*.*
Then,

$$I_{A_1}(K, \Phi_1, V) = I_{A_2}(K, \Phi_2, V) \ .$$

Proof. The proof follows immediately from 2.14 and 1.5.

4. Commutativity and mod-p property.

LEMMA 2.16. *Let* $\Phi_1 \colon K \to L$*,* $\Phi_2 \colon L \to M$ *be u.s.c., and let* $A(\Phi_1)$*,* $A(\Phi_2)$ *be A-systems for* Φ_1, Φ_2*. Then* $A \colon = A(\Phi_2) \circ A(\Phi_1)$ *is an A-system for* $\Phi_2 \Phi_1$*.*

Proof. We assume without loss of generality

$$n(A(\Phi_1)) = n(A(\Phi_2)) = 1 \ .$$

Let $j \in N$. Then for every $y \in K$ there is $j(y) \in N$ such that

$$\Phi_2(st^3(\Phi_1(y), j(y))) \subset st(\Phi_2\Phi_1(y), j) \ .$$

Since Φ_1 is u.s.c. and K is compact we find an integer $j' \in N$, $j' \geqq j$, such that for every $y \in K$ there is a point $u = u(y) \in K$ such that

$$y \in st(u, j)$$
$$st^2(\Phi_1(y), j') \subset st^3(\Phi_1(u), j(u)) \ ,$$

and therefore,

$$\Phi_2(st^2(\Phi_1(y), j')) \subset st(\Phi_2\Phi_1(u), j) \ .$$

For j' we obtain by 2.3 an integer $j_1 = j_1'(A(\Phi_1))$ and an integer $j_2 = j_1'(A(\Phi_2))$. Let $l, m \geqq \max\{j_1, j_2\}$.

$$C_*(K, l_2) \xrightarrow{\varphi_1} C_*(L, l) \xrightarrow{b} C_*(L, l_1) \xrightarrow{\varphi_1'} C_*(M, l)$$

$$\Big\uparrow \chi \quad \text{(I)} \quad \Big\uparrow \chi \quad \text{(II)} \quad \Big\uparrow \chi \quad \text{(III)} \quad \Big\uparrow \chi$$

$$C_*(K, m_2) \xrightarrow{\psi_1} C_*(L, m) \xrightarrow{b} C_*(L, m_1) \xrightarrow{\psi_1'} C_*(M, m)$$

$$\varphi = \varphi_1 b \in A(\Phi_1)_l , \quad \varphi' = \varphi_1' b \in A(\Phi_2)_l$$
$$\psi = \psi_1 b \in A(\Phi_1)_m , \quad \psi' = \psi_1' b \in A(\Phi_2)_m .$$

We show that $\varphi_1' b \varphi_1 \chi$ and $\chi \psi_1' b \psi_1$ are homotopic with a small homotopy.

Diagram (I) is homotopic commutative with a small homotopy D_1.

Diagram (III) is homotopic commutative with a small homotopy D_3.

Diagram (II) is homotopic commutative with a homotopy D_2 satisfying: carr $D_2 \sigma \subset$ carr $\chi(\sigma)$, for $\sigma \in \mu^m$.

Hence, $\varphi' \varphi_1 \chi$ and $\chi \psi' \psi_1$ are homotopic with the homotopy

$$D := \Delta_1 + \Delta_2 + \Delta_3, \quad \text{where} \quad \Delta_1 = \varphi_1' b D_1, \ \Delta_2 = \varphi_1' D_2 \psi_1$$
$$\text{and} \quad \Delta_3 = D_3 b \psi_1 .$$

We prove the smallness condition for Δ_1 (the others being similar):

For a simplex $\sigma \in \tau^{m_2}$ by 2.3 there is a point $z_1(\sigma) \in K$ such that

$$\sigma \subset st(z_1(\sigma), j')$$
$$\text{carr } D_1 \sigma \subset st(\Phi_1(z_1(\sigma)), j') .$$

By choice of j' for the point $z_1(\sigma)$ we can find a point $u = u(z_1(\sigma))$ such that

$$z_1(\sigma) \in st(u, j)$$
$$\Phi_2(st^2(\Phi_1(z_1(\sigma)), j')^2 \subset st(\Phi_2\Phi_1(u), j) .$$

Define $z(\sigma) := u(z_1(\sigma))$.

Then a straightforward calculation gives

$$\sigma \subset st^2(z(\sigma), j)$$
$$\text{carr } \Delta_1(\sigma) \subset st^2(\Phi_2\Phi_1(z(\sigma)), j) .$$

Thus, D satisfies the smallness condition, and A is an A-system for $\Phi_2\Phi_1$ with $n(A) = 2$.

PROPOSITION 2.17 (Commutativity). Let $W \subset K$ be open, and let $\Phi_1 \colon K \to L$, $\Phi_2 \colon L \to K$ be u.s.c.. Let $A_1 = A(\Phi_1)$, $A_2 = A(\Phi_2)$ be A-systems for Φ_1, Φ_2. Assume that $(K, \Phi_2\Phi_1, W)$ and $(L, \Phi_1\Phi_2, \Phi_2^{-1}(W))$ are admissible, and that the following condition is satisfied:

$$y \in \mathrm{Fix}(\varPhi_1 \varPhi_2) \backslash \overline{\varPhi_2^{-1}(W)} \implies \varPhi_2(y) \cap \mathrm{Fix}(\varPhi_2 \varPhi_1 |_W) = \varnothing \ .$$

Then

$$I_{A_1 A_2}(L, \ \varPhi_1 \varPhi_2, \ \varPhi_2^{-1}(W)) = I_{A_2 A_1}(K, \ \varPhi_2 \varPhi_1, \ W) \ .$$

Proof. We assume without loss of generality $n(A_1) = n(A_2) = 1$. Define

$$B_1 := \varPhi_2^{-1}(W) \cap \mathrm{Fix}(\varPhi_1 \varPhi_2)$$
$$B_2 := \mathrm{Fix}(\varPhi_1 \varPhi_2) \backslash \varPhi_2^{-1}(W)$$
$$B_3 := \mathrm{Fix}(\varPhi_2 \varPhi_1) \cap W \ .$$

There is an integer $k_1 \in N$ such that $st^2(\varPhi_2(B_1), k_1) \subset W$. Let $\mathscr{O} \subset L$ be an open set such that

$$B_1 \subset \mathscr{O} \ , \quad \varPhi_2(\mathscr{O}) \subset St(\varPhi_2(B_1), k_1) \ .$$

There is an integer $l_1 \in N$ such that $st^2(B_1, l_1) \subset \mathscr{O}$. Define

$$V := \mathrm{int} \ st(B_1, l_1) \ .$$

Since $\varPhi_2(B_2) \cap B_3 = \varnothing$, there is $k_2 \geqq k_1$ such that

$$st^2(\varPhi_2(B_2), k_2) \cap st(B_3, k_2) = \varnothing \ .$$

Define

$$U := \mathrm{int} \ st(B_3, k_2) \subset W \ .$$

There is an open set $\mathscr{O}_1 \supset B_2$ such that

$$\mathscr{O}_1 \cap \overline{\varPhi_2^{-1}(W)} = \varnothing, \ \varPhi_2(\mathscr{O}_1) \subset st(\varPhi_2(B_2), k_2) \ .$$

Therefore, there exists $l_2 \in N$, $l_2 \geqq l_1$ such that

$$st^2(B_2, l_2) \subset \mathscr{O}_1 \ .$$

Define $V_1 := st(B_2, l_2)$, $V_2 := L \backslash V_1$.

Since for all $x \in C := \overline{W \backslash U}$, $x \notin \varPhi_2 \varPhi_1(x)$ we obtain from 2.7 an integer l_0, and since for all $x \in C' := \overline{V_2 \backslash V}$, $x \notin \varPhi_1 \varPhi_2(x)$, we obtain from 2.7 an integer l_0' ($l_0 \geqq k_2$, $l_0' \geqq l_2$). Set $j := \max\{l_0, l_0'\}$, and define $\tilde{k} := \max\{j_1(A_1), j_1(A_2)\}$. Thus, the assumption of 1.6 is satisfied, and we obtain

$$I_{A_2 A_1}(K, \ \varPhi_2 \varPhi_1, \ U) = I_{A_1 A_2}(L, \ \varPhi_1 \varPhi_2, \ V) \ .$$

EXAMPLE 2.18. Let $W \subset K$ be open, and let $f: K \to L$, $g: L \to K$

be continuous and single-valued such that (K, gf, W) is admissible. Then

$$i(K, gf, W) = i(L, fg, g^{-1}(W)) \, .$$

Proof. Let $A_1 = A^*(f)$, $A_2 = A^*(g)$ be the A-systems induced by f and g. Then by 2.17

$$I_{A_2 A_1}(K, gf, U) = I_{A_1 A_2}(L, fg, V) \, .$$

(U and V are chosen as in the proof of 2.17.) Since $A^*(gf)_k \cap (A_2 \circ A_1)_k \neq \varnothing$ and $A^*(fg)_k \cap (A_1 \circ A_2)_k \neq \varnothing$ for all $k \in N$, this implies

$$i(K, gf, W) = i(L, fg, g^{-1}(W)) \, .$$

PROPOSITION 2.19 (Mod-p *Property*). *Let* $F = Z_p$, p *prime. Let* $W \subset K$ *be open, and let* $\Phi: K \to K$ *be u.s.c., such that* (K, Φ, W) *and* (K, Φ^p, W) *are admissible. Assume the following condition is satisfied:*

$$y \in \mathrm{Fix}(\Phi^p) \backslash W \Longrightarrow \Phi^k(y) \cap \mathrm{Fix}(\Phi^p|_{\overline{W}}) = \varnothing \ \text{ for } 1 \leqq k < p \, .$$

Let $A = A(\Phi)$ *be an* A-*system for* Φ. *Then*

$$I_A(K, \Phi, W) = I_{A^p}(K, \Phi^p, W) \quad (\text{in } F)$$

$(A^p = A \circ \cdots \circ A, p \text{ times})$.

Proof. Without loss of generality we assume $n(A) = 1$. Moreover, we give a proof only for $p = 2$, for $p > 2$ the proof runs along the same lines.
Define $B_1 := \mathrm{Fix}(\Phi^2) \cap W$, $B_2 := \mathrm{Fix}(\Phi^2) \backslash W$.
Since $\Phi(B_2) \cap B_1 = \varnothing$, there exists $k_1 \in N$ such that

$$st^2(\Phi(B_2), k_1) \cap st(B_1, k_1) = \varnothing \, .$$

Define

$$U := \mathrm{int}\, st(B_1, k_1)$$

Since Φ is u.s.c. there is an open set $\mathscr{O} \supset B_2$, $\mathscr{O} \subset K \backslash W$, such that

$$\Phi(\mathscr{O}) \subset st(\Phi(B_2), k_1) \, .$$

There exists $k_2 \geqq k_1$ such that $st^2(B_2, k_2) \subset \mathscr{O}$. Define

$$V_1 := st(B_2, k_2) \, .$$

Since for all $x \in C$: $= \overline{K \backslash V_1 \backslash U}$, $x \notin \Phi^2(x)$), we obtain from 2.7 an integer $l_0 \geqq k_2$. Set $j = l_0$ and define

$$\bar{k}: = j_1(A) .$$

Then the assumptions of 1.7 are satisfied, and we obtain

$$I_A(K, \Phi, U) = I_{A^2}(K, \Phi^2, U) \quad (\text{in } Z_2) .$$

EXAMPLE 2.20. Let $W \subset K$ be open, and let $f: K \to K$ be continuous and single-valued. Let (K, f, W) and (K, f^p, W) be admissible, p prime, and assume that for all $x \in \text{Fix}(f^p|_W)$,

$$x, f(x), \cdots, f^{p-1}(x) \in W .$$

Then

$$i(K, f, W) = i(K, f^p, W) \quad (\text{in } Z_p) .$$

Proof. Let $A = A^*(f)$ be the A-system induced by f. Then, by 2.19

$$I_A(K, f, W) = I_{A^p}(K, f^p, W) .$$

Since

$$(A^*(f))_k^p \cap A^*(f^p)_k \neq \varnothing \quad \text{for all} \quad k \in N ,$$

this implies

$$i(K, f, W) = i(K, f^p, W) \quad (\text{in } Z_p) .$$

REMARK 2.21. (1) The mod-p result of 2.20 was announced in 1971 by Zabreĭko and Krasnosel'skiĭ, however they gave only a sketch of proof [45]. A first complete proof (independent of [45]) was presented by Steinlein in [40]. Steinleins proof is of analytic nature and rather complicated. In the meantime, several proofs for the mod-p result (for single-valued maps) appeared [16], [25], [41], [42], [43], but all these proofs are based on certain transversality arguments. Thus, they can not be generalized to multivalued mappings.

(2) The essential ingredients for our proofs in Chapter II are a "simplicial structure" of the spaces and barycentric subdivision. These ingredients were axiomatized by Browder [5] in order to define 'semi-complexes'. Thus, our results should generalize to Browder's category of semi-complexes.

III. **Existence of A-systems.** In this chapter we construct A-

systems for certain classes of multivalued mappings $\Phi\colon K \to L$. It turns out that the existence of an A-system for acyclic maps is— implicitly—contained in Vietoris' (and Begle's) proof for the Vietoris-Begle mapping theorem [1], [44]. Also Eilenberg and Montgomery used chain approximations in their proof of the Eilenberg-Montgomery fixed point theorem [12].

In [35] O'Neill gave a construction of chain approximations for continuous (=upper- and lower semi continuous) multivalued mappings Φ such that $\Phi(x)$ is acyclic or consists of n acyclic components (n fixed) for all x. Moreover, he proved an "abstract" Lefschetz fixed point theorem for this class of mappings. Our approach of constructing A-systems is similar to O'Neills approach in [35]. However, we do not need continuity but upper semi continuity for the mapping. The basic ingredients for constructing A-systems are based more or less on Vietoris' work.

0. Preliminaries. In this paragraph H_* will denote Čech homology with coefficients in a field F, [13].

Let X, Y be compact Hausdorff spaces.

DEFINITION 3.1. Let $\Phi\colon X \to Y$ be a multivalued map. Φ is called **acyclic** (more precisely, F-acylic) provided

(i) Φ is u.s.c..

(ii) $\widetilde{H}_*(\Phi(x)) = 0$ for all $x \in X$ ($\widetilde{H}$ = reduced homology).

DEFINITION 3.2. Let $p\colon X \to Y$ be continuous. p is called *Vietoris mapping* (more precisely, F-Vietoris mapping) provided

(i) p is onto

(ii) $p^{-1}(y)$ is acyclic for all $y \in Y$.

REMARK 3.3. Let $\Phi\colon X \to Y$ be acyclic, let

$$\Gamma(\Phi)\colon = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}$$

be the graph of Φ, and let

$$X \xleftarrow{\ p\ } \Gamma(\Phi) \xrightarrow{\ q\ } Y$$
$$x \longleftarrow\!\!\!\mid (x, y) \longmapsto y$$

be the natural projections.
Then, $\Gamma(\Phi)$ is compact, and p is a Vietoris map.

DEFINITION 3.4. Let $\Phi\colon X \to Y$ be acyclic, and let p, q be the natural projections.

Because of the Vietoris-Begle mapping theorem [1], [44] the map $p_*\colon H_*(\Gamma(\Phi)) \to H_*(X)$ is an isomorphism.

The map

$$\Phi_*: = q_* p_*^{-1}: H_*(X) \longrightarrow H_*(Y)$$

is called the **induced homomorphism of** Φ.

1. Acyclic mappings and A-systems. In the following (K, τ) and (L, μ) will always denote compact polyhedra with triangulations τ and μ.

LEMMA AND DEFINITION 3.5. *Let* $\Phi: K \to L$ *be acyclic, and let* $l \in N$. *Then, there exists an integer* $l_0 \geqq l$ *and a chain map ("l-chain approximation of* Φ*")*

$$\varphi_1: C_*(K, l_0) \longrightarrow C_*(L, l)$$

with the following properties:

(1) φ_1 *is augmentation preserving, i.e.,* φ_1 *maps a vertex into a vertex.*

(2) *For any simplex* $\sigma \in \tau^{l_0}$ *there is a point* $p(\sigma) \in K$ *such that*

$$\sigma \subset st^2(p(\sigma), l)$$
$$\text{carr } \varphi_1 \sigma \subset st(\Phi(p(\sigma)), l) .$$

Any chain map satisfying (1) *and* (2) *is called an* **l-chain approximation of** Φ.

A proof of 3.5 will be given in the appendix.

REMARK 3.6. In analogy with simplicial approximations of single-valued mappings chain approximations of Φ are not unique.

DEFINITION 3.7. Let $\Phi: K \to L$ be acyclic. Then

$$A^*(\Phi): = \{A^*(\Phi)_j\}_{j \in N}$$

with $A^*(\Phi)_j: = \{\varphi = \varphi_1 b \mid \varphi_1 \ j\text{-chain approximation of } \Phi\}$ is called the **A-system induced by** Φ.

The following lemma justifies the definition above, i.e., it proves that $A^*(\Phi)$ is actually an A-system.

LEMMA 3.8. *Let* $\Phi: K \to L$ *be acyclic. For all* $j \in N$ *there is an integer* j_1 *such that for all* $l, m \geq j_1 (m \geq l)$ *and for all* $\varphi = \varphi_1 b \in A^*(\Phi)_l$, $\psi = \psi_1 b \in A^*(\Phi)_m$

$$\varphi_1 \chi \text{ and } \chi \psi_1 \text{ are homotopic with a small homotopy },$$

i.e.,

the diagram

$$\begin{array}{ccc} C_*(K, l_1) & \xrightarrow{\varphi_1} & C_*(L, l) \\ \Big\uparrow\chi & & \Big\uparrow\chi \\ C_*(K, m_1) & \xrightarrow[\psi_1]{} & C_*(L, m) \end{array}$$

is homotopy commutative with a chain homotopy D satisfying the following smallness condition,
for any simplex $\sigma \in \tau^{m_1}$ there is a point $c(\sigma) \in K$ such that

$$\sigma \subset st^3(c(\sigma), j)$$
$$\operatorname{carr} D\sigma \subset st(\Phi(c(\sigma)), j) \,.$$

Lemma 3.8 will be proved in the appendix.
The following lemma is proved in the same way as 3.8.

LEMMA 3.8'. *Let $\Phi\colon K \to L$ be acyclic. For all $j \in N$ there is an integer j_1 such that for all $l, m \geqq j_1 (m \geqq l)$ the following holds:*

If φ_1 is an $n - l$-approximation (see (2.1)) of Φ, and if ψ_1 is an $k - m$-approximation of Φ, and if φ_1 and ψ_1 are augmentation preserving, then

$\varphi_1\chi$ *and* $\chi\psi_1$ *are homotopic with a small homotopy D ,*

i.e., there is an integer $i \in N$ such that for any simplex $\sigma \in \tau^{m_1}$ there is a point $c(\sigma) \in K$ such that

$$\sigma \subset st^i(c(\sigma), j)$$
$$\operatorname{carr} D\sigma \subset st^i(\Phi(c(\sigma)), j) \,.$$

In view of the normalization property the following corollary is important.

COROLLARY 3.9. *Let $\Phi\colon K \to L$ be acyclic. There is an integer $k_0 \in N$ such that for all $k \geqq k_0$ and for all k-chain approximations φ_1 of Φ:*

$$(\varphi_1)_* = \Phi_*\colon H_*(K) \longrightarrow H_*(L) \,.$$

Proof. The proof follows from [1] and 3.8.

In the following we list some classes of multivalued mappings which have A-systems.

EXAMPLE 1. The class $\mathscr{T}$ [38] of all multivalued mappings $\Phi: K \to L$ which can be "factorized",

$$\Phi = \Phi_n \circ \cdots \circ \Phi_0 ,$$

where $\Phi_i: X_i \to X_{i+1}$ are acyclic maps of compact polyhedra ($X_0 = K$, $X_{n+1} = L$) has an A-system. (One can replace $X_1, \cdots, X_n$ by compact Hausdorff spaces.)

EXAMPLE 2. The class $\mathscr{A}$ of all 'admissible' [17] multivalued maps $\Phi: K \to L$ has an A-system.

EXAMPLE 3. The class of all u.s.c. multivalued maps $\Phi: K \to L$ for which there exists a "single-valued" approximation, i.e., for all $k \in N$ there is a continuous single-valued mapping $f_k: K \to L$ such that

$$f_k(x) \in st(\Phi(x), k) , \quad \text{for all} \quad x \in K ,$$

has an A-system, see e.g., [2], [8], [31], [32].

EXAMPLE 4. It can be shown that the class of all continuous (u.s.c. and l.s.c.) mappings $\Phi: K \to L$, where $\Phi(x)$ is acyclic or has n acyclic components (n fixed) [34], has an A-system.

REMARK 3.10. Any u.s.c. multivalued mapping for which there exists an A-system has an induced homomorphism in homology in the sense of O'Neill [34]. Because there are multivalued mappings which do not have an induced homomorphism, see [34], there are multivalued mappings which do not have an A-system.

IV. Fixed point index for acyclic maps. In this chapter we apply the results of Chapters II and III to define a fixed point index for acyclic maps. In the following (K, τ) and (L, μ) will always denote compact polyhedra with triangulations τ and μ.

1. Definition of fixed point index. Call an A-system for an u.s.c. mapping $\Phi: K \to L$ *regular* if all $\varphi \in A(\Phi)_k$, $k \in N$, are augmentation preserving.

PROPOSITION 4.1. *Let* $U \subset K$ *be open and polyhedral, and let* $\Phi: K \to K$ *be acyclic. Let* (K, Φ, U) *be admissible. Let further* $A_1 = A_1(\Phi)$, $A_2 = A_2(\Phi)$ *be two regular A-systems for* Φ.
Then, the induced I-systems $M(K, U; A_1)$ *and* $M(K, U; A_2)$ *are homotopic.*

Proof. Follows from 3.8′.

DEFINITION 4.2. Let $U \subset K$ be open, and let $\Phi: K \to K$ be acyclic such that (K, Φ, U) is admissible. Then

$$i(K, \Phi, U): = I_A(K, \Phi, U) \in F,$$

where $A = A(\Phi)$ is any regular A-system for Φ, is called the **fixed point index of Φ on U.** Because of 4.1 and 1.5 the definition of $i(K, \Phi, U)$ is independent of the choice of $A(\Phi)$.

2. Properties of the fixed point index. In order to obtain homotopy invariance for our fixed point index we need a lemma.

LEMMA 4.3. *Let $U \subset K$ be open and polyhedral. Let $\Phi_1, \Phi_2: K \to L$ be acyclic, and assume that $\Phi_1|_{\bar{v}}$ and $\Phi_2|_{\bar{v}}$ are homotopic with an acyclic homotopy $F: \bar{U} \times I \to L$. Then, for all $l \in N$ there is $l_0 \in N$ such that for all $k \geq l_0$ there are maps $\varphi = \varphi_1 b \in A^*(\Phi_1)_k$, $\psi = \psi_1 b \in A^*(\Phi_2)_k$,*

$$\varphi_1, \psi_1: C_*(K, k_0) \longrightarrow C_*(L, k)$$

such that $\bar{\varphi}_1: = \varphi_1|_{C_*(\bar{U}, k_0)}$ and $\bar{\psi}_1: = \psi_1|_{C_*(\bar{U}, k_0)}$ are homotopic with a chain homotopy D satisfying the following property,
for any simplex $\sigma \in \tau^{k_0}$, $\sigma \subset \bar{U}$ there is a point $d(\sigma) \in K$ such that

$$\sigma \subset st^2(d(\sigma), l)$$
$$\text{carr } D\sigma \subset st^2(F(d(\sigma) \times I), l).$$

Proof. Let $l \in N$. For $x \in \bar{U}$ there is an open neighborhood $\mathcal{O}_x$ such that $F(\mathcal{O}_x \times I) \subset st(F(x \times I), l)$. We can assume that the covering $\{\mathcal{O}_x\}_{x \in \bar{U}}$ is a refinement of the star covering $st(\tau^l)$. Let $\omega_1: = \{\mathcal{O}_{x_i}\}_{i=1}^s$ be a finite subcovering of $\bar{U}$.
Then there is $l_0 \in N$ such that

$$\{\text{int } st^2(\sigma, l_0) \,|\, \sigma \in \tau^{l_0}\}$$

is a refinement of ω_1.
Set

$$K_1 = K \times \{0\} \cup \bar{U} \times I \cup K \times \{1\}$$
$$\tau_1: = \tau \times \{0\} \cup \tau \times I \cup \tau \times \{1\},$$

and let $F_1: K_1 \to L$ be the extension of F such that $F_1|_{K \times \{0\}} = \Phi_1$, $F_1|_{K \times \{1\}} = \Phi_2$.
Choose $k \geq l_0$.

Because of 3.5 there is a k-chain approximation of F_1,

$$\vartheta_1 \colon C_*(K_1, k_0) \longrightarrow C_*(L, k) \ .$$

Define $\varphi_1 := \vartheta_1|_{C_*(K \times \{0\}, k_0)}$, $\psi_1 := \vartheta_1|_{C_*(K \times \{1\}, k_0)}$, ($\vartheta_1$ can be constructed in such a way that φ_1, ψ_1 are k-chain approximations of Φ_1, Φ_2) and define $D\sigma := \vartheta_1(\sigma \times I)$, where $\sigma \times I$ is a chain in $(\tau \times I)^{k_0}$.
Now let $\sigma \in \tau^{k_0}$, $\sigma \subset \bar{U}$. Write $\sigma \times I = \sum \sigma_i$ (Chain in $(\tau \times I)^{k_0}$).

Since ϑ_1 is a k-chain approximation, for any σ_i there is a point $p(\sigma_i) \in \bar{U} \times I$ such that

$$\sigma_i \subset st^2(p(\sigma_i), k)$$
$$\operatorname{carr} \vartheta_1 \sigma_i \subset st(F(p(\sigma_i)), k) \ .$$

$(p(\sigma_i) = (p_i, t_i), t_i \in I.)$

Denote by $\bar{\sigma}$ the simplex in τ^{l_0} such that $\sigma \subset \bar{\sigma}$. There is an open set $\mathscr{O}_x \in \omega_1$ such that $st^2(\bar{\sigma}, l_0) \subset \mathscr{O}_x$. Therefore,

$$F(st^2(\sigma, k) \times I) \subset st(F(x \times I), l) \ .$$

Because of $p_i \in st^2(\sigma, k)$, $\operatorname{carr} \vartheta_1 \sigma_i \subset st^2(F(x \times I), l)$. Thus, $\operatorname{carr} D\sigma \subset st^2(F(x \times I), l)$.
Set $d(\sigma) := x$, and the proof is finished.

THEOREM 4.4. *The fixed point index '$i(K, \Phi, U)$' for acyclic maps has the following properties:*

I. HOMOTOPY INVARIANCE: *Let $F \colon \bar{U} \times I \to K$ be an acyclic homotopy joining $\Phi_1|_{\bar{U}}$ and $\Phi_2|_{\bar{U}}$ such that (K, F_t, U) is admissible, $t \in I$. Then*

$$i(K, \Phi_1, U) = i(K, \Phi_2, U) \ .$$

II. ADDITIVITY. *Let $U_1, U_2 \subset U$ be open, disjoint subsets of U. If $\operatorname{Fix}(\Phi|_{\bar{U}}) \subset U_1 \cup U_2$, then*

$$i(K, \Phi, U) = i(K, \Phi, U_1) + i(K, \Phi, U_2) \ .$$

III. NORMALIZATION. $i(K, \Phi, K) = \Lambda(\Phi_* \colon H_*(K) \to H_*(K))$.

DEFINITION 4.5. Let $X_0 = K, X_1, \cdots, X_{n+1} = K$ be compact polyhedra, and let $\Phi_i \colon X_i \to X_{i+1}$, $i = 0, \cdots, n$ be acyclic. Let $U \subset K$ be open, and let (K, Φ, U) be admissible where $\Phi = \Phi_n \circ \cdots \circ \Phi_0$.
Then

$$i(K, \Phi, U) := I_A(K, \Phi, U) \ ,$$

where $A = A(\Phi_n) \circ \cdots \circ A(\Phi_0)$ for regular A-systems $A(\Phi_i)$ of Φ_i, is

called the *fixed point index of Φ in U*.

Because of 4.1 the definition is independent of the choice of A-systems $A(\Phi_i)$.

COMMUTATIVITY 4.6. Let $\Phi_1 \colon K \to L$, $\Phi_2 \colon L \to K$ be acyclic, and let $(K, \Phi_2\Phi_1, U)$ and $(L, \Phi_1\Phi_2, \Phi_2^{-1}(U))$ be admissible. Assume that for all $y \in \mathrm{Fix}(\Phi_1\Phi_2) \backslash \Phi_2^{-1}(U)$,

$$\Phi_2(y) \cap \mathrm{Fix}(\Phi_2\Phi_1|_{\bar{U}}) = \varnothing \ .$$

Then

$$i(K, \Phi_2\Phi_1, U) = i(L, \Phi_1\Phi_2, \Phi_2^{-1}(U)) \ .$$

COROLLARY 4.7 (*Reduction property*) [15]. *Let $U \subset K$ be open, and let $L \subset K$ be a polyhedron such that $\Phi(K) \subset L$. Let (K, Φ, U) be admissible. Then*

$$i(K, \Phi, U) = i(L, \Phi|_L, L \cap U) \ .$$

MOD-P PROPERTY 4.8. Let $p \in N$ be prime, and let (K, Φ, U) and (K, Φ^p, U) be admissible.
Assume that for all $y \in \mathrm{Fix}(\Phi^p) \backslash U$

$$\Phi^k(y) \cap \mathrm{Fix}(\Phi^p|_{\bar{U}}) = \varnothing \ , \quad 1 \leqq k < p \ .$$

Then

$$i(K, \Phi, U) = i(K, \Phi^p, U) \quad in \quad Z_p \ .$$

Proofs of 4.4–4.8. The homotopy invariance follows from 4.3, 2.14 and from 1.5. The additivity follows from 2.8. The normalization follows from 3.9 and the Hopf trace formula, e.g., [6, p. 6].

The commutativity follows from 2.17. and the mod-p property follows from 2.19.

Concluding remarks. (1) The results of the paper were proved for mappings on compact polyhedra. But, it should be possible to extend our fixed point index (including all properties) to compact metric ANR's by using retraction arguments, see [6], or by generalizing the theory to Browder's semi-complexes.*

(2) There are two natural questions which remain unsolved:

Problem I. How is the fixed point index introduced by Calvert

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* Added in proof: See G. Skordev, Fixed point index for ANR's preprint, University of Bremen, July 1981.

[7] related to the fixed point index presented here?

Since regularization techniques (e.g., the 'Hopf construction' [23], [6]) do not apply to acyclic maps, the problem can not be solved in the way known from single-valued maps.

Because of the same reason it is unclear whether fixed point index for acyclic mappings is uniquely determined by certain axioms:

Problem II. Is a fixed point index for acyclic maps (say on polyhedra) uniquely determined by the properties *Homotopy invariance, Additivity, Normalization* and *Reduction property*?

APPENDIX. In this appendix we give proofs of Lemma 3.5 and of Lemma 3.8. In order to prove Lemma 3.5 we need two further lemmas.

LEMMA A. *Let K be a compact polyhedron with triangulation τ, and let $C \subset K$ be a closed subset of K which is acyclic, $\widetilde{H}_*(C; F) = 0$ ($\widetilde{H}_* = $ reduced Čech-homology; $F = $ field).*
For $l \geq k$ let $i_{l,k}: C_l \hookrightarrow C_k$ be the inclusion, where

$$C_j := st(C, j), \quad j \in N .$$

Then for all $k \in N$ there is $k_0 \in N$ such that

$$(i_{k_0,k})_*: \widetilde{H}_*(C_{k_0}) \longrightarrow \widetilde{H}_*(C_k)$$

is trivial.

Proof. The proof follows from the continuity of Čech-homology, see [13].

LEMMA B. *Let (K, τ), (L, μ) be compact polyhedra with triangulations τ and μ, and let $\Phi: K \to L$ be acyclic. Let $k_0, n \in N$.*
There are numbers $k_1, \cdots, k_{n+1} \in N$ such that for any simplex $\sigma \in \tau^{k_i}$ there exists an integer $k_i(\sigma)$, $0 \leq i \leq n$, and there exists a point $a(\sigma) \in K$ such that
(1) $k_0 < k_1 < \cdots < k_{n+1}$

$$k_i > k_{i-1}(\sigma) \quad for \ all \quad \sigma \in \tau^{k_i} .$$

(2) *For any simplex $\sigma \in \tau^{k_i}$*

$$\sigma \subset st^2(a(\sigma), k_{i-1})$$
$$\Phi(st^3(\tau, k_i)) \subset st(\Phi(a(\sigma)), k_{i-1}(\sigma))$$

(3) $i_*\colon \tilde{H}_*(st^2(\Phi(a(\sigma))),\, k_{i-1}(\sigma)) \to \tilde{H}_*(st(\Phi(a(\sigma))),\, k_{i-1}))$ *is trivial*
($i = inclusion$).

Proof. The proof proceeds by induction on n. We give a proof only for $n = 0$.

Because of Lemma A for any $y \in K$ there is $k_0(y) \in N$ such that

$$i_*\colon \tilde{H}_*(st^2(\Phi(y)),\, k_0(y)) \longrightarrow \tilde{H}_*(st(\Phi(y),\, k_0))$$

is trivial ($i = inclusion$).

Moreover, for $y \in K$ there is a neighborhood $\mathcal{O}_y \subset K$ of y such that $\Phi(\mathcal{O}_y) \subset st(\Phi(y),\, k_0(y))$ and such that $\mathcal{O}_y$ is contained in some element of $st(\tau^{k_0})$. Let $\omega_1 = \{\mathcal{O}_{y_i}\}_{i=1}^s$ be a finite subcovering of $\omega = \{\mathcal{O}_y\}_{y \in K}$. Choose $k_1 \in N$ such that $k_1 > \max\{k_0,\, k_0(y_1),\, \cdots,\, k_0(y_s)\}$ and such that

$$\{int\, st^3(\sigma,\, k_1) \mid \sigma \in \tau^{k_1}\}$$

is a refinement of ω_1.

For $\sigma \in \tau^{k_1}$ define $a(\sigma) := y_i$, where $st^3(\sigma,\, k_1) \subset \mathcal{O}_{y_i}$, and $k_0(\sigma) := k_0(y_i)$.

Proof of Lemma 3.5. Let $l \in N$, and let $\Phi\colon K \to L$ be acyclic.

Applying Lemma B with $k_0 = l$ and $n = \dim K$ we obtain a sequence of numbers $k_0 < k_1 < \cdots < k_{n+1}$.
Define $l_0 := k_{n+1}$.
We will construct successively a chain map

$$\tilde{\varphi}_1\colon C_*(K,\, l_0) \longrightarrow C_*(L,\, l_0)$$

with the following properties:
(1) $\tilde{\varphi}_1$ is augmentation preserving
(2) For any simplex $\sigma \in \tau^{l_0}$ with $\dim \sigma = i$ there is a point $p(\sigma) \in K$ such that

$$\sigma \subset st^2(p(\sigma),\, k_{n-i})$$
$$carr\, \tilde{\varphi}_1\sigma \subset st(\Phi(p(\sigma)),\, k_{n-i})\,.$$

Composing $\tilde{\varphi}_1$ with $\chi\colon C_*(L,\, l_0) \to C_*(L,\, l)$ we obtain the desired chain approximation of Φ:

$$\varphi_1 := \chi\tilde{\varphi}_1\,.$$

Construction of $\tilde{\varphi}_1$:
$i = 0$: Let σ_0 be a vertex of τ^{l_0}. Because of Lemma B there is a point $a(\sigma_0) \in K$. Set $\tilde{\varphi}_1(\sigma_0) = a$, where a is a vertex of μ^{l_0} which is contained in $st(\Phi(a(\sigma_0)),\, k_n)$. Define $p(\sigma_0) := a(\sigma_0)$.

$i > 0$:   Assume $\widetilde{\varphi}_1$ is defined on simplices σ of dimension $i - 1$, and satisfies the conditions (1) and (2).

Let $\sigma \in \tau^{l_0}$ with $\dim \sigma = i$.

Let $\bar{\sigma} \in \tau^{k_{n+1-i}}$ such that $\dim \bar{\sigma} = i$ and $\bar{\sigma} \supset \mathcal{X}(\sigma)$.

Because of Lemma B there is a point $a(\bar{\sigma})$ and a number $k_{n-i}(\bar{\sigma})$.

Write $\partial \sigma = \sum \sigma_s (\dim \sigma_s = i - 1)$.

By assumption there are points $p(\sigma_s)$ such that

$$\sigma_s \subset st^2(p(\sigma_s), k_{n+1-i})$$
$$\mathbf{carr}\ \widetilde{\varphi}_1 \sigma_s \subset st(\Phi(p(\sigma_s)), k_{n+1-i})\ .$$

Since $\Phi(p(\sigma_s)) \subset st(\Phi(a(\bar{\sigma})), k_{n-i}(\bar{\sigma}))$

$$\mathbf{carr}\ \widetilde{\varphi}_1 \partial \sigma \subset st^2(\Phi(a(\bar{\sigma})), k_{n-i}(\bar{\sigma}))\ .$$

By our inductive assumption, the following is a cycle

$$\widetilde{\varphi}_1 \partial \sigma \in C_{i-1}(st^2(\Phi(a(\bar{\sigma})), k_{n-i}(\bar{\sigma})), l_0)\ .$$

Hence, by Lemma B there is a chain $c \in C_i(st(\Phi(a(\bar{\sigma})), k_{n-i}), l_0)$ such that $\partial c = \widetilde{\varphi}_1 \partial \sigma$.

Thus, define $\widetilde{\varphi}_1(\sigma) := c$, $p(\sigma) := a(\bar{\sigma})$.

Proof of Lemma 3.8.   Let $j \in N$, and let $\Phi : K \to L$ be acyclic. Applying Lemma B with $k_0 = j$ and $n = \dim K$ we obtain a sequence of numbers $k_0 < k_1 < \cdots < k_{n+1}$.

Define $j_1 := k_{n+1}$.

Let $m \geq l \geq j_1$, and let $\varphi = \varphi_1 b \in A^*(\Phi)_l$, $\psi = \psi_1 b \in A^*(\Phi)_m$,

$$\varphi_1 : C_*(K, l_1) \longrightarrow C_*(L, l)$$
$$\psi_1 : C_*(K, m_1) \longrightarrow C_*(L, m)\ .$$

We will construct successively a homotopy D,

$$D_i : C_i(K, m_1) \longrightarrow C_{i+1}(L, l)$$

such that

(1) $\mathcal{X}\psi_1 - \varphi_1\mathcal{X} = \partial D_i + D_{i-1}\partial$

(2)   For any simplex $\sigma \in \tau^{m_1}$, $\dim = i$, there is a point $c(\sigma) \in K$ such that

$$\sigma \subset st^3(c(\sigma), k_{n-i})$$
$$\mathbf{carr}\ D_i\sigma \subset st(\Phi(c(\sigma)), k_{n-i})\ .$$

Construction of D:

$i = 0$:   Let σ_0 be a vertex of τ^{m_1}, and set $\bar{\sigma} := \mathcal{X}(\sigma_0) \in \tau^{l_1}$, $\tilde{\sigma} := \mathcal{X}(\sigma_0) \in \tau^{j_1}$. Since φ_1 is a chain approximation we find a point $p(\bar{\sigma})$ satisfying the conditions of 3.5, and since ψ_1 is a chain approximation

we find a point $p(\sigma_0)$ satisfying the conditions of 3.5. Moreover, from Lemma B we find a point $a(\tilde{\sigma})$. $p(\bar{\sigma})$, $p(\sigma_0)$ and $a(\tilde{\sigma})$ have the following properties:

$$p(\bar{\sigma}),\ p(\sigma_0) \subset st^3(\tilde{\sigma},\ j_1)$$
$$\Phi(p(\bar{\sigma})) \cup \Phi(p(\sigma_0)) \subset st(\Phi(a(\tilde{\sigma})),\ k_n(\tilde{\sigma}))$$
$$\operatorname{carr} \mathcal{X}\psi_1\sigma_0 \cup \operatorname{carr} \varphi_1\mathcal{X}\sigma_0 \subset st^2(\Phi(a(\tilde{\sigma})),\ k_n(\tilde{\sigma}))\ .$$

Because of Lemma B,

$$i_*\colon \tilde{H}_*(st^2(\Phi(a(\tilde{\sigma})),\ k_n(\tilde{\sigma}))) \longrightarrow \tilde{H}_*(st(\Phi(a(\tilde{\sigma})),\ k_n))$$

is trivial (i = inclusion).

Since φ_1, ψ_1 are augmentation preserving, and since $l > k_n$, there is a chain $c_1 \in C_1(st(\Phi(a(\tilde{\sigma})),\ k_n),\ l)$ such that

$$\partial c_1 = \mathcal{X}\psi_1\sigma_0 - \varphi_1\mathcal{X}\sigma_0\ .$$

Set $D_0\sigma_0 := c_1$, $c(\sigma_0) := a(\tilde{\sigma})$.

$i > 0$: Assume D_j is defined for $j < i$ and satisfies the desired properties.
Let $\sigma \in \tau^{m_1}$, $\dim \sigma = i$.
Let $\bar{\sigma} \in \tau^{l_1}$ be the simplex with $\mathcal{X}(\sigma) \subset \bar{\sigma}$, $\dim \bar{\sigma} = i$, let $\tilde{\sigma} \in \tau^{k_{n+1-i}}$ be the simplex with $\mathcal{X}(\sigma) \subset \tilde{\sigma}$, $\dim \tilde{\sigma} = i$.

Since φ_1 is a chain approximation we find a point $p(\bar{\sigma})$, and since ψ_1 is a chain approximation we find a point $p(\sigma)$. Moreover, from Lemma B we find a point $a(\tilde{\sigma})$. $p(\bar{\sigma})$, $p(\sigma)$, and $a(\tilde{\sigma})$ satisfy the following properties:

$$p(\sigma),\ p(\bar{\sigma}) \subset st^3(\tilde{\sigma},\ k_{n+1-i})$$
$$\Phi(p(\sigma)) \cup \Phi(p(\bar{\sigma})) \subset st(\Phi(a(\tilde{\sigma})),\ k_{n-i}(\tilde{\sigma}))$$
$$\operatorname{carr} \mathcal{X}\psi_1\sigma \cup \operatorname{carr} \varphi_1\mathcal{X}\sigma \subset st^2(\Phi(a(\tilde{\sigma})),\ k_{n-i}(\tilde{\sigma}))\ .$$

Write $\partial\sigma = \sum \sigma_s$, $\dim \sigma_s = i - 1$.
By assumption there is a point $c(\sigma_s)$ such that

$$c(\sigma_s) \subset st^3(\tilde{\sigma},\ k_{n+1-i})\ .$$

Therefore,

$$\Phi(c(\sigma_s)) \subset st(\Phi(a(\tilde{\sigma})),\ k_{n-i}(\tilde{\sigma}))$$
$$\operatorname{carr} D_{i-1}\sigma_s \subset st^2(\Phi(a(\tilde{\sigma})),\ k_{n-i}(\tilde{\sigma}))\ .$$

Thus,

$$\operatorname{carr} D_{i-1}\partial\sigma \subset st^2(\Phi(a(\tilde{\sigma})),\ k_{n-i}(\tilde{\sigma}))\ .$$

Set $c\colon = \chi\psi_1\sigma - \varphi_1\chi\sigma - D_{i-1}\partial\sigma$.

Then $\operatorname{carr}(c) \subset st^2(\Phi(a(\tilde{\sigma})), k_{n-i}(\tilde{\sigma}))$ and $\partial c = 0$.

Because of Lemma B there is a chain

$$c_{i+1} \in C_{i+1}(st(\Phi(a(\tilde{\sigma})), k_{n-i}), l)$$

with

$$\partial c_{i+1} = c \ .$$

Set $D_i\sigma\colon = c_{i+1}, \ c(\sigma)\colon = a(\tilde{\sigma})$.

Note added in proof. This single-valued case was treated independently (using essentially O'Neills definition) by G. Fournier in [46].

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