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THE GEOMETRY OF THE JAMES-HOPF MAPS

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Snaith's splitting of the suspension spectrum of the space $\Omega^k\Sigma^k X$, for X path connected, into the wedge of the suspension spectra of spaces denoted $D_{k,q}X$, has been of considerable interest to homotopy theorists in recent years. If $\Sigma^\infty X$ denotes the suspension spectrum of a space X then this can be restated as

$$\Sigma^\infty\Omega^k\Sigma^k X \cong \bigvee_{q\geq 1} \Sigma^\infty D_{k,q}X .$$

Projection onto the qth wedge summand and adjunction yield James-Hopf maps $j_q\colon \Omega^k\Sigma^k X \to QD_{k,q}X$, where $QY = \varinjlim \Omega^k\Sigma^k Y$.

In this paper I study various compatibility relations which hold among the j_q as X is replaced by $\Sigma^n X$. In particular, I show that, for $k > n$, the iterated evaluation map $\varepsilon_n\colon \Sigma^n\Omega^k\Sigma^k X \to \Omega^{k-n}\Sigma^k X$ is naturally compatible with the stable splittings of these two spaces. This is done by exhibiting maps $\delta_{k,n}\colon \Sigma^n D_{k,q}X \to D_{k-n,q}\Sigma^n X$ making the following diagram of suspension spectra homotopy commute:

$$\begin{array}{ccc} \Sigma^n\Omega^k\Sigma^k X & \cong & \bigvee_{q\geq 1} \Sigma^n D_{k,q}X \\ \downarrow{\scriptstyle \varepsilon_n} & & \downarrow{\scriptstyle \delta_{k,n}} \\ \Omega^{k-n}\Sigma^n X & \cong & \bigvee_{q\geq 1} D_{k-n,q}\Sigma^n X . \end{array}$$

In certain cases, the maps $\delta_{k,n}$ are then identified as standard *projection maps*. *Consequences are then discussed.*

Special cases of James-Hopf maps were defined by James [7], Toda [17], and Milgram [13] who fit their maps into the E-H-P sequence and its generalization. Snaith [16] then proved the above mentioned splitting theorem. If X is a Thom space, Kosckorke and Sanderson [9] have given a geometric description of these maps as operations between bordism theories.

J. Caruso, F. Cohen, J. P. May, and L. Taylor [2, 5] note that a "Cartan formula" interrelating the maps j_q allow them to be defined even for nonconnected spaces. James' original combinatorial construction of the Hopf invariant is used as a model. This version can be used for all of the above applications, and the extension to nonconnected spaces leads to a quite elementary proof of the Kahn-Priddy theorem [2, 15].

Section 1 contains the statements of our compatibility results. In § 2 we apply these results in conjunction with the E-H-P sequence and the Kahn-Priddy theorem. In particular, we prove a "delooped"

Kahn-Priddy theorem. We also translate our results into Koschorke and Sanderson's geometric framework, yielding relationships among self-intersections of immersions. Sections 3 and 4 contain the proofs of the statements of §1.

Peter May and Larry Taylor deserve my thanks for stimulating discussions and critical readings of preliminary versions of this work.

1. The compatibility theorems. We review some notation and definitions.

Let $\mathscr{C}_k(q)$ be Boardman and Vogt's [1, 11] space of ordered q-tuples of little cubes disjointly embedded in I^k. The symmetric group Σ_q acts freely on $\mathscr{C}_k(q)$. Let $B_{k,q} = \mathscr{C}_k(q)/\Sigma_q$. Note that $B_{\infty,q}$ is a model for the classifying space $B\Sigma_q$. If X is a based space, C_kX is defined by

$$C_kX = \coprod_{q\geq 1} \mathscr{C}_k(q) \times_{\Sigma_q} X^q/(\sim)$$

where $[(c_1, \cdots, c_q), (x_1, \cdots, x_{q-1}, *)] \sim [(c_1, \cdots, c_{q-1}), (x_1, \cdots, x_{q-1})]$ generates the equivalence relation. The spaces C_kX serve as approximations to $\Omega^k\Sigma^kX$ in the following sense: there are natural maps $\alpha_k: C_kX \to \Omega^k\Sigma^kX$, preserving all additive structure, which are homotopy equivalences when X is connected, and group completions in general (see [11, 14]).

The space C_kX is filtered and the quotients of successive filtrations are the reduced extended power spaces $D_{k,q}X = \mathscr{C}_k(q)^+ \wedge_{\Sigma_q} X^{[q]}$. Here $X^{[q]}$ denotes the q-fold smash product and Y^+ denotes the union of a space Y with a disjoint basepoint. Let $CX = C_\infty X$ and $D_qX = D_{\infty,q}X$.

Let $\eta: \Omega^k\Sigma^kX \to \Omega^{k+n}\Sigma^{k+n}X$ be the usual map. There are maps $\eta_{k,n}: C_kX \to \Omega^nC_k\Sigma^nX$ and $\beta_{k,n}: C_kX \to \Omega^nC_{k-n}\Sigma^nX$ covered by η and the identity map respectively; that is, there are commutative diagrams:

$$\begin{array}{ccc} \Omega^k\Sigma^kX & \xrightarrow{\eta} & \Omega^{k+n}\Sigma^{k+n}X \\ \uparrow{\scriptstyle \alpha_k} & & \uparrow{\scriptstyle \alpha_k} \\ C_kX & \xrightarrow{\eta_{k,n}} & \Omega^nC_k\Sigma^nX \end{array} \qquad \begin{array}{ccc} \Omega^k\Sigma^kX & = & \Omega^k\Sigma^kX \\ \uparrow{\scriptstyle \alpha_k} & & \uparrow{\scriptstyle \alpha_{k-n}} \\ C_kX & \xrightarrow{\beta_{k,n}} & \Omega^nC_{k-n}\Sigma^nX\,. \end{array}$$

The adjoints, $\tilde{\eta}_{k,n}$ and $\tilde{\beta}_{k,n}$ are therefore covered by iterated evaluation maps. Moreover, the adjoints are filtration preserving and thus induce maps $\Delta_{k,n}: \Sigma^nD_{k,q} \to D_{k,q}\Sigma^nX$ and $\delta_{k,n}: \Sigma^nD_{k,q}X \to D_{k-n,q}\Sigma^nX$. By passage to limits, we obtain maps η_n and $\beta_n: CX \to \Omega^nC\Sigma^nX$ and maps Δ_n and $\delta_n: \Sigma^nD_qX \to D_q\Sigma^nX$.

We also have maps $i: C_kX \to C_{k+n}X$ and $i: D_{k,q}X \to D_{k+n,q}X$ induced by the inclusion $I^k \to I^{k+n}$ onto the first k coordinates.

In [5], F. Cohen, J. P. May, and L. Taylor define James-Hopf maps $j_q: C_kX \to CD_{k,q}X$. The space $\prod_{q\geq 0} QD_{k,q}X$ is a ring space where $D_{k,0}X$ is interpreted as S^0 and the multiplication is induced by natural pairings $\mathscr{M}: D_{k,p}X \wedge D_{k,q}X \to D_{k,p+q}X$ together with the smash product $\oplus: QY \times QZ \to Q(Y \wedge Z)$. Similarly the space $\prod_{q\geq 0} CD_{k,q}X$ is a ring space. The James-Hopf maps satisfy a "Cartan formula" [2, 9, 15] and in [2] this is interpreted as the statement that $\prod_{q\geq 0} j_q: C_kX \to \prod_{q\geq 0} CD_{k,q}X$ is an "exponential" H-map taking the additive structure in C_kX to the multiplicative one in $\prod_{q\geq 0} CD_{k,q}X$.

The next proposition is the key to our results.

PROPOSITION 1.1.

(1) *The following diagram commutes.*

$$\begin{array}{ccc} C_kX & \xrightarrow{j_q} & CD_{k,q}X \\ \downarrow{\scriptstyle \gamma_{k,n}} & & \downarrow{\scriptstyle \eta_n} \\ & & \Omega^n C\Sigma^n D_{k,q}X \\ & & \downarrow{\scriptstyle \Delta_{k,n}} \\ \Omega^n C_k\Sigma^n X & \xrightarrow{j_q} & \Omega^n CD_{k,q}\Sigma^n X \end{array}$$

(2) *The following diagram homotopy commutes.*

$$\begin{array}{ccc} C_kX & \xrightarrow{j_q} & CD_{k,q}X \\ & & \downarrow{\scriptstyle \eta_n} \\ \downarrow{\scriptstyle \beta_{k,n}} & & \Omega^n C\Sigma^n D_{k,q}X \\ & & \downarrow{\scriptstyle \delta_{k,n}} \\ \Omega^n C_{k-n}\Sigma^n X & \xrightarrow{j_q} & \Omega^n CD_{k-n,q}\Sigma^n X \end{array}$$

In fact, there is an "exponential" H-map

$$J: C_kX \longrightarrow F\left(I^+, \Omega^n \coprod_{q\geq 0} CD_{k-n,q}\Sigma^n X\right)$$

which is a homotopy between $J_0 = \delta_{k,n} \circ \eta_n \circ \prod_{q\geq 0} j_q$ *and* $J_1 = \prod_{q\geq 0} j_q \circ \beta_{k,n}$.

This will be proved in § 4. Statement (1) follows essentially from a check of the definitions while (2) requires more care.

As shown in [2], the "exponential" properties of the maps j_q allow them to be extended to $j_q: \Omega^k\Sigma^k X \to QD_{k,q}X$ and $j_q: QX \to QD_qX$.

THEOREM 1.2. *The following diagrams commute up to weak homotopy.*

(1)
$$\begin{array}{ccc} QX & \xrightarrow{j_q} & QD_qX \\ \| & & \| \\ & & \Omega^n Q\Sigma^n D_qX \\ & & \downarrow \Delta_n \\ \Omega^n Q\Sigma^n & \xrightarrow{\Omega^n j_q} & \Omega^n QD_q\Sigma^n X \end{array}$$

(2)
$$\begin{array}{ccc} \Omega^k\Sigma^k X & \xrightarrow{j_q} & QD_{k,q}X \\ \| & & \| \\ & & \Omega^n Q\Sigma^n D_{k,q}X \\ & & \downarrow \delta_{k,n} \\ \Omega^n\Omega^{k-n}\Sigma^{k-n}\Sigma^n X & \xrightarrow{\Omega^n j_q} & \Omega^n QD_{k-n,q}\Sigma_u X \end{array}$$

This will be proved in §4. By weak homotopy, we refer to neglect of $\lim^1$ terms.

Much of the usefulness of the theorem comes from an understanding of the maps $\Delta_{k,n}$ and $\delta_{k,n}$. The next proposition allows us to identify the maps $\delta_{k,n}$ in special cases,

PROPOSITION 1.3.

(1) *The composite* $\Sigma^n D_{n,q}X \xrightarrow{i} \Sigma^n D_{k+n,q}X \xrightarrow{\delta_{k+n,n}} D_{k,q}\Sigma^n X$ *is null-homotopic for all* $q > 1$.

(2) *The sequence* $\Sigma^n D_{n,2}X \xrightarrow{i} \Sigma^n D_{k+n,2}X \xrightarrow{\delta_{k+n,n}} D_{k,2}\Sigma^n X$ *is homotopic to a cofibration sequence. Furthermore, the sequence desuspends n times.*

(3) *If p is an odd prime, the sequence*

$$\Sigma^n D_{n,p}S^m \xrightarrow{i} \Sigma^n D_{n+k,p}S^m \xrightarrow{\delta_{k+n,n}} D_{k,p}S^{m+n}$$

is homotopic to a cofibration sequence, localized at p.

When $X = S^0$, $D_{k,2}X = \boldsymbol{RP}^{k-1+}$ and $D_{k,p}X = B^+_{k,p}$, hence the proposition has the following consequence.

COROLLARY 1.4. $D_{k,2}S^n \cong \Sigma^n \boldsymbol{RP}^{n+k-1}/\boldsymbol{RP}^{n-1}$ *and, localized at an odd prime p,* $D_{k,p}S^n \cong \Sigma^n B_{n+p,k}/B_{n,p}$. *Under these identifications,* $\delta_{k+n,n}$ *can be identified with the projections*

$$\Sigma^n \boldsymbol{RP}^{n+k-1} \longrightarrow \Sigma^n \boldsymbol{RP}^{n+k-1}/\boldsymbol{RP}^{n-1} \quad \textit{and}$$
$$\Sigma^n B_{n+k,p} \longrightarrow \Sigma^n B_{n+k,p}/B_{n,p}\ .$$

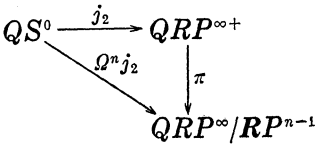
REMARK 1.5. Proposition 1.3(2) proves and generalizes Milgram's

Corollary 2.3 [13] in our setting.

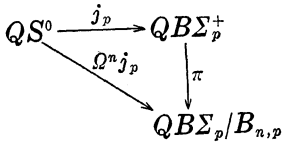
Theorem 1.2 and Corollary 1.4 imply the following.

COROLLARY 1.6.[1]

(1) *The following diagram commutes up to weak homotopy.*



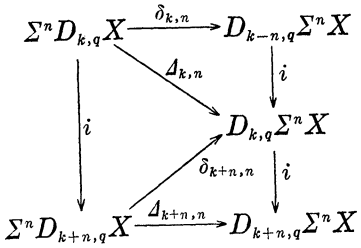
(2) *Localized at an odd prime p, the following diagram commutes up to weak homotopy.*



If n is odd then the finiteness of the homotopy groups of QRP^{∞}/RP^{n-1} and $QB\Sigma_p/B_{n,p}$ implies that the $\lim^1$ terms vanish and the diagrams actually homotopy commute.

Finally, the following proposition relates the maps $\Delta_{k,n}$ and $\delta_{k,n}$.

PROPOSITION 1.7. *In the following diagram, the quadrilaterals commute and the triangles commute up to homotopy.*



Thus, in the limit as k goes to ∞, $\delta_n \cong \Delta_n$: $\Sigma^n D_q X \to D_q \Sigma^n X$.

Propositions 1.3 and 1.7 will be proved in §3.

2. Applications. In our first application we apply Proposition 1.3 and Theorem 1.2 to the Kahn-Priddy theorem [8]. For a prime p, Segal [15] and Caruso, F. Cohen, May, and Taylor [2] find maps

[1] A variant of Corollary 1.6(1) appears in lecture notes by M. Mahowald and A. Unell, Northwestern University, 1977.

$j: Q_0S^0 \to QB\Sigma_p$ and $s: QB\Sigma_p \to Q_0S^0$ such that the composite is an equivalence localized at p. The maps j and s are defined by $j = \pi \circ j_p$ and $s = s_p \circ i$ in the diagram

$$\begin{array}{ccccc} QS^0 & \xrightarrow{j_p} & QB\Sigma_p^+ & \xrightarrow{s_p} & QS^0 \\ & & \pi \downarrow \uparrow i & & \\ & & QB\Sigma_p & & \end{array}$$

where j_p is the James-Hopf map as above and s_p is an infinite loop map. Thus s is also an infinite loop map.

Corollary 1.6 implies the following result.

PROPOSITION 2.1. *j is a loop map and thus the Kahn-Priddy theorem "deloops" once.*

REMARK 2.2. That j does not deloop twice follows from homology calculations in [10].

REMARK 2.3. In [3] it is shown that Proposition 2.1 is precisely what is needed to calculate the order of $\eta: \Omega_0^{2n+1}S^{2n+1} \to Q_0S^0$, localized at 2. From this can be computed the 2 torsion in the order of the bundles $\zeta_{k,q}: F(\boldsymbol{R}^k, q) \times_{\Sigma_q} \boldsymbol{R}^q \to F(\boldsymbol{R}^k, q)/\Sigma_q$. Note that the Thom complex of $n\zeta_{k,q}$ is precisely the space $D_{k,q}S^n$.

Our next application mixes our compatibility statements with the E-H-P sequence. We need the following version of Milgram's generalized E-H-P sequence.

LEMMA 2.4 [12].
If X is $m-1$ connected (and $\Sigma^n X$ is connected) then

$$\Omega^n\Sigma^n X \xrightarrow{\eta} QX \xrightarrow{\Omega^n j_2} \Omega^n QD_2\Sigma^n X$$

is homotopic to a fibration through dimensions $<2n+3m-1$ (the metastable range).

The corresponding long exact sequence of homotopy groups is known as the generalized E-H-P sequence.

COROLLARY 2.5. *If X is $m-1$ connected then, through dimensions $<2n+3m-1$, the following is a homotopy pullback diagram.*

$$\begin{array}{ccc} \Omega^n\Sigma^n X & \xrightarrow{j_2} & QD_{n,2}X \\ \downarrow i & & \downarrow i \\ QX & \xrightarrow{j_2} & QD_2X \end{array}$$

In particular, through dimensions $<2n-1$, *the following diagram is a pullback.*

$$\begin{array}{ccc} \Omega^n S^n & \xrightarrow{j} & QRP^{n-1} \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle i} \\ QS^0 & \xrightarrow{j} & QRP^{\infty} \end{array}$$

Thus the desuspension of 2-torsion through the metastable range is detected by the stable map i.

Proof. By Theorem 1.2, the following diagram commutes

$$\begin{array}{ccc} \Omega^n\Sigma^n X & \xrightarrow{j_2} & QD_{n,2}X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ QX & \xrightarrow{j_2} & QD_2X \\ \downarrow & & \downarrow{\scriptstyle \Delta_n} \\ \Omega^n QD_2\Sigma^n X & = & \Omega^n QD_2\Sigma^n X \end{array}$$

Proposition 1.3 implies that the right hand vertical sequence is a fibration and the lemma says that the left hand vertical sequence is a fibration through the desired dimension.

Finally we relate our results to Koschorke and Sanderson's theory of self intersections of immersions [9].

Using the notation of [9], let V be a manifold without boundary and let ξ be a vector bundle over a space B, Let $\mathscr{I}_k(V, \xi)$ be the bordism set of embeddings $g=(g_1, g_2)\colon M \to V \times \boldsymbol{R}^k$ for which $g_1\colon M \to V$ is an immersion with normal bundle ν expressed as a pullback $\bar{g}\colon \nu \to \xi$. Thus an element of $\mathscr{I}_k(V, \xi)$ is represented by a triple $(M, g, \bar{g})$. By abuse of notation, we will denote this class by $[M]$.

Let V_c denote the one point compactification of V. Let $T(\xi)$ denote the Thom space of ξ.

PROPOSITION 2.6 [9]. *There is a bijection*

$$\mathscr{I}_k(V, \xi) = [V_c, C_k T(\xi)] .$$

This is proved using a generalization of the classical Thom-Pontrjagin construction. See [9] for details.

To describe the geometric operations, we recall the following definition. Given a self transverse immersion $f\colon M \to V$, let $\tilde{M}(q) = \{(x_1, \cdots, x_q) \in M^q\colon f(x_1) = \cdots = f(x_q)$, and $x_i \neq x_j$ if $i \neq j\}$. This is a closed submanifold of M^q on which Σ_q acts freely. Let $M(q) = \tilde{M}(q)/\Sigma_k$, called the q-tuple point manifold.

If ξ is a bundle over B, let $\xi_{k,q}$ be the twisted power bundle over $\mathscr{C}_k(q)\times_{\Sigma_q}B^q$. Note that $T(\xi_{k,q}) = D_{k,q}T(\xi)$. Let ε^n denote the trivial bundle of dimension n.

Koschorke and Sanderson define operations $\theta_q\colon \mathscr{J}_k(V,\xi)\to\mathscr{J}_\infty(V,\xi_{k,q})$, roughly defined by $\theta_q[M] = [M(q)]$. These operations are related to the James-Hopf maps as follows.

PROPOSITION 2.7 [9]. *There is a commutative diagram*

$$\begin{array}{ccc} [V_c, C_kT(\xi)] & \xrightarrow{j_q} & [V_c, CD_{k,q}T(\xi)] \\ \| & & \| \\ \mathscr{J}_k(V,\xi) & \xrightarrow{\theta_q} & \mathscr{J}_\infty(V,\xi_{k,q})\,. \end{array}$$

Proposition 1.1 now has following consequences.

PROPOSITION 2.8. *The following diagrams commute.*

$$\begin{array}{ccc} \mathscr{J}_k(V,\xi) & \xrightarrow{\theta_q} & \mathscr{J}_\infty(V,\xi_{k,q}) \\ \downarrow & & \downarrow \\ & & \mathscr{J}_\infty(V\times\boldsymbol{R}^n, \xi_{k,q}\oplus\varepsilon^n) \\ & & \downarrow \Delta \\ \mathscr{J}_k(V\times\boldsymbol{R}^n, \xi\oplus\varepsilon^n) & \xrightarrow{\theta_q} & \mathscr{J}_\infty(V\times\boldsymbol{R}^n, (\xi\oplus\varepsilon^n)_{k,q}) \end{array}$$

Here Δ is induced by the diagonal inclusion of vector bundles:

$$\xi_{k,q}\oplus\varepsilon^n \longrightarrow (\xi\oplus\varepsilon^n)_{k,q}\,.$$

$$\begin{array}{ccc} \mathscr{J}_k(V,\xi) & \xrightarrow{\theta_q} & \mathscr{J}_\infty(V,\xi_{k,q}) \\ \downarrow & & \downarrow \\ & & \mathscr{J}_\infty(V\times\boldsymbol{R}^n, \xi_{k,q}\oplus\varepsilon^n) \\ & & \downarrow \delta \\ \mathscr{J}_{k-n}(V\times\boldsymbol{R}^n, \xi\oplus\varepsilon^n) & \xrightarrow{\theta_q} & \mathscr{J}_\infty(V\times\boldsymbol{R}^n, (\xi\oplus\varepsilon^n)_{k-n,q}) \end{array}$$

Here δ is induced by map $T(\xi_{k,q}\oplus\varepsilon^n)\to T((\xi\oplus\varepsilon^n)_{k-n,q})$.

REMARK 2.9. To understand the map Δ geometrically, note that any bundle inclusion $f\colon\xi\to\eta$ of bundles over B induces a map $f_*\colon\mathscr{J}_m(V,\xi)\to\mathscr{J}_m(V,\eta)$ which can be interpreted as follows. Given $g\colon M\to V\times\boldsymbol{R}^m$ and $\bar{g}\colon\nu\to\xi$, the composite $f\bar{g}\colon\nu\to\eta$ can be approximated by a map h transverse to the zero section of η. Then $f_*[M] = [h^{-1}(B)]$. Note that $h^{-1}(B)$ is a submanifold of some manifold

$M' \in [M]$.

Proposition 1.3(2) together with Proposition 2.6, yields the next result. This is just a long exact sequence associated to the fibration:

$$QT(\xi_{n,2} \oplus \varepsilon^n) \longrightarrow QT(\xi_{n+k,2} \oplus \varepsilon^n) \longrightarrow QT((\xi \oplus \varepsilon^n)_{k,2}) .$$

PROPOSITION 2.10. *The following sequence is exact.*

$$\longrightarrow \mathscr{J}_\infty(V \times \boldsymbol{R}, \xi_{k+n,2} \oplus \varepsilon^n) \longrightarrow \mathscr{J}_\infty(V \times \boldsymbol{R}, (\xi \oplus \varepsilon^n)_{k,2})$$
$$\longrightarrow \mathscr{J}_\infty(V, \xi_{n,2} \oplus \varepsilon^n) \longrightarrow \mathscr{J}_\infty(V, \xi_{k+n,2} \oplus \varepsilon^n) \xrightarrow{\delta} \mathscr{J}_\infty(V, (\xi \oplus \varepsilon^n)_{k,2}) .$$

3. The maps $\Delta_{k,n}$ and $\delta_{k,n}$. We first define the maps $\eta_{k,n}: C_kX \to \Omega^n C_k \Sigma^n X$ and $\beta_{k,n}: C_kX \to \Omega^n C_{k-n}\Sigma^n X$ used in § 1. Let $\eta_{k,n}: C_kX \to \Omega^n C_k \Sigma^n X$, with adjoint $\tilde{\eta}_{k,n}$, be defined by

$$\tilde{\eta}_{k,n}([(c_1, \cdots, c_q), (x_1, \cdots, x_q), t]) = [(c_1, \cdots, c_q), ((x_1, t), \cdots, (x_q, t))]$$

where $t \in S^n$, $(c_1, \cdots, c_q) \in \mathscr{C}_k(q)$, and $(x_1, \cdots, x_q) \in X^q$. $\beta_{k,1}$ is the map defined by May in [11, Prop. 5.4].

It is useful to view $\beta_{k,1}$ in the following way. Let $c_i = c_i' \times c_i''$ where $c_i': I \to I$, $c_i'': I^{k-1} \to I^{k-1}$. Then $\beta_{k,1}([(c_1, \cdots, c_q), (x_1, \cdots, x_q)])(t) = [(c_i'', \cdots, c_q''), ((x_1, s_1(t)), \cdots, (x_q, s_q(t)))]$ where $s_i(t)$ is defined by $c_i'(s_i(t)) = t$ if $t \in \operatorname{Im} c_i'$ and

$$s_i(t) = \begin{cases} 0 & \text{if } t < \operatorname{Im} c_i' \\ 1 & \text{if } t > \operatorname{Im} c_i' . \end{cases}$$

This definition makes sense because if the cubes $\{c_i', \cdots, c_q''\}$ are not all disjoint then exactly the right number of pairs $(x_i, s_i(t)) \in \Sigma X$ are the basepoint, by virtue of $s_i(t) = 0$ or 1.

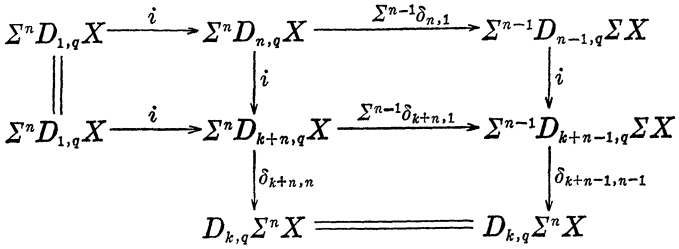
Iteration defines maps $\beta_{k,n}: C_kX \to \Omega^n C_{k-n}\Sigma^n X$, with adjoint $\tilde{\beta}_{k,n}$. As mentioned before, $\tilde{\eta}_{k,n}$ and $\tilde{\beta}_{k,n}$ are filtration preserving and thus induce $\Delta_{k,n}$ and $\delta_{k,n}$. Recall that $i: C_kX \to C_{k+n}X$ is the map induced by the inclusion $I^k \to I^{k+n}$ onto the first k coordinates. We will later make use of the fact that i is homotopic (by a filtration preserving homotopy) to the map $i': C_kX \to C_{k+n}X$ induced by the inclusion $I^k \to I^{k+n}$ onto the last k coordinates. For a proof of this, see [11, Lemma 4.9].

Proof of Proposition 1.3. In statements (1), (2), and (3) we first reduce the general case to the case $n = 1$.

Consider the following commutative diagram.

Inductively, if we assume statement (1) for $n - 1$, then the statement follows for n by consideration of the two vertical sequences.

Next, suppose $q = 2$ in the diagram. If we assume that statement



(2) of the proposition is true for $n = 1$, then the horizontal sequences are cofibrations, and inductively we assume that the right vertical sequence is a cofibration. Then, by diagram chasing, the left vertical sequence will be a cofibration.

Statement (3) reduces to the case $n = 1$ in a similar fashion, upon letting $q = p$ in the diagram. Note that the argument given above for statement (2) goes through here because the set of spheres is closed under suspension.

Thus to prove statement (1) we just need to show that the composite $\Sigma D_{1,q}X \xrightarrow{i} \Sigma D_{k+1,q}X \xrightarrow{\delta_{k+1,1}} D_{k,q}\Sigma X$ is nullhomotopic for $q > 1$. This follows immediately from the following observation.

LEMMA 3.1. *The composite* $\Sigma C_1 X \xrightarrow{i} \Sigma C_{k+1}X \xrightarrow{\beta_{k+1,1}} C_k \Sigma X$ *has image in filtration* 1.

To prove statement (2) we need to show that the following sequence is homotopic to a cofibration sequence.

$$\Sigma D_{1,2}X \xrightarrow{i} \Sigma D_{k+1,2}X \xrightarrow{\delta_{k+1,1}} D_{k,2}\Sigma X\,.$$

This sequence is equivalent to the following.

$$S^{0+} \wedge_{Z_2} (X \wedge X) \wedge S^1 \longrightarrow S^{k+} \wedge_{Z_2} (X \wedge X) \wedge S^1 \longrightarrow S^{k-1+} \wedge_{Z_2} (\Sigma X \wedge \Sigma X)\,.$$

Here $(\Sigma X \wedge \Sigma X)$ is equivariantly homeomorphic to $(X \wedge X) \wedge (S^1 \wedge S^1)$ where all spaces have the obvious permutation actions. $(S^1 \wedge S^1)$ is then equivalent to $S^1 \wedge S^1$ where S^1 has the trivial action and $S^1 = [-1, 1]/-1 \sim 1$ with Z_2 action given by $t \to -t$.

Let points in S^k be benoted by $(s_0, \cdots, s_k) \in \boldsymbol{R}^{k+1}$ with $\Sigma s_i^2 = 1$. Then our sequence desuspends to

$$S^{0+} \wedge_{Z_2} (X \wedge X) \xrightarrow{i} S^{k+} \wedge_{Z_2} (X \wedge X) \xrightarrow{p} S^{k-1+} \wedge_{Z_2} S^1 (X \wedge X)$$

where

$$p(s_0, \cdots, s_k, x_1, x_2) = \begin{cases} \left(\frac{s_1}{\sqrt{1-s_0^2}}, \cdots, \frac{s_k}{\sqrt{1-s_0^2}}, s_0, x_1, x_2\right) & \text{if } |s_0| < 1 \\ * & \text{otherwise} \end{cases}$$

and i is defined by $i: S^0 \to S^k$ with $i(\pm 1) = (\pm 1, 0, 0, \cdots, 0)$. Then p is homeomorphism of pairs,

$$p: (S^k \bigwedge_{Z_2} (X \wedge X), S^{0+} \bigwedge_{Z_2} (X \wedge X)) \cong (S^{k-1+} \bigwedge_{Z_2} S^1 \wedge (X \wedge X), *)$$

with inverse given by

$$(s_1, \cdots, s_k, t, x_1, x_2) \longmapsto (t, \sqrt{(1-t^2)}s_1, \cdots, \sqrt{(1-t^2)}s_k, x_1, x_2) .$$

REMARK 3.2. This last proof is essentially a generalization of the standard argument that the Thom space of $m\xi_k$, m copies of the canonical line bundle ξ_k over $\boldsymbol{RP}^k$, is homeomorphic to $\boldsymbol{RP}^{m+k}/\boldsymbol{RP}^{m-1}$. See [6, Example 15.1.7].

Finally, to prove statement (3) of Proposition 1.3, in the case $n = 1$, we need to show that the sequence

$$S^{pm+1} \xrightarrow{i} \Sigma D_{k+1,p} S^m \xrightarrow{\delta_{k+1,1}} D_{k,p} S^{m+1}$$

is a cofibration when localized at p. By virtue of statement (1), it suffices to show that the sequence acts homologically as a cofibration. Recall that $\delta_{k+1,1}$ is induced from the map $\tilde{\beta}_{k,1}: \Sigma C_{k+1} S^m \to C_k S^{m+1}$ which by definition induces the homology suspension σ_*. Let Q^i denote a Dyer-Lashof operation, let β denote the Bockstein, and let $\lambda_k(x, y)$ denote the Browder operation on the homology of a $\mathscr{C}_{k+1}$-space (see [4] for the definitions). In [4] it is shown that $\sigma_* Q^i x = Q^i \sigma_* x$ and $\sigma_* \lambda_k(x, y) = \lambda_{k-1}(\sigma_* x, \sigma_* y)$; while $\sigma_* \beta = \beta \sigma_*$ up to sign. We conclude that the above sequence induces a short exact homology sequence:

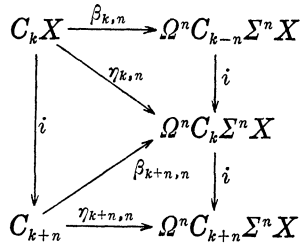
$$0 \longrightarrow H_*(S^{pm+1}; Z_p) \xrightarrow{i_*} H_*(\Sigma D_{k+1,p} S^m; Z_p) \xrightarrow{\delta_{k+1,1*}} H_*(D_{k,p} S^{m+1}; Z_n) \longrightarrow 0 .$$

The proposition follows.

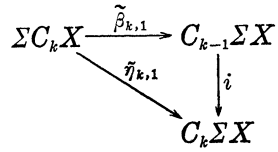
Proof of Proposition 1.7. It suffices to show that the quadrilaterals commute and the triangles commute up to a natural filtration preserving homotopy in the following diagram.

The quadrilaterals can be seen to commute by a check of definitions.

For the commutativity of the triangles it suffices to let $n = 1$. The left hand triangle commutes when i is replaced by $i': C_k X \to C_{k+1} X$, induced by the inclusion $\boldsymbol{R}^k \to \boldsymbol{R}^{k+1}$ onto the last k coordinates of $\boldsymbol{R}^{k+1}$. As mentioned before, i is homotopic to i'. To prove that



the right hand triangles commute, we show that



commutes up to a filtration preserving homotopy. Again replace i by i'.

If $[c, x] \in C_kX$ with $c = (c_1' \times c_1'', \cdots, c_q' \times c_q'')$ and $x = (x_1, \cdots, x_q)$ then we have following formulae:

$$i' \circ \tilde{\beta}_{k,1}(([c, x], t)) = [(1 \times c_1'', \cdots, 1 \times c_q''), ((x_1, s_1(t)), \cdots, (x_q, s_q(t)))]$$
$$\tilde{\eta}_{k,1}(([c, x], t)) = [(c_1' \times c_1'', \cdots, c_q' \times c_q''), ((x_1, t, \cdots, (x_q, t)))] .$$

To show that these are homotopic, we show that maps are homotopic to the map $h: \Sigma C_kX \to C_k\Sigma X$ defined by

$$h(([c, x], t)) = [(c_1' \times c_1'', \cdots, c_q' \times c_q''), ((x_1, s_1(t)), \cdots, (x_q, s_q(t)))] .$$

First, $\tilde{\eta}_{k,1} \cong h$ by shrinking id: $I \to I$ to s_i: $I \to I$ in some natural way (continuous in c_i'). Second, $h \cong i' \circ \tilde{\beta}_{k,1}$ by expanding each c_i' to id: $I \to I$ in a natural way.

4. The James maps and the compatibility theorems. In this section we prove Proposition 1.1 and Theorem 1.2.

We first note that a straightforward inductive argument reduces our statements to the case $n = 1$.

Proof of Proposition 1.1. The commutativity of diagram (1) in its adjoint form follows from a straightforward check of definitions.

We digress to note that this diagram is a special case of the following more general statement about maps j_q. We use the language of coefficient systems and Π-spaces from [5].

In general, there are James maps

$$j_q: CX \longrightarrow C'(D_q(\mathscr{C}, X))$$

where X is a Π-space and $\mathscr{C}$ and $\mathscr{C}'$ are coefficient systems with a James system $\{\xi_{qr}\}: \mathscr{C} \to \mathscr{C}'$. Abbreviate $D_q(\mathscr{C}, X)$ to D_qX.

LEMMA 4.1. *For any based space P, the diagonal maps $\Delta: P \to P^k$ induce following commutative diagram.*

$$\begin{array}{ccc} CX \wedge P & \xrightarrow{j_q \wedge 1} & C'(D_qX) \wedge P \\ \downarrow \Delta & & \downarrow \Delta \\ & & C'(D_qX \wedge P) \\ & & \downarrow \Delta \\ C(X \wedge P) & \xrightarrow{j_q} & C'(D_q(X \wedge P)) \end{array}$$

To see the specialization of this diagram to the adjoint of diagram (1) of Proposition 1.1 and to prepare for proof of (2), we recall the construction of the James-Hopf maps from [5].

Let $F(Y, q) \subset Y^q$ denote the configuration space of distinct q-tuples in Y^q. If X is based, let

$$C(Y, X) = \coprod_{q \geq 1} F(Y, q) \times_{\Sigma_q} X^q/(\sim)$$

with the equivalence relation generated as before. There is a natural equivalence $C_kX \cong C(\boldsymbol{R}^k, X)$, under the map which assigns to a set of cubes, their centers.

The James-Hopf maps are constructed as follows. We first define maps $j_q: C_kX \to C(B_{k,q}, D_{k,q}X)$. If $[c, X] \in C_kX$ is in filtration r, then $j_q((c, x))$ will be in filtration m, where m is the binomial coefficient $\binom{r}{q}$. Let $\{A_1, \cdots, A_m\}$ be the m subsets of c of cardinality q. There are corresponding subsets $\{B_1, \cdots, B_m\}$ of x. Let

$$j_q([c, x]) = [(A_1, \cdots, A_m), ((A_1, B_1), \cdots, (A_m, B_m))] .$$

This is well-defined and continuous.

An embedding $B_{k,q} \to \boldsymbol{R}^\infty$ induces a map

$$C(B_{k,q}, D_{k,q}X) \longrightarrow C(R^\infty, D_{k,q}X) ,$$

and the composite with the above maps defines $j_q: C_kX \to CD_{k,q}X$.

Proposition 1.1 (1) now follows from the application of the lemma to the special case in which $\mathscr{C} = \mathscr{C}_k$, the little cubes operad, $\mathscr{C}' = \mathscr{C}(B_{k,q})$, the coefficient system with tth space $F(B_{k,q}, t)$, and $P = S^1$.

The proof of statement (2) is a bit more delicate. We recall

that an embedding $B_{k,q} \subset \boldsymbol{R}^\infty$ is involved in defining the maps j_q. We can make these compatible as k varies by chossing a fixed embedding $B\Sigma_q \to \boldsymbol{R}^\infty$ and then embedding $B_{k,q}$ via the composite $B_{k,q} \to B\Sigma_q \to \boldsymbol{R}^\infty$. Then $j_q: C_kX \to C(\boldsymbol{R}^\infty, D_{k,q}X)$ factors as the composite $C_kX \to C(B_{k,q}, D_{k,q}X) \to C(B\Sigma_q, D_{k,q}X) \to C(\boldsymbol{R}^\infty, D_{k,q}X)$. The space $\prod_{q\geqq 0} C(B_{k,q}, D_{k,q}X)$ has an H-space structure induced from the maps $B_{k,q} \times B_{k,p} \to B_{k,q+p}$ and $D_{k,q}X \wedge D_{k,p}X \to D_{k,q+p}X$ [2]. It suffices to show that the following diagram commutes, and that, when the product is taken over all $q \geqq 0$, the homotopy is through compatible H-maps.

$$\begin{array}{ccccc}
C_kX & & \xrightarrow{\quad j_q \quad} & & C(B_{k,q}, D_{k,q}X) \\
\Big\downarrow{\scriptstyle \beta_{k,1}} & & & & \Big\downarrow{\scriptstyle \eta_1} \\
 & & & & \Omega C(B_{k,q}, \Sigma D_{k,q}X) \\
 & & & & \Big\downarrow{\scriptstyle \delta_{k,1}} \\
\Omega C_{k-1}\Sigma X & \xrightarrow{\Omega j_q} & \Omega C(B_{k-1,q}, D_{k-1,q}\Sigma X) & \xrightarrow{\quad i \quad} & \Omega C(B_{k,q}, D_{k-1,q}\Sigma X)
\end{array}$$

To show this we make use auxiliary spaces, lying between $\mathscr{C}_{k-1}(q)$ and $\mathscr{C}_k(q)$, which also form a coefficient system. Let $\mathscr{C}'_k(q) =$ ordered sets of q little k-cubes $(c_1, \cdots, c_q)$ such that $(c'_1, \cdots, c'_q)$ have disjoint images, where $c_i = c'_i \times c''_i$, $c'_i: I \to I$, $c''_i: I^{k-1} \to I^{k-1}$. Let $B'_{k,q} = \mathscr{C}'_k(q)/\Sigma_q$.

There is a projection map $\beta: \mathscr{C}'_k(q) \to \mathscr{C}_{k-1}(q)$ defined by

$$\beta((c_1, \cdots, c_q)) = (c''_1, \cdots, c''_q) .$$

Let $i_1: \mathscr{C}_{k-1}(q) \to \mathscr{C}'_k(q)$ be the inclusion map defined by $i_1(c_1, \cdots, c_q) = (1 \times c_1, \cdots, 1 \times c_q)$ and let $i_2: \mathscr{C}'_k(q) \to \mathscr{C}_k(q)$ be the obvious inclusion. Then $\beta \circ i_1 = id_{\mathscr{C}_{k-1}(q)}$ and $i_1 \circ \beta \cong id_{\mathscr{C}'_k(q)}$ equivariantly via a homotopy $H_t: \mathscr{C}'_k(q) \to \mathscr{C}'_k(q)$ "stretching" the c'_i to $id: I \to I$ in a natural way. Consider the following diagram. Once again, note that i has replaced by the homotopic map i'.

$$\begin{array}{ccccc}
\Sigma C_kX & & \xrightarrow{\quad j_q \quad} & & \Sigma C(B_{k,q}, D_{k,q}X) \\
 & \searrow^{j'_q} & & & \Big\downarrow{\scriptstyle \tilde{\eta}_1} \\
\Big\downarrow{\scriptstyle \tilde{\beta}_{k,1}} & & C(B'_{k,q}, D_{k-1,q}\Sigma X) & & C(B_{k,q}, \Sigma D_{k,q}X) \\
 & & \Big\downarrow{\scriptstyle \beta} \quad \uparrow{\scriptstyle i_1} & \searrow^{i_2} & \Big\downarrow{\scriptstyle \delta_{k,1}} \\
C_{k-1}\Sigma X & \xrightarrow{j_q} & C(B_{k-1,q}, D'_{k-1,q}\Sigma X) & \xrightarrow{\quad i' \quad} & C(B_{k,q}, D_{k-1,q}\Sigma X)
\end{array}$$

By inspection, $\delta_{k,1} \circ \tilde{\eta}_1 \circ j_q$ factors through $C(B'_{k,q}, D_{k-1,q}\Sigma X)$, thus defining the map j''_q. Thus the upper part of the diagram commutes. By a

check of definitions, one can see that the left hand quadralateral commutes, i.e., $\beta \circ j = j_q \circ \tilde{\beta}_{k,1}$.

Finally we check that the triangle commutes up to homotopy. $i' = i_2 \circ i_1$ so that $i' \circ \beta = i_2 \circ i_1 \circ \beta \cong i_2 \circ 1 = i_2$ Thus diagram (2) of Proposition 1.1 commutes up to homotopy.

To conclude that this homotopy is an H-map we first note that $\prod_{q \geq 0} C(B'_{k,q}, D_{k-1,q}X)$ also has a natural H-space structure. In the language of [2], $\mathscr{C}'$ has a separated sum. Proposition 1.1 (2) now follows from the following lemma.

LEMMA 4.2. *The following maps are strict H-maps.*

(a) $\prod_{q \geq 0} j_q \colon C_k X \to \prod_{q \geq 0} C(B_{k,q}, D_{k,q}X)$

(b) $\beta_{k,1} \colon C_k X \to \Omega C_{k-1} \Sigma X$

(c) $i_1 \colon \prod_{q \geq 0} C(B_{k-1,q}, D_{k-1,q}\Sigma X) \to \prod_{q \geq 0} C(B_{k,q}, D_{k-1,q}\Sigma X)$

(d) $i_2 \colon \prod_{q \geq 0} C(B'_{k,q}, D_{k-1,q}\Sigma X) \to \prod_{q \geq 0} C(B_{k,q}, D_{k-1,q}\Sigma X)$

(e) $H \colon \prod_{q \geq 0} C(B'_{k,q}, D_{k-1,1q}\Sigma X) \to F(I^+, \prod_{q \geq 0} C(B'_{k,q}, D_{k-1,q}\Sigma X))$

(f) *The homotopy between* i *and* $i' \colon \prod_{q \geq 0} C(B_{k-1,q}, D_{k-1,q}\Sigma X) \to \prod_{q \geq 0} C(B_{k,q}, D_{k-1,q}\Sigma X)$

(g) *The composite*

$$\coprod_{q \geq 0} C(B_{k,q}, D_{k,q}X) \xrightarrow{\eta_1} \Omega \coprod_{q \geq 0} C(B_{k,q}, \Sigma D_{k,q}X) \xrightarrow{\delta_{k,1}} \Omega \coprod_{q \geq 0} C(B_{k,q}, D_{k-1,q}\Sigma X)$$

(h) $\prod_{q \geq 0} j'_q \colon C_k X \to \prod_{q \geq 0} C(B'_{k,q}, D_{k-1,q}\Sigma X)$.

That (a) is a strict H-map is shown in [2]. That the maps (b)–(g) are strict H-maps follows from a straightforward check of the definitions. Statement (h) follows from statements (a), (d), and (g) and the fact that i_2 is an injection.

Proof of Theorem 1.2. As remarked before, it suffices to prove the theorem when $n = 1$. We will just show that the second diagram homotopy commutes; that the first diagram commutes follows from a slightly simpler but essentially identical argument.

To extend the James-Hopf maps $j_q \colon C_k X \to CD_{k,q}X \to QD_{k,q}X$ to $j_q \colon \Omega^k \Sigma^k X \to QD_{k,q}X$ we note that $\prod_{q \geq 0} j_q \colon C_k X \to \prod_{q \geq 0} QD_{k,q}X$ is an H-map and that $\prod_{q \geq 0} QD_{k,q}X$ is grouplike. A universal property of group completions [2] then allows $\prod_{q \geq 0} j_q$ to be extended to $\Omega^k \Sigma^k X$, uniquely up to weak homotopy, i.e., disregrading $\lim^1$ terms.

Similarly the homotopy $J \colon C_k X \to F(I^+, \Omega \prod_{q \geq 0} QD_{k-1,q}\Sigma X)$ of Proposition 1.1 extends, uniquely up to weak homotopy, to give a homotopy of weak H-maps $\tilde{J} \colon \Omega^k \Sigma^k X \to F(I^+, \Omega \prod_{q \geq 0} Q_{k-1,q}\Sigma X)$. Then $\tilde{J}_1 = \Omega \prod_{q \geq 0} j_q$ because $\Omega \prod_{q \geq 0} j_q$ does cover J_1 and is an H-map. As before, $\delta_{k,1} \colon \prod_{q \geq 0} QD_{k,q}X \to \prod_{q \geq 0} QD_{k-1,q}\Sigma X$ is an H-map and thus so also is $\delta_{k,1} \circ \prod_{q \geq 0} j_q$. This composite covers J_0 and thus $\tilde{J}_0 = \delta_{k,1} \circ \prod_{q \geq 0} j_q$.

Theorem 1.2 follows.

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