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IRREDUCIBLE REPRESENTATIONS OF FINITE GROUPS OF LIE TYPE THROUGH BLOCK THEORY AND SPECIAL CONJUGACY CLASSES

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This paper is concerned with the study of certain irreducible representations, over the field of complex numbers, of finite groups of Lie type, and especially with the characters afforded by these representations. The methods used are based on the theory of blocks with cyclic defect groups for certain prime different from the characteristic, called special primes, relative to which the groups have cyclic Sylow subgroups. Character values are obtained on certain regular semisimple classes, and all Deligne-Lusztig virtual characters relative to certain maximal tori are decomposed.

0. Introduction. Consider a pair (G, T) where G is a finite group of Lie type and T a maximal torus of G whose order is divisible by at least one special prime. Let Ω be a complete set of orbit representatives for the action of $N = N_G(T)$ on the set of irreducible characters of T whose orders in the character group $T^\wedge$ are relatively prime to each special prime dividing $|T|$. Then for each $\psi \in \Omega$, a family of irreducible characters of G is constructed. These families, which are pairwise disjoint, resemble closely the organization of the irreducible characters of G into blocks, and the behavior of their members reflects in a number of ways the character theory of N (see Theorem (5.2)).

That special primes exist for a wide class of pairs (G, T) is established in §2, where they are seen to arise as the primitive divisors of Zsigmondy [14]. Let (G, T) be such a pair, let π be the set of special primes dividing $|T|$, and let X be the set of elements of T having order divisible by some $r \in \pi$. Then the Brauer-Dade theory and Suzuki's theory of special conjugacy classes, as adapted to the peculiarities of this setting in §4 and §3 respectively, are employed in §5 to show (Theorem (5.2)) that the sets of nonexceptional characters in certain r-blocks of G are independent of $r \in \pi$, and that the irreducible characters of G which are of interest, namely those not vanishing on X, arise as follows. If $\theta \in T^\wedge$ and $e = [\mathrm{stab}_N(\theta): T]$, then there exist irreducible characters $\chi_1, \cdots, \chi_e$ of G and signs $\varepsilon_1, \cdots, \varepsilon_e = \pm 1$ such that for all i,

$$\chi_i|_X = \frac{\varepsilon_i}{e}\theta^N|_X \,.$$

It is shown also that $\chi_i(1) \equiv (\varepsilon_i/e)\theta^N(1) \pmod s$ where s is the π-part of $|G|$.

One of the primary objectives in the study of representations of finite groups of Lie type is the decomposition of the Deligne-Lusztig virtual characters (Deligne-Lusztig [5]) in cases where this has not yet been accomplished. In §6 such decompositions are given (Theorem (6.1)) for all Deligne-Lusztig virtual characters arising from certain maximal tori. Indeed, let $\mathscr{G}$ be a connected reductive affine algebraic group giving rise to a finite group G of Lie type, let $\mathscr{T}$ be a maximal torus of $\mathscr{G}$ giving rise to a maximal torus T of G such that $|T|$ is divisible by a special prime, and let $\theta \in T^\wedge$. Then using the previous notation (subject, however, to a possible relabeling if a certain transitivity condition holds), the result obtained is that

$$R_{\mathscr{T}}^{\mathscr{G}}(\theta) = \sum_{i=1}^{e} \varepsilon_i \chi_i ,$$

where $R_{\mathscr{T}}^{\mathscr{G}}(\theta)$ is the Deligne-Lusztig virtual character of G corresponding to $\mathscr{T}$ and $\theta \in T^\wedge$. In case $G = GL(n, q)$, this decomposition is given by Fong and Srinivasan in [7]. Finally, the Deligne-Lusztig theory is applied in §6 to obtain improvements in §5.

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1. Preliminaries.

NOTATION. We adopt the exponential notation $x^g = g^{-1}xg$ where x and g are elements of a group. If X is the union of a set of conjugacy classes of a finite group G and $\chi: X \to \boldsymbol{C}$ is constant on conjugacy classes, then for $g \in G$, $\chi^g: X^g \to \boldsymbol{C}$ is defined by $\chi^g(x^g) = \chi(x)$ for all $x \in X$.

Let $GF(q)$ be a finite field of characteristic $p > 0$ and order q, viewed as a subfield of its algebraic closure K. For greater detail in what follows, we refer the reader to [2], [11], and [13]. Let $\mathscr{G}$ be a connected reductive affine algebraic group over K with coordinate ring $\mathscr{A}$, and let $\mathscr{A}_{GF(q)} \subseteq \mathscr{A}$ be a $GF(q)$-rational structure for $\mathscr{G}$ such that the induced Frobenius morphism $F: \mathscr{G} \to \mathscr{G}$ is a homomorphism of abstract groups. Denote by G the corresponding *finite group of Lie type*, by which is meant the finite group $\mathscr{G}_F$ of fixed points of F in $\mathscr{G}$.

F-stable maximal tori of $\mathscr{G}$ are known to exist, and if $\mathscr{T}$ is such a torus, then the abelian subgroup $T = \mathscr{T}_F$ of G is called a *maximal torus* of G. Furthermore, there exists an F-stable maximal torus $\mathscr{T}'$ of $\mathscr{G}$ which is contained in an F-stable Borel subgroup $\mathscr{B}'$ of $\mathscr{G}$, and the pair $(\mathscr{T}', \mathscr{B}')$ is unique up to G-conjugacy. Therefore the classification of the maximal tori of G, which we outline below, does not depend on our choice of $(\mathscr{T}', \mathscr{B}')$. Let $W(\mathscr{T}')$ be the Weyl group $\mathscr{N}'/\mathscr{T}'$, where $\mathscr{N}' = N_{\mathscr{G}}(\mathscr{T}')$. Given an element $w = n\mathscr{T}'$ where $n \in \mathscr{N}'$, Lang's theorem guarantees the existence of an element $a \in \mathscr{G}$ such that $n = a(Fa)^{-1}$. It follows that $\mathscr{T} = a^{-1}\mathscr{T}'a$ is an F-stable maximal torus of $\mathscr{G}$, and hence that $T = \mathscr{T}_F$ is a maximal torus of G, said to be obtained by *twisting* by w. Moreover, using the action induced by F on $W(\mathscr{T}')$, we may define an equivalence relation on $W(\mathscr{T}')$, called *F-conjugacy*, whereby w_1 and w_2 are related if and only if there is an element $w_3 \in W(\mathscr{T}')$ such that $w_2 = w_3w_1(Fw_3)^{-1}$. Then the assignment

$$w \longmapsto (a^{-1}\mathscr{T}'a)_F$$

induces a bijection between the F-conjugacy classes of $W(\mathscr{T}')$ and the G-conjugacy classes of maximal tori of G.

If F acts trivially on $W(\mathscr{T}')$, then F-conjugacy degenerates to the usual notion of conjugacy, in which case we may speak (abusively) of the *Coxeter torus* of G, namely any maximal torus of G obtained by twisting by a member of the conjugacy class of Coxeter elements in $W(\mathscr{T}')$.

To obtain information about irreducible characters of G, we shall make use of Suzuki's theory of special conjugacy classes and Brauer's theory of blocks.

DEFINITION 1.1. Let N be a subgroup of G, let $\mathscr{C}_1, \cdots, \mathscr{C}_m$ be distinct conjugacy classes of N represented by the respective elements $n_1, \cdots, n_m$, and assume that the following conditions hold:

(a) For all i, $C_G(n_i) \leqq N$.

(b) If $i \neq j$, then n_i and n_j are not conjugate in G.

(c) If for some i, $n \in N$ satisfies $\langle n \rangle = \langle n_i \rangle$, then $n \in \mathscr{C}_j$ for some j.

Then $\mathscr{C}_1, \cdots, \mathscr{C}_m$ form a set of *special conjugacy classes* of N in G.

PROPOSITION 1.2 (*Suzuki, Higman*). *Let G, N, and $n_i \in \mathscr{C}_i$ $(1 \leqq i \leqq m)$ be as in* (1.1). *Then*

(a) *$X = \cup\{\mathscr{C}_i : 1 \leqq i \leqq m\}$ is a T.I. set in G and $N_G(X) = N$ (see Dornhoff* [6], p. 60).

(b) *There is a basis $\theta_1, \cdots, \theta_m$ of virtual characters of N for the complex vector space of class functions of N which vanish off X.*

(c) *Let* $\operatorname{Irr}(G) = \{\chi_1, \cdots, \chi_u\}$, *let* $\operatorname{Irr}(N) = \{\varphi_1, \cdots, \varphi_v\}$, *and set*

$\theta_i = \sum_{j=1}^{v} a_{ij}\varphi_j$ *and* $\theta_i^G = \sum_{j=1}^{u} b_{ij}\chi_j$ $(1 \leqq i \leqq m)$. *Then*

$$\sum_{k=1}^{u} b_{ik}b_{jk} = \sum_{k=1}^{v} a_{ik}a_{jk}$$

for all $i, j \in \{1, \cdots, m\}$.

(d) *There exist complex numbers* c_{jk} *determined uniquely by the equations* $\varphi_i(n_j) = \sum_{k=1}^{m} c_{jk}a_{ki}$ $(1 \leqq i \leqq v, 1 \leqq j \leqq m)$; *moreover, these numbers also satisfy the equations* $\chi_i(n_j) = \sum_{k=1}^{m} c_{jk}b_{ki}$ $(1 \leqq i \leqq u, 1 \leqq j \leqq m)$.

Proof. See Dornhoff [6], p. 149.

Detailed accounts of block theory may be found in Curtis and Reiner [3], Dornhoff [6], or Isaacs [8]. Given a prime number r and a subgroup H of G, we adopt the viewpoint that a member $B = B(r)$ of the set $\mathcal{B}\ell(H) = \mathcal{B}\ell_r(H)$ of all r-blocks of H is a subset of the disjoint union

$$\mathrm{Irr}\,(H) \cup \mathrm{IBr}(H)\,,$$

where $\mathrm{Irr}\,(H)$ denotes the set of irreducible complex characters of H and $\mathrm{IBr}\,(H)$ denotes the set of irreducible Brauer characters of H relative to r (see [8], Chapter 15). We denote by B' (resp. B'') the set $B \cap \mathrm{Irr}\,(H)$ (resp. $B \cap \mathrm{IBr}\,(H)$).

If D is a subgroup of G, then the actions of $N_G(D)$ by conjugation on $\mathrm{Irr}\,(C_G(D))$ and on $\mathrm{IBr}\,(C_G(D))$ induce an obvious action of $N_G(D)$ on $\mathcal{B}\ell(C_G(D))$.

Our primary block theoretic tool is the following portion of Dade's results on blocks with cyclic defect groups.

PROPOSITION 1.3 (*Dade* [4]). *Let* $B \in \mathcal{B}\ell(G)$ *have nontrivial cyclic defect group* D *of order* r^a. *For each* $k \in \{0, \cdots, a\}$, *let* D_k, C_k, *and* N_k *be the subgroups of* G *defined respectively by* $[D: D_k] = r^k$, $C_k = C_G(D_k)$, *and* $N_k = N_G(D_k)$. *There exists a block* b_0 *of* C_0 *satisfying* $b_0^G = B$. *Let* $E = \mathrm{Stab}_{N_0}(b_0)$, *and* $e = [E: C_0]$. *Then the following assertions hold:*

(a) *If* $b \in \mathcal{B}\ell(C_0)$, *then* b^G *is defined and* $b^G = B$ *if and only if* $b^n = b_0$ *for some* $n \in N_0$.

(b) *For each* $k \in \{0, \cdots, a-1\}$, *the block* $b_k = b_0^{C_k}$ *is defined and* $(b_k)'' = \{\varphi_k\}$ *for some* $\varphi_k \in \mathrm{IBr}\,(C_k)$.

(c) B' *contains certain distinct irreducible characters* $\chi_1, \cdots, \chi_e$ *of* G *such that there exist signs* $\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_e, \gamma_0, \gamma_1, \cdots, \gamma_{a-1} = \pm 1$ *satisfying the property that if* $k \in \{0, \cdots, a-1\}$, *then*

$$\chi_i(xy) = \frac{\varepsilon_i\gamma_k}{|EC_k|}\sum_{n \in N_k} \varphi_k(y^n)$$

for each i, where $D_k = \langle x \rangle$ and $y \in C_k$ is r-regular.

(d) *The signs ε_i and γ_k may be chosen so that $\gamma_0 = 1$. Under this assumption, replace G by C_{a-1}. Since $b_{a-1} \in B\ell(C_{a-1})$ has defect group D,* (a)–(d) *hold for C_{a-1} and b_{a-1}, giving us new signs $(\gamma_0)', \cdots, (\gamma_{a-1})'$, where we may assume that $(\gamma_0)' = 1$. Then $\gamma_0 = (\gamma_0)', \cdots, \gamma_{a-1} = (\gamma_{a-1})'$.*

LEMMA 1.4. *Retaining the notation of* (1.3), *assume that $G = C_0$. Then $e = 1$ and the proposition holds with $\varepsilon_1 = \gamma_0 = \gamma_1 = \cdots = \gamma_{a-1} = 1$.*

Proof. This is Proposition 2.1 of [4].

After the terminology of Brauer (see Dade [4]), the characters $\chi_1, \cdots, \chi_e$ in (1.3c) are called the *nonexceptional characters* of B. We remark however that if the action of E on the nontrivial irreducible characters of D is transitive, then (1.3c) holds independently of which characters of B' are labeled $\chi_1, \cdots, \chi_e$.

In [5], Deligne and Lusztig establish the existence of a set of virtual representations of G over $\boldsymbol{C}$, parameterized by pairs $(\mathscr{T}, \theta)$ where $\mathscr{T}$ is an F-stable maximal torus of $\mathscr{G}$ and θ is an irreducible character of $\mathscr{T}_F$ over $\boldsymbol{C}$, the set of which we denote by $(\mathscr{T}_F)^\wedge$ to emphasize that it forms a group isomorphic to $\mathscr{T}_F$. We shall confine our attention to the corresponding virtual characters of G, the one associated with $(\mathscr{T}, \theta)$ being denoted by $R^{\mathscr{G}}_{\mathscr{T}}(\theta)$ (or by $R_{\mathscr{T}}(\theta)$ if the reference to $\mathscr{G}$ is clear).

If $\mathscr{T}$ is an F-stable maximal torus of $\mathscr{G}$, then $N_{\mathscr{G}}(\mathscr{T})$ is F-stable, and the action of $N_{\mathscr{G}}(\mathscr{T})_F$ by conjugation on $(\mathscr{T}_F)^\wedge$ lifts to $N_{\mathscr{G}}(\mathscr{T})_F/\mathscr{T}_F$. $\theta \in (\mathscr{T}_F)^\wedge$ is said to be *in general position* if $\{w \in N_{\mathscr{G}}(\mathscr{T})_F/\mathscr{T}_F \colon \theta^w = \theta\} = \{1\}$.

For any closed connected reductive F-stable subgroup $\mathscr{H}$ of $\mathscr{G}$, denote by $\sigma(\mathscr{H})$ the common dimension of all maximal $GF(q)$-split tori of $\mathscr{H}$. If $\mathscr{T}$ is an F-stable maximal torus of $\mathscr{H}$ and $\mathscr{U}$ is the set of all unipotent elements of $\mathscr{H}$, then the function $Q^{\mathscr{H}}_{\mathscr{T}} \colon \mathscr{U}_F \to \boldsymbol{C}$, defined by $Q^{\mathscr{H}}_{\mathscr{T}}(u) = R^{\mathscr{H}}_{\mathscr{T}}(1_{\mathscr{T}_F})(u)$ for all $u \in \mathscr{U}_F$, is called *Green's function* of $\mathscr{H}$ relative to $\mathscr{T}$.

PROPOSITION 1.5 (*Deligne-Lusztig* [5]). *Let $\mathscr{T}$ be an F-stable maximal torus of $\mathscr{G}$ with $\theta, \theta' \in (\mathscr{T}_F)^\wedge$. Then the following assertions hold:*

(a) $(R_{\mathscr{T}}(\theta), R_{\mathscr{T}}(\theta'))_G = |\{w \in N_{\mathscr{G}}(\mathscr{T})_F/\mathscr{T}_F \colon \theta^w = \theta\}|$.

(b) *If $R_{\mathscr{T}}(\theta)$ is irreducible up to sign, then $\varepsilon R_{\mathscr{T}}(\theta) \in \mathrm{Irr}\,(G)$ where $\varepsilon = (-1)^{\sigma(\mathscr{T})}(-1)^{\sigma(\mathscr{G})}$.*

(c) *If $x = su$ (s semisimple, u unipotent) is the Jordan decom-*

position in $\mathscr{G}$ *of* $x \in G$, *then for any* $g \in G$, $\mathscr{C}(g) = C_{\mathscr{G}}(gsg^{-1})^{\circ}$ *is a closed connected reductive F-stable subgroup of* $\mathscr{G}$, *and*

$$R_{\mathscr{T}}^{\mathscr{G}}(\theta)(x) = |\mathscr{C}(s)_F|^{-1} \sum_{g \in G} Q_{\mathscr{T}}^{\mathscr{C}(g)}(gug^{-1})\theta^{\circ}(gsg^{-1})$$

where θ° *coincides with* θ *on* $\mathscr{T}_F$ *and is zero on* $G - \mathscr{T}_F$.

(d) $R_{\mathscr{T}}(\theta)(x) = \theta^G(x)$ *for all* $x \in \mathscr{T}_F$ *satisfying* $C_{\mathscr{G}}(x)^{\circ} = \mathscr{T}$.

2. Regular semisimple elements and special primes. If $x \in G$ is semisimple, then x is contained in some F-stable maximal torus $\mathscr{T}$ of $\mathscr{G}$, and if in addition $C_{\mathscr{G}}(x)^{\circ} = \mathscr{T}$, then $\mathscr{T}$ is clearly the unique maximal torus of $\mathscr{G}$ containing x (see Springer-Steinberg [11]).

DEFINITION 2.1. Let $x \in G$ be semisimple and let $\mathscr{T}$ be an F-stable maximal torus of $\mathscr{G}$ such that $x \in T = \mathscr{T}_F$. Let $N = N_G(T)$. Then we call x *regular* if $C_{\mathscr{G}}(x)^{\circ} = \mathscr{T}$, and we call x *locally regular* (relative to T) if $C_N(x) = T$.

By our preceding remark, the notion of regularity is well-defined.

LEMMA 2.2. *Let* $T = \mathscr{T}_F$ *be a maximal torus of* G, *and let* $x \in T$. *Then the following assertions hold*:

(a) *If* x *is locally regular* (*relative to* T), *then* x *is regular.*

(b) x *is locally regular* (*relative to* T) *if and only if* $C_G(x) = T$.

Proof. (a) is proved by Springer in [10], Lemma 6.11. Plainly $C_G(x) = T$ implies that $C_N(x) = T$, so only the converse of this remains to be proved. If $C_N(x) = T$, then by (a), $C_{\mathscr{G}}(x)^{\circ} = \mathscr{T}$. The connected component of an affine algebraic group is a normal subgroup, so $\mathscr{T} \lhd C_{\mathscr{G}}(x)$, whence $C_{\mathscr{G}}(x) \leqq N_{\mathscr{G}}(\mathscr{T})$. Now $N_{\mathscr{G}}(\mathscr{T})_F \leqq N_G(T)$, therefore $C_G(x) = C_{\mathscr{G}}(x)_F$ forces $C_G(x) \leqq N_G(T)$. Hence $C_G(x) = C_N(x) = T$, and the proof is complete.

It is implied by (2.2a) that if a semisimple element $x \in G$ is locally regular (relative to T), then $\mathscr{T}$ is the unique maximal torus of $\mathscr{G}$ which contains x. Hence the phrase "relative to T" is superfluous and we shall omit it.

DEFINITION 2.3. Let T be a maximal torus of G. A prime number r is called a *special prime* of G relative to T (or simply a special prime when the references to G and T are understood) if the following conditions hold:

(a) $r \mid |T|$.

(b) For all $x \in T$, $r \mid |x|$ implies that x is locally regular.

We denote by $\mathscr{S}(G, T)$ the set of all special primes of G relative to T.

The next result, for the proof of which the author is indebted to Gary Seitz, implies that blocks of G relative to a special prime r have cyclic defect groups, thereby enabling us to invoke 1.3.

PROPOSITION 2.4. *Let $T = \mathscr{T}_F$ be a maximal torus of G. Assume that $\mathscr{T} \neq \mathscr{G}$, and let $r \in \mathscr{S}(G, T)$. Then each $R \in \mathrm{Syl}_r(G)$ is cyclic, and there exists a unique such R contained in T.*

Proof. We may choose $x \in T$ such that $|x| = r$. Then $x \in R$ for some $R \in \mathrm{Syl}_r(G)$. We show first that $R \leqq T$. Let $1 \neq z \in Z(R)$. $C_G(x) = T$ by (2.2b) since x is locally regular. Therefore z, which centralizes x, must lie in T. Now z, which has order divisible by r, is locally regular, so $R \leqq C_G(z) = T$ by (2.2b). Since T is abelian, R is the unique element of $\mathrm{Syl}_r(G)$ contained in T.

Suppose now that R is not cyclic. Recalling that $\mathscr{T} \neq \mathscr{G}$, let $\beta\colon \mathscr{T} \to K^*$ be a root, $\mathscr{U}_\beta$ the corresponding root group of $\mathscr{G}$, and $x_\beta\colon K \to \mathscr{U}_\beta$ an isomorphism of affine algebraic groups (where K is viewed additively). Then for all $t \in \mathscr{T}$ and all $a \in K$, $tx_\beta(a)t^{-1} = x_\beta(\beta(t)a)$. $\beta(R)$ is a finite subgroup of K^*, hence it is cyclic, so since R is not cyclic there exists a nontrivial element $y \in R \cap \ker \beta$. It follows that $yx_\beta(a)y^{-1} = x_\beta(a)$ for all $a \in K$, so that $\mathscr{U}_\beta \leqq C_{\mathscr{G}}(y)$. Now $\mathscr{U}_\beta \leqq C_{\mathscr{G}}(y)^\circ$ since $\mathscr{U}_\beta$ is connected. But y is locally regular, so (2.2a) implies that $\mathscr{U}_\beta \leqq \mathscr{T}$, a contradiction. Therefore R is cyclic, thus concluding the proof.

If $\mathscr{G} = SL(2, K)$ and F is the map which raises matrix entries to the power q, then $G = SL(2, q)$, the Coxeter torus T of G is cyclic of order $q + 1$ (see, for example, 1.10 in Chapter II of Springer-Steinberg [11]), and the only elements of T which are not locally regular are ± 1 (see Theorem 38.1 and Step 1 of its proof in Dornhoff [6]). Therefore the case $q = 3^3$ shows that not every locally regular element of T need have order divisible by some $r \in \mathscr{S}(G, T)$, the case $q = 5^3$ shows that generally speaking $|\mathscr{S}(G, T)| > 1$, yet the case $q = 3$ shows that it can occur that $\mathscr{S}(G, T) = \varnothing$.

We give now conditions which lead to the existence of special primes in a variety of cases. In particular, the last example above will be seen to be deviant.

DEFINITION 2.5. Given a power $\eta \geqq 2$ of a prime p and an integer $v > 1$, the pair (η, v) is said to be *compatible* if neither of the following hold:

(a) $\eta = 2$ and $v = 6$.

(b) η is a prime of the form $2^k - 1$ for some integer $k \geqq 1$, and $v = 2$.

LEMMA 2.6 (*Zsigmondy* [15]). *If (η, v) is a compatible pair, then there exists a prime number r such that $r \mid \eta^v - 1$ and for all positive integers $b < v$, $r \nmid \eta^b - 1$.*

For each positive integer v we denote by $f_v(x)$ the vth cyclotomic polynomial.

PROPOSITION 2.7. *In order that $\mathscr{S}(G, T) \neq \varnothing$, it is sufficient that N/T be cyclic with a generator nT $(n \in N)$ of order m and that there exist an integer $s \geqq 1$ such that the following conditions hold:*
(a) $t^n = t^{q^s}$ *for all* $t \in T$.
(b) $f_{ms}(q) \mid |T|$.
(c) (q, ms) *is a compatible pair.*

Proof. By (c), (2.6) implies the existence of a prime r such that $r \mid q^{ms} - 1$ and for all positive integers $b < ms$, $r \nmid q^b - 1$. Since $q^{ms} - 1 = \prod f_b(q)$, the product being taken over all positive integers b dividing ms, r must divide $f_b(q)$ for some b. If $b < ms$, then $r \mid q^b - 1$ contradicts the property defining r. Therefore $r \mid f_{ms}(q)$, and by (b), $r \mid |T|$.

To complete the proof, we must show that if $x \in T$ satisfies $r \mid |x|$, then $C_N(x) = T$. But if such is not the case, then the cardinality d of the class of x in N satisfies $d < m$. It follows by (a) that $x^{q^{sd}} = x$, whence $x^{q^{sd}-1} = 1$, thus forcing $r \mid q^{sd} - 1$ contrary to the property defining r. Therefore $C_N(x) = T$.

We discuss some examples now where $\mathscr{S}(G, T) \neq \varnothing$. Let $G = \mathscr{G}_F$ where $\mathscr{G}$ is a connected semisimple affine algebra group over K and F is the usual Frobenius morphism induced on $\mathscr{G}$ by the field automorphism $a \mapsto a^q$ of K. Then G is a finite (untwisted) Chevalley group over $GF(q)$ (see Steinberg [14]). We assume for convenience that the root system associated with this group is indecomposable, and we base our classification of the maximal tori of G on the diagonal subgroup $\mathscr{T}'$ of $\mathscr{G}$, which is an F-stable maximal torus of $\mathscr{G}$ contained in an F-stable Borel subgroup of $\mathscr{G}$. Note that F acts trivially on the Weyl group $W = W(\mathscr{T}')$, so that the conjugacy classes of W parameterize the G-conjugacy classes of maximal tori of G.

Consider a maximal torus $T = \mathscr{T}_F$ of G obtained from $\mathscr{T}'$ by twisting by $w \in W$. We make the further assumptions that $N_G(\mathscr{T})_F = N_G(T)$ and that $(q, |w|)$ is a compatible pair. The failure of either of these conditions is incidental and rare ((2.7) of Seitz [9] implies that the first holds whenever $q \geqq 4$). Let $P_w(X)$ be the characteristic polynomial of w, w being viewed as a linear transformation of the $\boldsymbol{R}$-space $X(\mathscr{T}') \otimes_Z \boldsymbol{R}$, where $X(\mathscr{T}')$ is the group $\mathrm{Hom}(\mathscr{T}', K^*)$ of characters of $\mathscr{T}'$. We assert that $\mathscr{S}(G, T) \neq \varnothing$ if the following conditions hold:

$$C_W(w) = \langle w \rangle \; ; \tag{2.8}$$

$$f_{|w|}(q) \,|\, P_w(q) \, . \tag{2.9}$$

Indeed, in the presence of (2.8) it follows from §2 of Srinivasan [12] that $N/T \cong \langle w \rangle$, that $T = T_1^g$ and $N = N_1^g$ for some $g \in \mathscr{G}$ where $T_1 = \{t_1 \in \mathscr{T}' \colon t_1^n = Ft_1\}$ and $N_1 = \{n_1 \in N_{\mathscr{G}}(\mathscr{T}') \colon n_1^n = Fn_1\}$, and that there exists an element $n_1 \in N_1$ such that $t_1^{n_1} = Ft_1$ for all $t_1 \in T_1$. Since $Fx = x^q$ for all $x \in \mathscr{T}'$, the above facts imply that there exists an element $n \in N$ such that $t^n = t^q$ for all $t \in T$. Now by 1.7 in Chapter II of Springer-Steinberg [11], $|T| = |P_w(q)|$, so in the presence of (2.9) the assertion follows from (2.7) with $m = |w|$ and $s = 1$.

By Proposition 30 and Table 3 of Chapter [1], (2.8) and (2.9) always hold if w is a Coxeter element of W. Moreover, using Carter's terminology and tables in [1], we conclude that (2.8) and (2.9) also hold, for example, if w corresponds to one of the admissible diagrams $E_6(a_1)$, $E_7(a_1)$, $E_8(a_1)$, or $E_8(a_2)$, in which cases w is not a Coxeter element.

The algebraic groups considered above are all semisimple. However, by essentially the same discussion, $\mathscr{S}(G, T)$ can be shown to be nonempty for certain finite groups $G = \mathscr{G}_F$ of Lie type where $\mathscr{G}$ is not semisimple. For example, let $G = GL(m, q)$ and $\mathscr{G} = GL(m, K)$, let F be the usual Frobenius morphism of $\mathscr{G}$, and let $T = \mathscr{T}_F$ be the Coxeter torus of G. Then $G = \mathscr{G}_F$ is a finite group of Lie type, and we may conclude as before that $\mathscr{S}(G, T) \neq \varnothing$ provided that $N_{\mathscr{G}}(\mathscr{T})_F = N_G(T)$ and that (q, m) is compatible. (The order of a Coxeter element of the Weyl group associated with $\mathscr{G}$ is m.)

We return now to the case where $G = \mathscr{G}_F$ is an arbitrary finite group of Lie type, and we observe that our requirements in the above examples that $N_G(T)/T$ be cyclic and that $N_{\mathscr{G}}(\mathscr{T})_F = N_G(T)$ are not accidents.

PROPOSITION 2.10. *Let $T = \mathscr{T}_F$ be a maximal torus of G such that $\mathscr{T} \neq \mathscr{G}$ and $\mathscr{S}(G, T) \neq \varnothing$. Then*

(a) *$N_G(T)/T$ is cyclic.*

(b) *$N_{\mathscr{G}}(\mathscr{T})_F = N_G(T)$.*

Proof. Let $r \in \mathscr{S}(G, T)$, and let $x \in T$ have order r. Set $N =$

$N_G(T)$. Since x is locally regular, $C_N(\langle x\rangle) = C_N(x) = T$. Now $\langle x\rangle \lhd N$ by (2.4), so N/T is embedded in $\operatorname{Aut}(\langle x\rangle)$. But $\langle x\rangle$ is cyclic of prime order, hence $\operatorname{Aut}(\langle x\rangle)$ is cyclic, and (a) follows.

As for (b), we observe that $N_{\mathscr{G}}(\mathscr{T})_F \leqq N$ always holds. Now if $n \in N$, then $\mathscr{T}$ and $\mathscr{T}^n$ are both maximal tori of $\mathscr{G}$ containing x, which is regular by (2.2a). Therefore $\mathscr{T} = \mathscr{T}^n$ and $n \in N_{\mathscr{G}}(\mathscr{T})_F$, as desired.

3. Special conjugacy classes and special primes. Henceforth we fix a maximal torus $T = \mathscr{T}_F$ of G and we set $N = N_G(T)$. We assume that the set $\mathscr{S}(G, T) = \{r_1, \cdots, r_m\}$ of special primes of G relative to T is not empty. As justification for invoking (2.4), we assume also that $\mathscr{T} \neq \mathscr{G}$.

N acts by conjugation on $T^\wedge$, and we denote by N_θ the stabilizer in N of $\theta \in T^\wedge$. Note that by (2.10b), θ is in general position if and only if $N_\theta = T$.

For each j $(1 \leqq j \leqq m)$, let R_j be the unique r_j-Sylow subgroup of G contained in T whose existence is guaranteed by (2.4). Set $R = R_1 \times \cdots \times R_m$, and let Q be the unique subgroup of T satisfying $T = Q \times R$. Set $Q^\sim = \{\psi \in T^\wedge : R \leqq \ker \psi\}$ and $R^\sim = \{\lambda \in T^\wedge : Q \leqq \ker \lambda\}$. Denote by Y the set of regular elements in T, and by $Y^\wedge$ the set of characters in $T^\wedge$ which are in general position. Set

$$X = \{x \in T : r_j \,||\, |x| \text{ for some } j\}\,, \quad \text{and}$$
$$X^\wedge = \{\theta \in T^\wedge : r_j \,||\, |\theta| \text{ for some } j\}\,.$$

In view of (2.2a), $\varnothing \neq X \subseteqq Y$. Each element $x \in T$ can be expressed uniquely in the form $x = ab$ $(a \in Q, b \in R)$, and $x \in X$ if and only if $b \neq 1$.

Analogously, $T^\wedge = Q^\sim \times R^\sim$, so that each character $\theta \in T^\wedge$ can be written uniquely in the form $\theta = \psi\lambda$ $(\psi \in Q^\wedge, \lambda \in R^\wedge)$, and $\theta \in X^\wedge$ if and only if $\lambda \neq 1_T$.

LEMMA 3.1. $\varnothing \neq X^\wedge \subseteqq Y^\wedge$.

Proof. Since $T^\wedge \cong T$ as abstract groups, $\varnothing \neq X^\wedge$ is clear. Let $\theta = \psi\lambda \in X^\wedge$ $(\psi \in Q^\sim, 1_T \neq \lambda \in R^\sim)$, and choose $n \in N_\theta$. It follows that $n \in N_\lambda$, whence $x^n x^{-1} \in \ker \lambda|_R$ where x is a generator of the cyclic group R. Since $\lambda \neq 1_T$, we may choose $k \geqq 1$ such that $(x^n x^{-1})^k = 1 \neq x^k$. It follows that n centralizes x^k, whose order is divisible by some special prime. This forces $n \in T$, thus concluding the proof.

LEMMA 3.2. *X is the union of a set of special conjugacy classes of N in G.*

Proof. We prove first that the assertion holds with X replaced by Y. Indeed, Y is a union of conjugacy classes of N, for if $y \in Y$ and $n \in N$, then since by (2.10b) $N = N_{\mathscr{G}}(\mathscr{T})_F \leqq N_{\mathscr{G}}(\mathscr{T})$, we obtain $C_{\mathscr{G}}(y^n)^\circ = (C_{\mathscr{G}}(y)^\circ)^n = \mathscr{T}^n = \mathscr{T}$, thus forcing y^n to be regular. Let $\{y_1, \cdots, y_u\}$ be a complete set of representatives for the classes of N contained in Y.

For each i, $\mathscr{T} = C_{\mathscr{G}}(y_i)^\circ \lhd C_{\mathscr{G}}(y_i)$ implies that $C_{\mathscr{G}}(y_i) \leqq N_{\mathscr{G}}(\mathscr{T})$, hence $C_G(y_i) = C_{\mathscr{G}}(y_i)_F \leqq N_{\mathscr{G}}(\mathscr{T})_F = N$.

Now if $y_i^g = y_k$ for some i, some k, and some $g \in G = \mathscr{G}_F$, then $\mathscr{T}^g = (C_{\mathscr{G}}(y_i)^\circ)^g = C_{\mathscr{G}}(y_i^g)^\circ = C_{\mathscr{G}}(y_k)^\circ = \mathscr{T}$, so that $g \in N_{\mathscr{G}}(\mathscr{T})_F = N$.

Finally, suppose that $\langle y \rangle = \langle y_i \rangle$ for some $y \in N$ and some i. Then $C_{\mathscr{G}}(y)^\circ = C_{\mathscr{G}}(\langle y \rangle)^\circ = C_{\mathscr{G}}(\langle y_i \rangle)^\circ = C_{\mathscr{G}}(y_i)^\circ = \mathscr{T}$, therefore y is conjugate in N to some y_k. We have proved the assertion for Y.

Now X is clearly a union of classes of N, and we represent these classes by $x_1, \cdots, x_s \in X$. (a) and (b) of (1.1) hold for the x_i by inheritance from Y. As for (1.1c), if $x \in N$ satisfies $\langle x \rangle = \langle x_i \rangle$ for some i, then $x \in T$ and $|x| = |x_i|$, so the lemma follows.

In view of (3.2), (1.2b) guarantees the existence of a basis of virtual characters of N for the $\boldsymbol{C}$-space of class functions of N which vanish off X. Our goal is the construction of such a basis. Since $Q \lhd N$, N acts by conjugation on $Q^\check{}$. Fix a complete set Ω of orbit representatives for this action. $R \lhd N$, so for each $\psi \in \Omega$, N_ψ acts by conjugation on $R^\check{} - \{1_T\}$. Fix a complete set $\Lambda(\psi)$ of representatives for this action.

DEFINITION 3.3. For each $\psi \in \Omega$ and each $\lambda \in \Lambda(\psi)$, set $\theta_{\psi,\lambda} = \psi^N - (\psi\lambda)^N$.

N plainly acts by conjugation on $X^\wedge$.

LEMMA 3.4. *$\{\psi\lambda : \psi \in \Omega, \lambda \in \Lambda(\psi) \cup \{1_T\}\}$ is a complete set of representatives for the orbits of $T^\wedge$ under the action of N.*

Proof. By the uniqueness of the expression for each of the members of $T^\wedge$ in the form $\psi\lambda$ $(\psi \in Q^\check{}, \lambda \in R^\check{})$ and by the identity $(\psi\lambda)^n = \psi^n\lambda^n$ $(\psi \in Q^\check{}, \lambda \in R^\check{}, n \in N)$, one may deduce easily that $\{\psi\lambda : \psi \in \Omega, \lambda \in \Lambda(\psi)\}$ is a complete set of representatives for the N-orbits of $X^\wedge$. The lemma then follows from the disjoint union $T^\wedge = Q^\check{} \cup X^\wedge$.

We are prepared now to discuss the irreducible characters of N. Henceforth, for each $\theta \in T^\wedge$, we denote by $C(\theta)$ the set of irreducible constituents of θ^{N_θ}, and by n_θ the index $[N_\theta : T]$.

PROPOSITION 3.5. (a) *For each* $\psi \in \Omega$, *the maps* $\delta \mapsto \delta^N$ $(\delta \in C(\psi))$ *and* $\psi\lambda \mapsto (\psi\lambda)^N$ $(\lambda \in \Lambda(\psi))$ *are injective. Moreover,*

$$\mathrm{Irr}\,(N) = \bigcup_{\psi \in \Omega} (\{\delta^N \colon \delta \in C(\psi)\} \cup \{(\psi\lambda)^N \colon \lambda \in \Lambda(\psi)\}) ,$$

all of the indicated unions being disjoint.

(b) *Let* $\psi \in \Omega$. *Then* $|C(\psi)| = n_\psi$, $\delta|_T = \psi$ *for each* $\delta \in C(\psi)$, *and* $\psi^N = \sum_{\delta \in C(\psi)} \delta^N$.

(c) *For each* $\psi \in \Omega$, $\psi^N|_X = -n_\psi \sum_{\lambda \in \Lambda(\psi)} (\psi\lambda)^N|_X$.

Proof. (a) Set $\Delta = \Omega \cup \{\psi\lambda \colon \psi \in \Omega, \lambda \in \Lambda(\psi)\}$. Since $T \lhd N$, and by (3.4), the Clifford theory (see, for example, Isaacs [8]) implies that all irreducible characters of N are obtained, each once, in the form $\delta(\theta)^N$, where θ ranges over Δ, and for each $\theta \in \Delta$, $\delta(\theta)$ ranges over $C(\theta)$. By (3.1), each $\theta \in \Delta$ of the form $\psi\lambda$ $(\psi \in \Omega, \lambda \in \Lambda(\psi))$ has stabilizer T, hence $\delta(\theta)^N = (\psi\lambda)^N$ is irreducible. This proves (a).

(b) By (2.10a), N/T is cyclic. Let $N/T = \langle nT \rangle$ where $n \in N$, and set $m = |N/T|$. Then $n^m = t_0$ for some $t_0 \in T$. Given $\psi \in \Omega$, let $\zeta_1, \cdots, \zeta_{n_\psi}$ be the n_ψ distinct zeros in $\boldsymbol{C}$ of $X^{n_\psi} - \psi(t_0)$. Now $N_\psi = \langle T, n^{m/n_\psi} \rangle$, and each of its elements may be expressed uniquely in the form tn^{im/n_ψ} $(t \in T, 0 \leqq i < n_\psi)$. For each $j \in \{1, \cdots, n_\psi\}$, define $\delta_j \colon N_\psi \to \boldsymbol{C}$ by

$$\delta_j(tn^{im/n_\psi}) = \psi(t)\zeta_j^i .$$

Then the δ_j are distinct linear characters of N_ψ satisfying $\delta_j|_T = \psi$. Now Frobenius reciprocity implies that $(\delta_j, \psi^{N_\psi})_{N_\psi} = 1$ for all j, hence by comparing degrees we obtain $\psi^{N_\psi} = \sum_j \delta_j$. It follows that $\psi^N = \sum_j \delta_j^N$, thus proving (b).

(c) Fix $\psi \in \Omega$. Let ρ be the regular character of R. Then by decomposing elements of X relative to the decomposition $T = Q \times R$, it is straightforward that $\sum_{\lambda \in \Lambda(\psi)} (\psi\lambda)^{N_\psi}$ coincides on X with $\psi|_Q \otimes (\rho - 1_R)$, hence with $-\psi$. Therefore, choosing a right transversal D of N_ψ in N, we compute that for each $x \in X$,

$$\begin{aligned} n_\psi &\sum_{\lambda \in \Lambda(\psi)} (\psi\lambda)^N(x) \\ &= n_\psi \sum_{n \in D} \sum_{\lambda \in \Lambda(\psi)} (\psi\lambda)^{N_\psi}(x^n) \\ &= -n_\psi \sum_{n \in D} \psi(x^n) \\ &= -\psi^N(x) . \end{aligned}$$

This concludes the proof of the proposition.

PROPOSITION 3.6. *The set* $\{\theta_{\psi,\lambda} \colon \psi \in \Omega, \lambda \in \Lambda(\psi)\}$ *forms a basis of virtual characters for the* $\boldsymbol{C}$*-space of class functions of* N *which* *vanish off X.*

Proof. If $\psi \in \Omega$ and $\lambda \in \Lambda(\psi)$, then $\theta_{\psi,\lambda} = (\psi - \psi\lambda)^N$ is clearly a virtual character of N, and it vanishes off X since $\psi - \psi\lambda$ vanishes off X. Moreover, the set $S = \{\theta_{\psi,\lambda}: \psi \in \Omega, \lambda \in \Lambda(\psi)\}$ is easily seen to be linearly independent.

The dimension of the space V of class functions of N which vanish off X is equal to the number of classes of N contained in X, and since each such class consists of locally regular elements of T, we conclude that $\dim V = |X|/[N: T]$. Similarly, the number of orbits in $X^\wedge$ under the action of N is $|X^\wedge|/[N: T]$. Therefore, since $|X^\wedge| = |X|$, the result follows from the fact that $|S|$ is equal to the order of $\{\psi\lambda: \psi \in \Omega, \lambda \in \Lambda(\psi)\}$, which by (3.4) is a complete set of representatives for the orbits of $X^\wedge$ under the action of N.

4. Block theory for special primes.

LEMMA 4.1. *Fix $r_j \in \mathscr{S}(G, T)$ and let $B(r_j) \in B\ell_{r_j}(G)$ have non-trivial defect. Then R_j is a defect group of $B(r_j)$.*

Proof. Let D be a defect group of $B(r_j)$ satisfying $D \leqq R_j \leqq T$. By inheritance from R_j, D is generated by a locally regular element of T, thus $C_G(D) = T$. Now by Brauer's theory (see Theorem 64.10 of Dornhoff [6]), there exists a block $b(r_j) \in B\ell_{r_j}(T)$ with defect group D and satisfying $b(r_j)^G = B(r_j)$. But since T is abelian, $D = R_j$ follows, as desired.

Our objective now is the application of Dade's results (1.3) to our present setting. For each j, let Q_j be the unique subgroup of T satisfying $T = Q_j \times R_j$. Set $Q_j^\sim = \{\theta \in T^\wedge: R_j \leqq \ker\theta\}$ and $R_j^\sim = \{\theta \in T^\wedge: Q_j \leqq \ker\theta\}$. Since $Q_j \lhd N$, N acts on $Q_j^\sim$ by conjugation. Let Ω_j be a complete set of orbit representatives for this action. We may, and henceforth we shall, assume that $\Omega \subseteqq \Omega_j$. Set $X_j = \{x \in T: r_j \mid |x|\}$. Then clearly $X_j \subseteqq X$. If $B(r_j) \in B\ell_{r_j}(G)$, we denote by $\delta(B(r_j))$ the defect of $B(r_j)$.

LEMMA 4.2. (a) *For each j, the set $\{b(r_j)': b(r_j) \in B\ell_{r_j}(T)\}$ coincides with the set of cosets of $R_j^\sim$ in $T^\wedge$. Moreover, if $b(r_j) \in B\ell_{r_j}(T)$, then $b(r_j)''$ contains precisely one element φ which is determined uniquely by the property that for all $\theta \in b(r_j)'$, $\varphi = \theta|_{Q_j}$.*

(b) *For each j there is a bijection*

$$\Omega_j \longleftrightarrow \{B(r_j) \in B\ell_{r_j}(G): \delta(B(r_j)) \neq 0\}$$

given by $\psi \leftrightarrow b(r_j)^G$, where $b(r_j)$ is the unique r_j-block of T such that $\psi \in b(r_j)'$.

Proof. We omit the proof of (a), which is straightforward and holds when T denotes an arbitrary abelian group and r_j an arbitrary prime.

As for (b), fix $j \in \{1, \cdots, m\}$. Then $T = C_G(R_j)$ since R_j is generated by a locally regular element of T. Therefore, in view of (4.1), (a), and the fact that since T is abelian each r_j-block of T has defect group R_j, the result follows from a version of Brauer's first main theorem (see Theorem 64.10 of [6]).

DEFINITION 4.3. For each $j \in \{1, \cdots, m\}$ and each $\psi \in \Omega_j$, denote by $b_\psi(r_j)$ the unique r_j-block of T satisfying $b_\psi(r_j)' = \psi R_j^\sim$, and denote by $B_\psi(r_j)$ the r_j-block of G given by $B_\psi(r_j) = b_\psi(r_j)^G$.

PROPOSITION 4.4. *Fix $j \in \{1, \cdots, m\}$ and $\psi \in \Omega_j$. Then there are n_ψ distinct nonexceptional characters $\chi_{\psi,1}, \cdots, \chi_{\psi,n_\psi}$ in $B_\psi(r_j)$, and there exist signs $\varepsilon_{\psi,1}, \cdots, \varepsilon_{\psi,n_\psi} = \pm 1$ such that*

$$\chi_{\psi,i}|_{X_j} = (\varepsilon_{\psi,i}/n_\psi)\psi^N|_{X_j}$$

for all $i \in \{1, \cdots, n_\psi\}$.

Proof. By (4.2b) and (4.1), $B_\psi(r_j)$ has nontrivial cyclic defect group R_j, so we may apply (1.3) with $r = r_j$, $B = B_\psi(r_j)$, and $D = R_j$.

We observe that if $0 \leqq k < a$, then $C_k = T$ and $N_k = N$ in (1.3). The first equality holds because D_k is generated by a locally regular element of T. The second follows from the facts that N normalizes every subgroup of R_j, and if N_k normalizes a subgroup H of G, then it must also normalize $C_G(H)$. Therefore, for $0 \leqq k < a$, we replace C_k by T and N_k by N in (1.3).

Now $b_\psi(r_j)^G = B$, so we may take $b_0 = b_\psi(r_j)$ in (1.3). By (4.2a), and since $T^\wedge = Q_j^\sim \times R_j^\sim$ and $N_0 = N$, we conclude that $E = N_\psi$ in (1.3), hence $e = [N_\psi : T] = n_\psi$.

Let $\chi_{\psi,1}, \cdots, \chi_{\psi,n_\psi}$ be the nonexceptional characters of B and $\varepsilon_{\psi,1}, \cdots, \varepsilon_{\psi,n_\psi}$, $\gamma_0, \gamma_1, \cdots, \gamma_{a-1}$ the signs given in (1.3c). We show that these signs may be chosen so that $\gamma_0 = \gamma_1 = \cdots = \gamma_{a-1} = 1$. Indeed, invoking (1.3d), we choose the signs so that $\gamma_0 = 1$, and we apply (1.3) to C_{a-1} and b_{a-1}. But $C_{a-1} = T = C_0$ and $b_{a-1} = b_\psi(r_j)$ by our previous observation, so in effect we are applying (1.3) to C_0 and $b_\psi(r_j)$. Thus by (1.4) we obtain new signs $(\gamma_0)' = (\gamma_1)' = \cdots = (\gamma_{a-1})' = 1$, which, by (1.3d), forces $\gamma_0 = \gamma_1 = \cdots = \gamma_{a-1} = 1$.

With this choice of signs, and in view of (4.2a), (1.3c) enables us to compute that if $x \in R_j^*$ and $y \in Q_j$, then for each $i \in \{1, \cdots, n_\psi\}$,

$$\chi_{\psi,i}(xy) = (\varepsilon_{\psi,i}/|N_\psi T|) \sum_{n \in N} \varphi_k(y^n)$$

$$\begin{aligned}&= (\varepsilon_{\psi,i}/|N_\psi|) \sum_{n\in N} \psi(y^n)\\ &= (\varepsilon_{\psi,i}/n_\psi)\,|T|^{-1} \sum_{n\in N} \psi((xy)^n)\\ &= (\varepsilon_{\psi,i}/n_\psi)\psi^N(xy)\ .\end{aligned}$$

Since xy is a typical element of X_j, this concludes the proof of the proposition.

COROLLARY 4.5. *With notation as in* (4.4), *if* $\psi \in \Omega$ *and* $\lambda \in \Lambda(\psi)$, *then* $(\chi_{\psi,i}, \psi^G - (\psi\lambda)^G)_G = \varepsilon_{\psi,i}$ *for all* $i \in \{1, \cdots, n_\psi\}$.

Proof. In the following calculation we employ Frobenius reciprocity, the fact that $\psi^N - (\psi\lambda)^N$ vanishes on $N - X_j$, the proposition, and (3.5):

$$\begin{aligned}(\chi_{\psi,i}, \psi^G - (\psi\lambda)^G)_G &= (\chi_{\psi,i}|_N, \psi^N - (\psi\lambda)^N)_N\\ &= (\varepsilon_{\psi,i}/n_\psi)(\psi^N, \psi^N - (\psi\lambda)^N)_N\\ &= \varepsilon_{\psi,i}\ .\end{aligned}$$

5. Irreducible characters of G. We now assimilate the information of §4 in such a way that all primes in $\mathscr{S}(G, T)$ are dealt with simultaneously. Denote by E the set

$$\{g \in G\colon r_j \,|\, |g| \text{ for some } j \in \{1, \cdots, m\}\}\ .$$

LEMMA 5.1. (a) $E = \bigcup_{y\in G} X^y$, *the union being disjoint in the sense that* $X^y \cap X^z \neq \emptyset$ *if and only if* $X^y = X^z$.

(b) $\gamma^G|_X = \gamma|_X$ *for all complex-valued class functions* γ *of* N.

Proof. (a) $X^y \subseteqq E$ is plain for all $y \in G$, and the other inclusion follows by applications of Sylow theory and (2.2b) to the r_j-part of an element $g \in E$, where $r_j \,|\, |g|$. The assertion on disjointness holds since by (3.2) and (1.2a) X is a T. I. set in G.

(b) It suffices to show that if $x \in X$ and $g \in G$, then $x^g \in N$ implies that $g \in N$. But again by applications of Sylow theory and (2.2b) to the r_j-part of x^g in N, where $r_j \,|\, |x|$, we conclude that $x^g \in X$, thereby forcing $X \cap gXg^{-1} \neq \emptyset$. Since X is a T. I. set in G with normalizer N, it follows that $g \in N$, as desired.

In view of (5.1a), the values of a class function of G are known on E if they are known on X.

THEOREM 5.2. *For each* $\psi \in \Omega$, *there exists a sign* $\varepsilon_\psi = \pm 1$, *together with an irreducible character* $\chi_{\psi,i}$ *of* G *and a sign* $\varepsilon_{\psi,i} =$

± 1 *for each* $i \in \{1, \cdots, n_\psi\}$, *and an irreducible character* $\chi_{\psi,\lambda}$ *of* G *for each* $\lambda \in \Lambda(\psi)$, *such that the following assertions hold*:

(a) *For each* $\psi \in \Omega$ *and each* $\lambda \in \Lambda(\psi)$,

$$\psi^G - (\psi\lambda)^G = \left(\sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}\chi_{\psi,i}\right) - \varepsilon_\psi \chi_{\psi,\lambda} .$$

(b) *The map*

$$f\colon \{(\psi, \lambda)\colon \psi \in \Omega, \lambda \in \Lambda(\psi)\} \,\dot{\cup}\, \{(\psi, i)\colon \psi \in \Omega, 1 \leqq i \leqq n_\psi\} \longrightarrow \operatorname{Irr}(G) ,$$

given by $f((\psi, \beta)) = \chi_{\psi,\beta}$, *is injective.*

(c) *Given* $\psi \in \Omega$, *the set* $\{\chi_{\psi,i}\colon 1 \leqq i \leqq n_\psi\}$ *coincides with the set of nonexceptional characters in* $B_\psi(r_j)$ *for all* $r_j \in \mathscr{S}(G, T)$.

(d) *For each* $\psi \in \Omega$ *and each* $i \in \{1, \cdots, n_\psi\}$,

$$\varepsilon_{\psi,i}\chi_{\psi,i}|_X = \frac{1}{n_\psi}\psi^N|_X = -\varepsilon_\psi \sum_{\lambda \in \Lambda(\psi)} \chi_{\psi,\lambda}|_X .$$

(e) *For each* $\psi \in \Omega$ *and each* $\lambda \in \Lambda(\psi)$,

$$\chi_{\psi,\lambda}|_X = \varepsilon_\psi(\psi\lambda)^N|_X \quad \textit{and} \quad \varepsilon_\psi\chi_{\psi,\lambda}|_{G-E} = \sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}\chi_{\psi,i}|_{G-E} .$$

(f) *Let* $\chi \in \operatorname{Irr}(G)$ *be distinct from all* $\chi_{\psi,i}$ *and all* $\chi_{\psi,\lambda}$. *Then* χ *vanishes on* E.

REMARK. We shall see in §6 that ε_ψ is independent of $\psi \in \Omega$, and that (e), (f), and the first equality of (d) all hold if X is replaced by Y and E by $\bigcup_{g \in G} Y^g$.

Proof. (a) Let $\psi \in \Omega$ and $\lambda \in \Lambda(\psi)$. Since $\Omega \subseteqq \Omega_1$, (4.4) and (4.5) imply that there exist signs $\varepsilon_{\psi,1}, \cdots, \varepsilon_{\psi,n_\psi}$ and distinct irreducible characters $\chi_{\psi,1}, \cdots, \chi_{\psi,n_\psi} \in B_\psi(r_1)'$ (i.e., the nonexceptional characters in $B_\psi(r_1)$) such that

$$\chi_{\psi,i}|_{X_1} = \frac{\varepsilon_{\psi,i}}{n_\psi}\psi^N|_{X_1} \quad \text{and} \quad (\chi_{\psi,i}, \theta^G_{\psi,\lambda})_G = \varepsilon_{\psi,i} . \tag{5.3}$$

For each $\zeta \in \operatorname{Irr}(N)$ and each $\chi \in \operatorname{Irr}(G)$, let $a_\zeta = (\zeta, \theta_{\psi,\lambda})_N$ and $b_\chi = (\chi, \theta^G_{\psi,\lambda})_G$. Then by (3.2) and (3.6), we may invoke (1.2c) to obtain

$$\sum_{\chi \in \operatorname{Irr}(G)} b_\chi^2 = \sum_{\zeta \in \operatorname{Irr}(N)} a_\zeta^2 ,$$

which by (3.5) forces $\sum_\chi b_\chi^2 = n_\psi + 1$. Now by (5.3), $\sum_{i=1}^{n_\psi} b^2_{\chi_{\psi,i}} = n_\psi$, and it follows that there exists a sign $\varepsilon_{\psi,\lambda}$ and an irreducible character $\chi_{\psi,\lambda}$ of G, distinct from $\chi_{\psi,i}$ for each i, such that

$$\theta^G_{\psi,\lambda} = \left(\sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}\chi_{\psi,i}\right) - \varepsilon_{\psi,\lambda}\chi_{\psi,\lambda} .$$

The signs $\varepsilon_{\psi,\lambda}$ are independent of $\lambda \in \Lambda(\psi)$, for if $\psi \in \Omega$ and $\lambda_1, \lambda_2 \in \Lambda(\psi)$, then (1.2c) and (3.5) imply that $(\sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}^2) + \varepsilon_{\psi,\lambda_1}\varepsilon_{\psi,\lambda_2} = n_\psi + 1$, thus forcing $\varepsilon_{\psi,\lambda_1} = \varepsilon_{\psi,\lambda_2}$. Therefore we are justified in replacing $\varepsilon_{\psi,\lambda}$ by ε_ψ.

(b) We have remarked already that for each $\psi \in \Omega$ and each $\lambda \in \Lambda(\psi)$, the characters $\chi_{\psi,1}, \cdots, \chi_{\psi,n_\psi}, \chi_{\psi,\lambda}$ are distinct. Moreover, if $\psi \in \Omega$ and $\lambda_1, \lambda_2 \in \Lambda(\psi)$, then $\chi_{\psi,\lambda_1} = \chi_{\psi,\lambda_2}$ implies by (a) that $\theta^G_{\psi,\lambda_1} = \theta^G_{\psi,\lambda_2}$. Now it follows from (5.1b) that induction is an isometry, hence a monomorphism, from the $\boldsymbol{C}$-space of class functions of N vanishing off X into the $\boldsymbol{C}$-space of class functions of G. We conclude that $\theta_{\psi,\lambda_1} = \theta_{\psi,\lambda_2}$, whence $(\psi\lambda_1)^N = (\psi\lambda_2)^N$. (3.5a) now forces $\lambda_1 = \lambda_2$.

Therefore in view of (a) we must show that if $\psi_1, \psi_2 \in \Omega$ are distinct with $\lambda_1 \in \Lambda(\psi_1)$ and $\lambda_2 \in \Lambda(\psi_2)$, then no irreducible constituent of $\theta^G_{\psi_1,\lambda_1}$ can be an irreducible constituent of $\theta^G_{\psi_2,\lambda_2}$. But by the proof of (a), $\{\chi_{\psi_1,i}\colon 1 \leqq i \leqq n_{\psi_1}\} \subseteq B_{\psi_1}(r_1)'$ and $\{\chi_{\psi_2,i}\colon 1 \leqq i \leqq n_{\psi_2}\} \subseteq B_{\psi_2}(r_1)'$. And by (4.2b), $B_{\psi_1}(r_1)' \cap B_{\psi_2}(r_1)' = \varnothing$, so it suffices to show that $\chi_{\psi_1,i} \neq \chi_{\psi_2,\lambda}$ for all $i \in \{1, \cdots, n_{\psi_1}\}$ and all $\lambda \in \Lambda(\psi_2)$, because then by (a), $\chi_{\psi_1,\lambda_1} = \chi_{\psi_2,\lambda_2}$ ($\lambda_j \in \Lambda(\psi_j)$, $j = 1, 2$) cannot occur since $(\theta^G_{\psi_1,\lambda_1}, \theta^G_{\psi_2,\lambda_2})_G = (\theta_{\psi_1,\lambda_1}, \theta_{\psi_2,\lambda_2})_N = 0$.

In order to do this we invoke (1.2d). The class functions $\theta_{\psi,\lambda}$ may be indexed by pairs (ψ, λ) ($\psi \in \Omega$, $\lambda \in \Lambda(\psi)$). Let C be a complete set of representatives of the classes of N contained in X. By (1.2d) there exist uniquely determined complex numbers $c_{x;\psi,\lambda}$ ($x \in C$, $\psi \in \Omega$, $\lambda \in \Lambda(\psi)$) satisfying

$$\zeta(x) = \sum_{\psi \in \Omega} \sum_{\lambda \in \Lambda(\psi)} c_{x;\psi,\lambda} a_{\psi,\lambda;\zeta} \tag{5.4}$$

for all $\zeta \in \mathrm{Irr}(N)$ and all $x \in C$, where $a_{\psi,\lambda;\zeta} = (\zeta, \theta_{\psi,\lambda})_N$. Moreover, the numbers $c_{x;\psi,\lambda}$ also satisfy

$$\chi(x) = \sum_{\psi \in \Omega} \sum_{\lambda \in \Lambda(\psi)} c_{x;\psi,\lambda} b_{\psi,\lambda;\chi} \tag{5.5}$$

for all $\chi \in \mathrm{Irr}(G)$ and all $x \in C$, where $b_{\psi,\lambda;\chi} = (\chi, \theta^G_{\psi,\lambda})_G$. Now by (3.5), for all $\psi \in \Omega$ and all $\lambda \in \Lambda(\psi)$,

$$a_{\psi,\lambda;\zeta} = \begin{cases} 1 & \text{if} \quad \zeta = \delta^N \quad \text{for some} \quad \delta \in C(\psi) \\ -1 & \text{if} \quad \zeta = (\psi\lambda)^N \\ 0 & \text{if} \quad \zeta \in \mathrm{Irr}(N) \quad \text{is otherwise}\,. \end{cases} \tag{5.6}$$

And by (a), for all $\psi \in \Omega$ and all $\lambda \in \Lambda(\psi)$,

$$b_{\psi,\lambda;\chi} = \begin{cases} \varepsilon_{\psi,i} & \text{if} \quad \chi = \chi_{\psi,i} \quad \text{for some} \quad i \in \{1, \cdots, n_\psi\} \\ -\varepsilon_\psi & \text{if} \quad \chi = \chi_{\psi,\lambda} \\ 0 & \text{if} \quad \chi \in \mathrm{Irr}(G) \quad \text{is otherwise}\,. \end{cases} \tag{5.7}$$

Fix $\psi' \in \Omega$ and $i \in \{1, \cdots, n_{\psi'}\}$, and set

$$B = \{\psi \in \Omega\colon \chi_{\psi',i} = \chi_{\psi,\lambda} \quad \text{for some} \quad \lambda \in \Lambda(\psi)\} .$$

We must show that $B = \emptyset$. By our previous discussion, $\psi' \notin B$ and if $\psi \in B$, then there is a unique element $\lambda_\psi \in \Lambda(\psi)$ such that $\chi_{\psi',i} = \chi_{\psi,\lambda_\psi}$. Let $x \in X_1 \cap R^*$. Then by (5.3), (5.5), (5.7), (5.6), (5.4), and (3.5c) respectively,

$$\begin{aligned}(\varepsilon_{\psi',i}/n_{\psi'})\psi'^N(x) &= \chi_{\psi',i}(x)\\ &= \sum_{\psi \in \Omega}\sum_{\lambda \in \Lambda(\psi)} c_{x;\psi,\lambda} b_{\psi,\lambda;\chi_{\psi',i}}\\ &= \sum_{\lambda \in \Lambda(\psi')} c_{x;\psi',\lambda}\varepsilon_{\psi',i} + \sum_{\psi \in B} c_{x;\psi,\lambda_\psi}(-\varepsilon_\psi)\\ &= -\varepsilon_{\psi',i}\sum_{\lambda \in \Lambda(\psi')} c_{x;\psi',\lambda} a_{\psi',\lambda;(\psi'\lambda)^N}\\ &\quad + \sum_{\psi \in B}\sum_{\lambda \in \Lambda(\psi)} c_{x;\psi,\lambda} a_{\psi,\lambda;(\psi\lambda_\psi)^N}(\varepsilon_\psi)\\ &= -\varepsilon_{\psi',i}\sum_{\lambda \in \Lambda(\psi')}(\psi'\lambda)^N(x) + \sum_{\psi \in B}\varepsilon_\psi(\psi\lambda_\psi)^N(x)\\ &= \frac{\varepsilon_{\psi',i}}{n_{\psi'}}\psi'^N(x) + \sum_{\psi \in B}\varepsilon_\psi(\psi\lambda_\psi)^N(x) .\end{aligned}$$

It follows that $\sum_{\psi \in B}\varepsilon_\psi(\psi\lambda_\psi)^N(x) = 0$.

Next we observe that for each $\psi \in B$, $\varepsilon_\psi(\psi\lambda_\psi)^N(x) = \chi_{\psi,\lambda_\psi}(x)$. Indeed, it follows from (a), (5.3), and (5.1b) that

$$\begin{aligned}\varepsilon_\psi\chi_{\psi,\lambda_\psi}(x) &= \left(\sum_{k=1}^{n_\psi}\varepsilon_{\psi,k}\chi_{\psi,k}(x)\right) - (\psi - \psi\lambda_\psi)^G(x)\\ &= \left(\sum_{k=1}^{n_\psi}\frac{1}{n_\psi}\psi^N(x)\right) - (\psi^N(x) - (\psi\lambda_\psi)^N(x))\\ &= (\psi\lambda_\psi)^N(x) ,\end{aligned}$$

and the observation follows. Therefore $0 = \sum_{\psi \in B}\chi_{\psi,\lambda_\psi}(x) = |B|(\varepsilon_{\psi',i}/n_{\psi'})\psi'^N(x)$ by (5.3). Finally, $\psi'^N(x) = [N\colon T]$ since $\psi' \in Q^{\sim}$ and $x \in R$, so $0 = |B|\varepsilon_{\psi',i}[N\colon N_{\psi'}]$, whence $B = \emptyset$.

(c) In the proof of (a), the characters $\chi_{\psi,i}$ $(1 \leqq i \leqq n_\psi)$ arose as the nonexceptional characters in $B\psi(r_1)$. So if $\mathscr{S}(G, T) = \{r_1\}$ there is nothing to prove. Therefore, let $r_j \in \mathscr{S}(G, T)$ be distinct from r_1. Let χ_i $(1 \leqq i \leqq n_\psi)$ be the nonexceptional characters in $B_\psi(r_j)$ (see (4.4)). As in the proof of (a), χ_i is an irreducible constituent of $\psi^G - (\psi\lambda)^G$ for all i and all $\lambda \in \Lambda(\psi)$. Suppose $\chi_i \notin \{\chi_{\psi,k}\colon 1 \leqq k \leqq n_\psi\}$ for some i. It follows from (a) that $\chi_i = \chi_{\psi,\lambda}$ for all $\lambda \in \Lambda(\psi)$, and then from (b) that $|\Lambda(\psi)| = 1$. Since $R_1^{\sim}$ is invariant under the action of N on $R^{\sim}$, this forces $R^{\sim} = R_1^{\sim}$, and $\mathscr{S}(G, T) = \{r_1\}$ follows, in violation of our choice of r_j.

(d), (e) In view of (c), the proof of (a) may be adapted to show that for each $\psi \in \Omega$ and each $i \in \{1, \cdots, n_\psi\}$,

$$\chi_{\psi,i}|_{X_j} = \frac{\varepsilon_{\psi,i}}{n_\psi}\psi^N|_{X_j}$$

for all $j \in \{1, \cdots, m\}$. Since $X = \bigcup_j X_j$, the first equality of (d) follows. This equality, together with (a), implies the first equality of (e), which in turn, together with (3.5c), implies the second equality of (d). Finally, for each $\psi \in \Omega$ and each $\lambda \in \Lambda(\psi)$, $\psi^G - (\psi\lambda)^G$ vanishes on $G - E$ since $\psi - \psi\lambda$ vanishes on $T - X$. Therefore the last part of (e) follows from (a).

(f) By (a), $b_{\psi,\lambda;\chi} = 0$ for all $\psi \in \Omega$ and all $\lambda \in \Lambda(\psi)$. Thus (5.5) implies that χ vanishes on E, and this concludes the proof of the theorem.

COROLLARY 5.8. *The following congruences* (mod $|R|$) *hold in* $\boldsymbol{Z}$:

(a) $\chi_{\psi,i}(1) \equiv (\varepsilon_{\psi,i}/n_\psi)\psi^N(1)$ $(\psi \in \Omega,\ 1 \leqq i \leqq n_\psi)$.

(b) $\chi_{\psi,\lambda}(1) \equiv \varepsilon_\psi(\psi\lambda)^N(1)$ $(\psi \in \Omega,\ \lambda \in \Lambda(\psi))$.

(c) $\chi(1) \equiv 0$ *whenever* $\chi \in \operatorname{Irr}(G)$ *is distinct from all* $\chi_{\psi,i}$ *and all* $\chi_{\psi,\lambda}$.

Proof. Fix $\psi' \in \Omega$ and $i \in \{1, \cdots, n_{\psi'}\}$. Set $\Omega' = \Omega - \{\psi'\}$. From Frobenius reciprocity and (5.2a, b), it follows for all $\psi \in Q^\sim$ and all $\lambda \in R^\sim - \{1_T\}$ that

$$(\psi - \psi\lambda,\ \chi_{\psi',i}|_T)_T = \begin{cases} \varepsilon_{\psi',i} & \text{if } \psi^n = \psi' \text{ for some } n \in N \\ 0 & \text{otherwise}\ . \end{cases}$$

Therefore, setting $a_\psi = (\psi, \chi_{\psi',i}|_T)_T$ for all $\psi \in \Omega$, we conclude that for all $n \in N$ and all $\lambda \in R^\sim - \{1_T\}$,

$$(\psi^n\lambda,\ \chi_{\psi',i}|_T)_T = \begin{cases} a_{\psi'} - \varepsilon_{\psi',i} & \text{if } \psi = \psi' \\ a_\psi & \text{if } \psi \neq \psi'\ . \end{cases}$$

Now for each $\psi \in \Omega$, let d_ψ be the order $[N: N_\psi]$ of the orbit of ψ under the action of N on $T^\wedge$. Then by the uniqueness of the expression $\theta = \psi\lambda$ $(\psi \in Q^\sim, \lambda \in R^\sim)$ for each $\theta \in T^\wedge$, we compute that

$$\begin{aligned} \chi_{\psi',i}(1) &= \sum_{\theta \in T^\wedge} (\theta, \chi_{\psi',i}|_T)_T\theta(1) \\ &= \sum_{\psi \in \Omega} d_\psi \sum_{\lambda \in R^\sim} (\psi\lambda, \chi_{\psi',i}|_T)_T \\ &= |R|\Big(\sum_{\psi \in \Omega'} d_\psi a_\psi\Big) + d_{\psi'}[a_{\psi'} + (|R| - 1)(a_{\psi'} - \varepsilon_{\psi',i})] \\ &= |R|\Big[\Big(\sum_{\psi \in \Omega} d_\psi a_\psi\Big) - d_{\psi'}\varepsilon_{\psi',i}\Big] + \frac{\varepsilon_{\psi',i}}{n_{\psi'}}\psi'^N(1)\ . \end{aligned}$$

This proves (a), from which, in view of (5.2a), (b) follows. The proof of (c) is similar to but easier than that of (a).

Since R^* is a union of conjugacy classes of N, each of order $[N: T]$, we conclude that $(|R|, [N: T]) = 1$. Therefore for all $\chi \in \operatorname{Irr}(G)$ it follows from (5.8) that $|R| \mid \chi(1)$ if χ is distinct from all $\chi_{\psi,i}$ and all $\chi_{\psi,\lambda}$, whereas $\chi(1)$ is relatively prime to $|R|$ otherwise.

6. The connection with the Deligne-Lusztig theory. Let $\varepsilon = (-1)^{\sigma(\mathscr{G})}(-1)^{\sigma(\mathscr{T})}$, as in (1.5b).

THEOREM 6.1. *For each $\psi \in \Omega$, each $i \in \{1, \cdots, n_\psi\}$, and each $\lambda \in \Lambda(\psi)$, let $\chi_{\psi,i}$, $\varepsilon_{\psi,i}$, $\chi_{\psi,\lambda}$, and ε_ψ be as in* (5.2). *Then $\varepsilon_\psi = \varepsilon$. Moreover, (after a possible relabeling of the characters $\chi_{\psi,1}, \cdots, \chi_{\psi,n_\psi}$, $\chi_{\psi,\lambda}$ and the signs $\varepsilon_{\psi,1}, \cdots, \varepsilon_{\psi,n_\psi}$, ε in case $|\Lambda(\psi)| = 1$)*

$$R^{\mathscr{G}}_{\mathscr{T}}(\psi\lambda) = \varepsilon\chi_{\psi,\lambda} \quad \textit{and} \quad R^{\mathscr{G}}_{\mathscr{T}}(\psi) = \sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}\chi_{\psi,i} .$$

Proof. For $\theta \in T^\wedge$, we write $R^{\mathscr{G}}_{\mathscr{T}}(\theta)$ in the abbreviated form $R(\theta)$. Fix $\psi \in \Omega$. Since by (2.2a) each element of X is regular, (1.5d) implies that for all $\lambda \in \Lambda(\psi)$, $R(\psi) - R(\psi\lambda)$ and $\psi^G - (\psi\lambda)^G$ agree on $E = \bigcup_{g \in G} X^g$. From the fact that $\psi - \psi\lambda$ vanishes on $T - X$ for all $\lambda \in \Lambda(\psi)$, it follows that $\psi^G - (\psi\lambda)^G$ vanishes on $G - E$, and the character formula (1.5c) implies that $R(\psi) - R(\psi\lambda)$ vanishes on $G - E$ as well. Thus by (5.2a),

$$R(\psi) - R(\psi\lambda) = \left(\sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}\chi_{\psi,i}\right) - \varepsilon_\psi\chi_{\psi,\lambda} \tag{6.2}$$

for all $\lambda \in \Lambda(\psi)$.

Now using (1.5a, b), it suffices to show that for all $\psi \in \Omega$ and $\lambda \in \Lambda(\psi)$, $R(\psi\lambda) = \varepsilon\chi_{\psi,\lambda}$. Fix $\psi \in \Omega$. (1.5) implies for all $\lambda \in \Lambda(\psi)$ that $\varepsilon R(\lambda) \in \operatorname{Irr}(G)$ and $R(\psi), R(\psi\lambda))_G = 0$, If $\lambda_0 \in \Lambda(\psi)$ satisfies $R(\psi\lambda_0) \neq \varepsilon\chi_{\psi,\lambda_0}$, then (6.2) implies that $\varepsilon R(\psi\lambda_0) = \chi_{\psi,k}$ for some k. It follows that $\chi_{\psi,k}$ is not an irreducible constituent of $R(\psi)$. But then by (6.2) again, $\chi_{\psi,k} = \varepsilon R(\psi\lambda)$ for all $\lambda \in \Lambda(\psi)$. By (5.2b) this forces $|\Lambda(\psi)| = 1$. So again we conclude that $R = \langle x \rangle$ where $|x| = r$ and r is the unique element of $\mathscr{S}(G, T)$.

It follows (see Dade [4]) that $B_\psi(r)'$ consists precisely of $\chi_{\psi,1}, \cdots, \chi_{\psi,n_\psi}$ (the nonexceptional characters of $B_\psi(r)$), and χ_{ψ,λ_0}. Since $\Lambda(\psi) = \{\lambda_0\}$, the action of N on $R^\wedge$-$\{1_T\}$ is transitive, hence (1.3c) is independent of which characters in $B_\psi(r)'$ are called the nonexceptional ones. Therefore all of our previous results remain valid if we relabel the elements of $B_\psi(r)'$ and the corresponding signs in such a way that $\chi_{\psi,\lambda_0} = \varepsilon R(\psi\lambda_0)$. Then (6.2) implies that

$R(\psi) = \sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}\chi_{\psi,i}$, as desired. This concludes the proof of the theorem.

It should be remarked that the information contained in (6.1) is complete in the sense that

$$\{R_{\mathscr{T}}^{\mathscr{G}}(\theta)\colon \theta \in T^{\wedge}\} = \{R_{\mathscr{T}}^{\mathscr{G}}(\psi\lambda)\colon \psi \in \Omega,\ \lambda \in \Lambda(\psi) \cup \{1_T\}\}\ .$$

This follows by (3.4) and the character formula (1.5c).

Thanks to (6.1), the multiplicity of each $\chi \in \mathrm{Irr}\,(G)$ in each $R_{\mathscr{T}}^{\mathscr{G}}(\theta)$ is known (up to sign). Therefore for $\chi \in \mathrm{Irr}\,(G)$, we may apply the formula 7.6.2 of Deligne-Lusztig [5], which states that for all regular elements y in T,

$$\chi(y) = \sum_{\theta \in T^{\wedge}} (\chi, R_{\mathscr{T}}^{\mathscr{G}}(\theta))_G \theta(y)\ .$$

Familiar arguments then show that (5.2e), (5.2f), and the first equality of (5.2d) remain valid if X is replaced by Y and E by $\bigcup_{g \in G} Y^g$.

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