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THE SPLITTING OF OPERATOR ALGEBRAS, II

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Let $\{A_\alpha: \alpha \in A\}$ be a family of C^*-algebras (resp., W^*-algebras). For $\alpha_0 \in A$, we let $P_{\alpha_0}: \oplus_\alpha A_\alpha \to A_{\alpha_0}$ denote the canonical coordinate projection of $\oplus_\alpha A_\alpha$ onto A_{α_0}. If B is a C^*-(resp., W^*-) subalgebra of $\oplus_\alpha A_\alpha$, we say that B *splits* if $B = \oplus_\alpha P_\alpha(B)$. In this note, we give conditions both necessary and sufficient for B to split. In the C^*-category, these conditions are given in terms of separation properties of the spectrum and primitive ideal space of B, and in the W^*-category, the conditions are expressed in terms of disjointness of certain subsets of the center of B. We also give examples to show that these conditions cannot be weakened, and are hence the best possible of their kind.

In [4], Sze-kai Tsui and the author obtained several results on the splitting of singly-generated operator algebras. Theorems 2.1 and 3.4 of [4] are the principle results of that paper, and it is the purpose of this paper to present results which both improve and generalize the main results of [4].

If A is a C^*-algebra (resp., W^*-algebra) and $a \in A$, then $C^*(a)$ (resp., $W^*(a)$) denotes the C^*-subalgebra (resp., W^*-subalgebra) of A generated by a. Let π be a representation of A_{α_0}, for some fixed $\alpha_0 \in A$. We define a representation $\tilde{\pi}$ of $\bigoplus_\alpha A_\alpha$ by

$$\tilde{\pi}: \bigoplus_\alpha a_\alpha \longrightarrow \pi(a_{\alpha_0}) , \qquad \bigoplus_\alpha a_\alpha \in \bigoplus_\alpha A_\alpha .$$

The sets

$$\sum\nolimits_{n_0} = \{\ker(\tilde{\rho}|_{C^*(\oplus_\alpha a_\alpha)}): \rho \text{ an irreducible representation of } C^*(a_{\alpha_0})\}$$

are subsets of the primitive ideal space of $C^*(\bigoplus_\alpha a_\alpha)$. The first main result of [4] asserted that $C^*(a_1 \oplus a_2)$ splits if and only if $\sum_1$ and $\sum_2$ disconnect the pimitive ideal space of $C^*(a_1 \oplus a_2)$ equipped with the hull-kernel topology. In Theorem 2.2 of this paper, we improve and generalize this to arbitrary C^*-subalgebras of arbitrary direct sums of C^*-algebras.

Let N be W^*-algebra with predual N_* and let τ be a $\sigma(N, N_*)$-continuous representation of N. We set $\operatorname{supp} \tau =$ complement of the central support projection of $\ker \tau$ in N. We denote the class of all nonzero $\sigma(N, N_*)$-continuous representations of N by $\operatorname{Rep}_\sigma(N)$. If S and T are subsets of N, we say that S and T are orthogonal if $st = ts = 0$, for $s \in S$ and $t \in T$.

Let $N_\alpha: \alpha \in \mathscr{A}$ be a family of W^*-algebras, with $\bigoplus_\alpha n_\alpha$ a fixed element of $\bigoplus_\alpha N_\alpha$. We set

$$S_{\alpha_0} = \{\mathrm{supp}\,(\tilde{\tau}|_{W^*(\oplus_\alpha n_\alpha)}): \tau \in \mathrm{Rep}_\sigma\,(W^*(n_{\alpha_0}))\}\,.$$

The second main theorem of [4] asserted that $W^*(n_1 \oplus n_2)$ splits if and only if S_1 and S_2 are orthogonal and $\sup (S_1 \cup S_2) =$ identity in $W^*(n_1) \oplus W^*(n_2)$. In Theorem 2.4 of this paper we improve and generalize this to arbitrary W^*-subalgebras of arbitrary direct sums of W^*-algebras.

2. Solution of the splitting problem. Let $\{A_\alpha: \alpha \in \mathfrak{A}\}$ be a family of C^*-algebras, and let $P_\alpha: \bigoplus_\alpha A_\alpha \to A_\alpha$ denote the canonical coordinate projection of $\bigoplus_\alpha A_\alpha$ onto A_α. A C^*-subalgebra B of $\bigoplus_\alpha A_\alpha$ is said to be *substantial* in $\bigoplus_\alpha A_\alpha$ if $P_\alpha(B) = A_\alpha$, for each $\alpha \in \mathfrak{A}$. A C^*-subalgebra B of $\bigoplus_\alpha A_\alpha$ is said to *split* if $B = \bigoplus_\alpha P_\alpha(B)$. The question that concerns us asks: when does a C^*-subalgebra of $\bigoplus_\alpha A_\alpha$ split?

The following lemma, the key to our answer to this question, is a trivial modification of a result kindly suggested to us by Don Hadwin, who in turn heard it from T. B. Hoover:

LEMMA 2.1. *Let $\{A_1, \cdots, A_n\}$ be C^*-algebras, with B a substantial C^*-subalgebra of $A_1 \oplus \cdots \oplus A_n$. Then $B = A_1 \oplus \cdots \oplus A_n$ if and only if the following condition holds: there exist no distinct indices i and j and irreducible representations ρ_α of A_α, $\alpha = i, j$, for which $\tilde{\rho}_i|_B = \tilde{\rho}_j|_B$.*

Proof. ($\Rightarrow$). This is clear.

($\Leftarrow$). Fix $i \neq j$. It sufficies to show that $(P_i \oplus P_j)(B) = A_i \oplus A_j$, and hence we may assume with no loss of generality that $n = 2$. Set $J_i = B \cap \ker (P_i)$, $i = 1, 2$. Then $J_1 + J_2$ is a closed, two-sided ideal in B. Let $a_1 \in A_1$. Since $P_1(B) = A_1$, there exists $a' \in A_2$ such that $a_1 \oplus a' \in B$. Define the $*$-homomorphism $\sigma_1: A_1 \to B/(J_1 + J_2)$ by $\sigma_1: a_1 \to a_1 \oplus a' + (J_1 + J_2)$. Let $a_2 \in A_2$. Since $P_2(B) = A_2$, there exists $a'' \in A_1$ such that $a'' \oplus a_2 \in B$. Define the $*$-homomorphism $\sigma_2: A_2 \to B/(J_1 + J_2)$ by $\sigma_2: a_2 \to a'' \oplus a_2 + (J_1 + J_2)$. One easily checks that $\tilde{\sigma}_1|_B = \tilde{\sigma}_2|_B$. Suppose $\tilde{\sigma}_1(B) \neq (0)$. Let ρ be an irreducible representation of $\tilde{\sigma}_1(B)$. Since $\sigma_1(A_1) = \tilde{\sigma}_1(B) = \tilde{\sigma}_2(B) = \sigma_2(A_2)$, $\rho_i = \rho \circ \sigma_i$ is an irreducible representation of A_i, $i = 1, 2$, and we thus have $\tilde{\rho}_1|_B = \tilde{\rho}_2|_B$, contrary to assumption. Thus $\tilde{\sigma}_1(B) = \tilde{\sigma}_2(B) = (0)$, whence $\sigma_1 = \sigma_2 = 0$. It follows that $J_1 = (0) \oplus A_2$, $J_2 = A_1 \oplus (0)$, whence $B = A_1 \oplus A_2$. □

We now introduce some notation and terminology for the statement and proof of our principle result.

Let A be a C^*-algebra. We let A^{**} denote the enveloping W^*-algebra of A, realized as the ultraweak closure of the image of A under its universal representation. If S is a subset of A^{**}, we will denote the ultraweak closure of S by S^-. If I is a closed, two-sided ideal in A then I^- is an ultraweakly closed, two-sided ideal in A^{**}, so there is a central projection p of A^{**} such that $I^- = A^{**}p$. We set $s(I) = p$.

If p is a central projection of A^{**}, the representation of A defined by $a \to ap$, $a \in A$, will be denoted by π_p.

If B is a C^*-subalgebra of A, we will write $B/B \cap I = A/I$ to indicate that the canonical injection of $B/B \cap I$ into A/I is surjective.

The class of all irreducible representations of A will be denoted by $\mathrm{Irr}\,(A)$, and we identify $\mathrm{Irr}\,(A/I)$ with $\{\rho \in \mathrm{Irr}\,(A) \colon I \subseteq \ker(\rho)\}$.

We recall that two representations of A are *disjoint* if they have no nonzero, unitarily equivalent subrepresentations.

Finally, we need to consider the *restricted* direct sum $\hat{\bigoplus}_\alpha A_\alpha$ of a family $A_\alpha \colon \alpha \in \mathfrak{A}$ of C^*-algebras. By definition, $\hat{\bigoplus}_\alpha A_\alpha$ is the closed, two-sided ideal of $\bigoplus_\alpha A_\alpha$ consisting of all elements $\bigoplus_\alpha a_\alpha$ for which the sets $\{\alpha \in \mathfrak{A} \colon \|a_\alpha\| \geqq \varepsilon\}$ are finite for each $\varepsilon > 0$.

We can now present our solution of the splitting problem for arbitrary families of C^*-algebras:

THEOREM 2.2. *Let $A_\alpha \colon \alpha \in \mathfrak{A}$ be a family of C^*-algebras, B a C^*-subalgebra of $\bigoplus_\alpha A_\alpha$. Let $A = \bigoplus_\alpha P_\alpha(B)$, $I = \hat{\bigoplus}_\alpha P_\alpha(B)$. The following are equivalent*:

(i) *B splits*;

(ii) *$B/B \cap I = A/I$, and the sets*

$$\{\ker(\rho|_B) \colon \rho \in \mathrm{Irr}\,(A/I)\}\,, \qquad \{\ker(\tilde{\rho}|_B) \colon \rho \in \mathrm{Irr}\,(P_\alpha(B))\}\,, \qquad \alpha \in \mathfrak{A}\,,$$

are pairwise disjoint subsets of the primitive ideal space of B;

(iii) *$B/B \cap I = A/I$, and the following condition holds: for each fixed $\alpha \in \mathfrak{A}$ and $(\alpha_1, \alpha_2) \in \mathfrak{A} \times \mathfrak{A}$ with $\alpha_1 \neq \alpha_2$, and for each ordered pair (ρ_1, ρ_2) in* $\mathrm{Irr}\,(P_{\alpha_1}(B)) \times \mathrm{Irr}\,(P_{\alpha_2}(B))$ *(resp.,* $\mathrm{Irr}\,(A/I) \times \mathrm{Irr}\,(P_\alpha(B))$*), we have $\tilde{\rho}_1|_B \neq \tilde{\rho}_2|_B$ (resp., $\rho_1|_B \neq \tilde{\rho}_2|_B$).*

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear.

(iii) $\Rightarrow$ (i). We may assume with no loss of generality that B is substantial in $A = \bigoplus_\alpha A_\alpha$. Let $p = s(I)$, so that $I^- = A^{**}p$. The map $a + I \to a(1-p)$ of A/I into $A^{**}(1-p)$ extends to an isomorphism of $(A/I)^{**}$ onto $A^{**}(1-p)$. Since $B/B \cap I = A/I$, we conclude that $B^-(1-p) = A^{**}(1-p)$.

Let Σ denote the set of all finite subsets of the indexing set

$\mathfrak{A}$. For each $\sigma \in \Sigma$ and $a = \bigoplus_\alpha a_\alpha \in \bigoplus_\alpha A_\alpha$, set $a_\sigma = \bigoplus\{a_\alpha : \alpha \in \sigma\}$, $A_\sigma = \bigoplus\{A_\alpha : \alpha \in \sigma\}$, $P_\sigma = \bigoplus\{P_\alpha : \alpha \in \sigma\}$, and $B_\sigma = P_\sigma(B)$. It follows from the hypothesis that B_σ is a substantial C^*-subalgebra of A_σ which satisfies the conditions of Lemma 2.1, so by that lemma, $B_\sigma = A_\sigma$. Thus $P_\sigma|_B$ implements a $*$-isomorphism of $B/\ker(P_\sigma|_B)$ onto A_σ, and since this isomorphism is an isometry, it follows that B has the following property:

(*) for each $a = \bigoplus_\alpha a_\alpha \in \bigoplus_\alpha A_\alpha$ and $\sigma \in \Sigma$, there exists $b_\sigma = \bigoplus_\alpha b_\alpha^\sigma \in B$ such that $\|b_\sigma\| \leqq 1 + \|a\|$ and $b_\alpha^\sigma = a_\alpha$, for each $\alpha \in \sigma$.

Set $P_\alpha =$ support projection of A_α^{**} in A^{**}. Then $\{p_\alpha : \alpha \in \mathfrak{A}\}$ is a family of pairwise orthogonal projections of I^- such that $\bigoplus_\alpha p_\alpha = p$. Letting $p_\sigma = \bigoplus\{p_\alpha : \alpha \in \sigma\}$ for each $\sigma \in \Sigma$, and considering Σ as a net, ordered by inclusion, we have $\lim_\sigma \|x - xp_\sigma\| = 0$, for each $x \in I$.

Fix $x \in I$. By (*), for each $\sigma \in \Sigma$ there exists $b_\sigma \in B$ such that $xp_\sigma = b_\sigma p_\sigma$ and $\|b_\sigma\| \leqq 1 + \|x\|$. Since $\{b \in B^- : \|b\| \leqq 1 + \|x\|\}$ is ultraweakly compact, $\{b_\sigma\}$ has an ultraweak accumulation point $b \in B^-$. Passing if necessary to a cofinal subnet, we may assume that ultraweak-$\lim_\sigma b_\sigma = b$, and we hence have

$$x = \lim_\sigma xp_\sigma = \lim_\sigma b_\sigma p_\sigma = \text{ultraweak-}\lim_\sigma b_\sigma p_\sigma = bp .$$

Thus $I \subseteqq B^-p$, whence $A^{**}p = I^- = B^-p$.

We assert that $\pi_p|_B$ and $\pi_{1-p}|_B$ are disjoint. If they are not, we can find irreducible representations ρ_1 and ρ_2 of A with $I \nsubseteqq \ker(\rho_1)$, $I \subseteqq \ker(\rho_2)$, such that $\rho_1|_B = \rho_2|_B$. Since $\rho_1 = \tilde{\rho}$ for $\rho \in \mathrm{Irr}(A_\alpha)$ for some $\alpha \in \mathfrak{A}$, this contradicts (iii).

Let $q =$ support projection of B^- in A^{**}. Since $A^{**} = B^-p \oplus B^-(1-p)$, $q = 1$, and so $1 \in B^-$. Thus by the disjointness of $\pi_p|_B$ and $\pi_{1-p}|_B$ and Proposition 5.2.1 of [1], we have (with $'$ denoting the commutant):

$$\begin{aligned} B^- &= (\pi_p \oplus \pi_{1-p})(B)'' = (Bp)'' \oplus (B(1-p))'' \\ &= B^-p \oplus B^-(1-p) \\ &= A^{**} . \end{aligned}$$

If $\iota : B \to A$ denotes the inclusion map, then B^- can be identified with $\iota^{**}(B^{**})$ in A^{**}. We have hence shown that ι^{**} is a surjection of B^{**} onto A^{**}. By duality and the Hahn-Banach theorem, we therefore conclude that $B = A$. □

If instead of the full direct sum $\bigoplus_\alpha A_\alpha$, we consider C^*-subalgebras of restricted direct sums $\hat{\bigoplus}_\alpha A_\alpha$, then $I^{**} = A^{**}$ in the above proof, and so we immediately deduce:

COROLLARY 2.3. *Let $A_\alpha: \alpha \in \mathfrak{A}$ be a family of C^*-algebras, B a C^*-subalgebra of $\hat{\bigoplus}_\alpha A_\alpha$. The following are equivalent:*

(i) *B splits;*

(ii) *The sets $\{\ker(\tilde{\rho}|_B): \rho \in \mathrm{Irr}(P_\alpha(B))\}$, $\alpha \in \mathfrak{A}$, are pairwise disjoint subsets of the primitive ideal space of B;*

(iii) *For each $(\alpha_1, \alpha_2) \in \mathfrak{A} \times \mathfrak{A}$ with $\alpha_1 \neq \alpha_2$ and for each $(\rho_1, \rho_2) \in \mathrm{Irr}(P_{\alpha_1}(B)) \times \mathrm{Irr}(P_{\alpha_2}(B))$, we have $\tilde{\rho}_1|_B \neq \tilde{\rho}_2|_B$.*

The reasoning of Theorem 2.2 can be applied to easily obtain a solution to the splitting problem for an arbitrary direct sum of W^*-algebras. Indeed, recalling the notation of the introduction, we have:

THEOREM 2.4. *Let $N_\alpha: \alpha \in \mathfrak{A}$ be a family of W^*-algebras, M a W^*-subalgebra of $\bigoplus_\alpha N_\alpha$. The following are equivalent:*

(i) *M splits;*

(ii) *The subsets $\{\mathrm{supp}(\tilde{\tau}|_M): \tau \in \mathrm{Rep}_\sigma(P_\alpha(M))\}$, $\alpha \in \mathfrak{A}$, of the center of M are pairwise disjoint;*

(iii) *For each $(\alpha_1, \alpha_2) \in \mathfrak{A} \times \mathfrak{A}$ with $\alpha_1 \neq \alpha_2$ and for each $(\tau_1, \tau_2) \in \mathrm{Rep}_\sigma(P_{\alpha_1}(M)) \times \mathrm{Rep}_\sigma(P_{\alpha_2}(M))$, we have $\tilde{\tau}_1|_M \neq \tilde{\tau}_2|_M$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear.

(iii) $\Rightarrow$ (i). Lemma 2.1 holds with C^*-(sub)algebra (resp., irreducible representation) replaced by W^*-(sub)algebra (resp., nonzero, $\sigma(A_\alpha, (A_\alpha)_*)$-continuous representation). Thus the argument of the first part of the implication (iii) $\Rightarrow$ (i) of Theorem 2.2, appropriately modified, together with the fact that the net $\{p_\sigma: \sigma \in \Sigma\}$ (where $p_\alpha =$ identity of N_α) converges in the *-strong topology to the identity of $\bigoplus_\alpha N_\alpha$ now finishes the proof. □

REMARKS. (1) The splitting phenomenon is much more likely to occur in the W^*-category than in the C^*-category, to which Theorems 2.2 and 2.4 attest. In fact, an example of two diagonal operators T_1 and T_2 acting on a separable Hilbert space is given in [4] for which $W^*(T_1 \oplus T_2)$ splits, while neither $W^*(\mathrm{Re}\, T_1 \oplus \mathrm{Re}\, T_2)$, $W^*(\mathrm{Im}\, T_1 \oplus \mathrm{Im}\, T_2)$, nor $C^*(T_1 \oplus T_2)$ splits.

(2) Theorems 1.4 and 2.2 of [3] can be combined with Lemma 2.1 to give an alternate proof of Theorem 2.2. The proof given here seems more natural in the present context, quickly gives a solution to the splitting problem for W^*-algebras, and avoids the fairly complicated machinery of algebras of operator fields and regularized dual spaces used in [3].

(3) In closing, we present some simple examples which show

that the conditions of Theorem 2.2 cannot be weakened. More specifically, we give examples of a proper, substantial C^*-subalgebra B of l_∞ for which $B/B \cap c_0 = l_\infty/c_0$ and for which

$$(**) \qquad (P_1 \oplus \cdots \oplus P_n)(B) = \boldsymbol{C}^n, \text{ for each positive integer } n,$$

and a proper, substantial C^*-subalgebra C of l_∞ which satisfies the second part of condition (ii) of Theorem 2.2 and for which $C/C \cap c_0$ has codimension 1 in l_∞/c_0.

We identify l_∞ with the C^*-algebra $C(X)$ of continuous, complex-valued functions on the Stone-Čech compactification X of the positive integers Z_+ with discrete topology. Z_+ is a discrete, dense, open subset of X. Set $E = X \backslash Z_+$. Then c_0 can be identified with the ideal of functions in $C(X)$ which vanish on E.

Choose $x \in Z_+$, $y \in E$, and set $B = \{f \in C(X): f(x) = f(y)\}$. B is a proper C^*-subalgebra of $C(X)$. Let $\{x_1, \cdots, x_n\}$ be a fixed finite subset of Z_+, $(a_1, \cdots, a_n)$ a fixed n-tuple of complex numbers. Then by the Tietze extension theorem ([2], Theorem 5.1, p. 149), we can find an $f \in C(X)$ such that $f(x_i) = a_i$, $i = 1, \cdots, n$, and $f(x) = f(y)$. Thus B is substantial in $C(X)$ and satisfies (**). Let g be a fixed element of $C(X)$. Again by the Tietze extension theorem, there exists $f \in C(X)$ such that $f = g$ on E and $f(x) = g(y)$. Thus $f \in B$, and since $f - g = 0$ on E, $f - g \in c_0$. Hence $B/B \cap c_0 = l_\infty/c_0$.

To obtain C, simply choose distinct elements x and y of E and set $C = \{f \in C(X): f(x) = f(y)\}$. Since elements of $\operatorname{Irr}(l_\infty)$ of the form $\tilde{\rho}$, ρ an irreducible representation of some coordinate algebra, correspond to evaluation at points of Z_+ and elements of $\operatorname{Irr}(l_\infty/c_0)$ correspond to evaluation at points of E, the previous reasoning shows that C satisfies the second part of condition (ii) of Theorem 2.2. Now l_∞/c_0 can be identified with the C^*-algebra $C(E)$ of continuous, complex-valued functions on E, and $C/C \cap c_0$ can be identified with the subalgebra D of all $f \in C(E)$ for which $f(x) = f(y)$. Since D is the kernel of the linear functional $f \to f(x) - f(y)$ on $C(E)$, it follows that $C/C \cap c_0$ has codimension 1 in l_∞/c_0.

These arguments can clearly be used to construct similar examples for an arbitrary infinite direct sum of commutative C^*-algebras.

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