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ON PROPERTY (*Q*) AND OTHER SEMICONTINUITY PROPERTIES OF MULTIFUNCTIONS

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Different upper semicontinuity properties of multifunctions in general topological spaces are presented and their interrelationships are expounded in detail. In particular criteria are given for Cesari's property (Q) for multifunctions f: $X \to E$ where X is a general topological space and E is a locally convex space. Among them are that either f is mild upper semicontinuous, or f is maximal monotone.

Introduction. In addition to their intrinsic mathematical interest, the study of upper semicontinuity properties of multifunctions has been motivated by numerous applications in different fields, for instance, in optimal control problems (Cesari [5], [6], [7]), in mathematical programming (Zangwill [20]), and in nonlinear functional analysis (Brézis [2]).

Several concepts of upper semicontinuity (u.s.c.) have been introduced in the past at various levels of generality. In the present paper we present these concepts in general topological spaces: the closed graph property, the u.s.c. property, and property (Q). Moreover, for the sake of comparison, we also introduce the concept of mild u.s.c. The comparison of these concepts at this level of generality does not seem to have been done before. Also, we prove here a number of implications which seem to be new. For instance, we prove in § 2 that mild u.s.c. implies property (Q) and that mild u.s.c. also implies the closed graph property. The latter is a slightly more general statement than the essentially known result that u.s.c. implies the closed graph property. In particular we prove in § 6 a result which seems to be of some relevance, that is, a maximal monotone multifunction in any locally convex space E satisfies all the upper semicontinuity properties discussed in this paper if it is locally bounded. Thus a maximal monotone multifunction necessarily has property (Q) if it is locally bounded. In the case that E is a Frechét space, maximal monotonicity alone is sufficient to imply property (Q). This last result extends a previous one of Suryanarayana [17].

All these results seem to indicate how property (Q) appears to be a generalization of a number of different concepts introduced in different fields. The following diagram summarizes some of the interrelationships among all these concepts.

Let $f: X \to Y$ be a multifunction concerning two topological spaces X and Y; Y a locally convex topological vector space when appropriate.

$$
\begin{tikzcd}[column sep=large]
\text{property } (Q) \arrow[rrr, bend right=30] & \text{mild } u.s.c. \arrow[l, "(1)"'] \arrow[rr, bend left=40, "(2)"] \arrow[r, bend right=40, "(5)"'] & u.s.c. \arrow[l] \arrow[r, shift right, "(3)"'] & \text{closed graph} \arrow[l, shift right, "(4)"']
\end{tikzcd}
$$

Numbers on arrows indicate additional assumptions referred to below.

(1) f has closed convex values.

(2) f has closed values.

(3) f has closed values and Y is regular, or f has compact values.

(4) f is locally compact.

(5) f has compact values.

1. Closedness of multifunctions. Let X, Y be nonempty sets. A multifunction or set valued map $f: X \to Y$ is a point to set correspondence from X to Y. Equivalently, f can be viewed as a function from the set X into the power set 2^Y of Y. Let P be an attribute defined for the subsets of Y. A multifunction $f: X \to Y$ is said to be P-valued or to have P-values if, for every x of X, $f(x)$ is P.

For sets $A \subseteq X$, $B \subseteq Y$, it is customary to write $f(A) \equiv \bigcup_{x \in A} f(x)$, $f^-(B) \equiv \{x \in X: f(x) \cap B \neq \emptyset\}$ and $f^+(B) \equiv \{x \in X: f(x) \subseteq B\}$. We have the following useful relations:

(a) $X \backslash f^-(B) = f^+(Y \backslash B)$

(b) $f^-(\cup B_i) = \cup f^-(B_i)$.

Also we denote the graph of f by $\mathrm{Gr}(f) \equiv \{(x, y) \in X \times Y: y \in f(x)\}$.

We shall write $\mathrm{co}\, A$, $\mathrm{cl}\, A$ and $\mathrm{clco}\, A$ for the convex hull, closure and convex closure of a set A. Also the notations cl_w and cl_{w^*} will be employed to mean the closure in the weak topology $\sigma(E, E^*)$ and in the weak-* topology $\sigma(E^*, E)$ for a Hausdorff locally convex space E and its dual E^* respectively.

DEFINITION 1.1. Let X, Y be topological spaces. A multifunction $f: X \to Y$ is said to be closed at a point $x_0 \in X$ if, for every net $\{(x_\alpha, y_\alpha)\}$ in $\mathrm{Gr}(f)$, $(x_\alpha, y_\alpha) \to (x_0, y_0)$ implies that $y_0 \in f(x_0)$. Also f is said to be closed if it is closed at every point of X.

If the closedness property is only true for sequences in $\mathrm{Gr}(f)$, we agree to call f sequential closed.

The following result is immediate from Definition 1.1.

PROPOSITION 1.2. *A multifunction $f: X \to Y$ is closed on X if and only if its graph is a closed subset of $X \times Y$.*

We have another characterization of closedness of multifunctions.

THEOREM 1.3. *A multifunction f: $X \to Y$ is closed at a point $x \in X$ if and only if, for every convergent net $\{x_\alpha: \alpha \in J\}$ in X with limit x, the inclusion $\bigcap_{\alpha \in J} \mathrm{cl} \bigcup_{\beta \geqq \alpha} f(x_\beta) \subseteqq f(x)$ holds.*

REMARK. The property $\bigcap_{\alpha \in J} \mathrm{cl} \bigcup_{\beta \geqq \alpha} f(x_\beta) \subseteqq f(x)$ for convergent sequence $x_\alpha \to x$ is called property (K) in the literature (cf. Cesari [6]).

Proof. Suppose f is closed at x. Let $\{x_\alpha: \alpha \in J\}$ be a net in X with limit x, and $y \in \bigcup_{\alpha \in J} \mathrm{cl} \bigcup_{\beta \geqq \alpha} f(x_\beta)$. Define $D \equiv \{(\alpha, G): G$ is a neighborhood of y and $\alpha \in J$ with $x_\alpha \in f^-(G)\}$. Clearly D is nonempty. Order D by defining $(\alpha, G) \geqq (\alpha', G')$ to mean $\alpha \geqq \alpha'$ in J and $G \subseteqq G'$. This makes D a directed set. Indeed, given (a, G), (a_1, G_1) in D, there exists $a_2 \in J$ with $a_2 \geqq a$, $a_2 \geqq a_1$. Since $y \in \bigcap_{\alpha \in J} \mathrm{cl} \bigcup_{\alpha \geqq \beta} f(x_\beta)$ and $G \cap G_1$ is a neighborhood of y, we can find $a' \geqq a_2$ such that $x_{\alpha'} \in f^-(G \cap G_1)$. Then $(a', G \cap G_1) \geqq (a, G)$ and $(a', G \cap G_1) \geqq (a_1, G_1)$ proving D is directed. Now define a map $u: D \to J$ by $u_{(a,G)} \equiv a$. Then u is cofinal since given any $a \in J$, $(a'', G'') \geqq (a, G)$ implies $a'' \geqq a$.

For any $(a, G) \in D$ define $y_{u(a,G)}$ to be some point in $f(x_a) \cap G$. This is possible since $x_a \in f^-(G)$ implying $f(x_a) \cap G \neq \emptyset$. Also define $x_{u(a,G)} \equiv x_a$. We note that $\{x_{u(a,G)}$: $(a, G) \in D\}$ is a subnet of $\{x_\alpha: a \in J\}$ as the map u is cofinal. Thus $x_{u(a,G)} \to x$.

Next we claim: $y_{u(a,G)} \to y$. Let N be a neighborhood of y. Since $y \in \bigcap_{a \in J} \mathrm{cl} \bigcup_{\beta \geqq \alpha} f(x_\beta)$, there exists $\alpha' \in J$ such that $x_{\alpha'} \in f^-(N)$. Then any $(a, G) \geqq (\alpha', N)$ implies $y_{u(a,G)} \in G \subseteqq N$, proving the claim.

With $x_{u(a,G)} \to x$, $y_{u(a,G)} \to y$, $(x_{u(a,G)}, y_{u(a,G)}) \in \mathrm{Gr}(f)$ and f being closed at x, we have $y \in f(x)$ implying $\bigcap_{\alpha \in J} \mathrm{cl} \bigcup_{\beta \geqq \alpha} f(x_\beta) \subseteqq f(x)$.

Conversely, suppose $\{(x_\alpha, y_\alpha): \alpha \in J\} \subset \mathrm{Gr}(f)$ with $(x_\alpha, y_\alpha) \to (x, y)$ and the inclusion $\bigcap_{\alpha' \in J} \mathrm{cl} \bigcup_{\beta \geqq \alpha'} f(x_\beta) \subseteqq f(x)$ is true. Then for each $\alpha' \in J$, $\{y_\alpha\}$ is eventually in $\mathrm{cl} \bigcup_{\beta \geqq \alpha'} f(x_\beta)$ and hence so is y. Therefore $y \in \bigcap_{\alpha' \in J} \mathrm{cl} \bigcup_{\beta \geqq \alpha'} f(x_\beta) \subseteqq f(x)$ implying f is closed at x. $\square$

The next result shows in particular that a closed multifunction is necessarily closed valued.

PROPOSITION 1.4. *Let $f\colon X \to Y$ be a closed multifunction and let K be a compact set of X. Then $f(K)$ is closed in Y.*

Proof. Let $y \in \operatorname{cl} f(K)$. Choose a net $\{y_\alpha\}$ in $f(K)$ with $y_\alpha \to y$ and $\{x_\alpha\}$ in K such that $y_\alpha \in f(x_\alpha)$. Since K is compact, we may extract a subnet of $\{x_\alpha\}$ converging to some $x \in K$. Thus $y \in f(x) \subseteq f(K)$ as f has closed graph, and the proposition follows. □

A similar statement to Theorem 1.3 holds for sequential closed multifunctions.

THEOREM 1.5. *Let $f\colon X \to Y$ be a multifunction. If $\bigcap_{n=1}^{\infty} \operatorname{cl} \bigcup_{k=n}^{\infty} f(x_k) \subseteq f(x_0)$ for any convergent sequence $x_n \to x_0$, then f is* seq. *closed at x_0. The converse is true if, in addition, Y is a first countable space.*

Proof. Let $\{(x_n, y_n)\}$ be a sequence in $\operatorname{Gr}(f)$ with $(x_n, y_n) \to (x_0, y_0)$. We show that $y_0 \in f(x_0)$. Indeed, for each positive integer m, $\{y_n\}$ is eventually in $\bigcup_{k=m}^{\infty} f(x_k)$ and whence $y_0 \in \operatorname{cl} \bigcup_{k=m}^{\infty} f(x_k)$. This shows $y_0 \in \bigcap_{m=1}^{\infty} \operatorname{cl} \bigcup_{k=m}^{\infty} f(x_k) \subseteq f(x_0)$.

For the converse, suppose that Y is first countable, f seq. closed at x_0 and $\{x_n\}$ is a sequence convergent to x_0. Let $z \in \bigcap_{m=1}^{\infty} \operatorname{cl} \bigcup_{k=m}^{\infty} f(x_k)$. We claim: $z \in f(x_0)$. Indeed, since Y is first countable, there exists a countable shrinking local base $\{V_n\}$ at z (i.e., $V_{n+1} \subseteq V_n$ for all n). Thus for each m and n, $V_n \cap \bigcup_{k=m}^{\infty} f(x_k) \neq \varnothing$. We can then choose a sequence of positive integers $k_1 < k_2 < \cdots$ and points $z_{k_n} \in V_n \cap f(x_{k_n})$. Obviously $z_{k_n} \to z$ as $n \to \infty$. Therefore $z \in f(x_0)$ by virtue of the seq. closedness of f at x_0. □

Obviously a closed multifunction is necessarily sequential closed. The converse may not be true unless, extra condition is imposed.

DEFINITION 1.6. A topological space Z is called closure sequential if for every set $A \subseteq Z$ and $x \in \operatorname{cl} A$, A contains a sequence converging to x (cf. Wilansky [19]).

A first countable space, in particular a metric space, is closure sequential. Also the weak closure of a bounded set in a reflexive Banach space is closure sequential by the following lemma of Kaplansky (Brezis et al. [3]):

If B is a bounded subset of a reflexive Banach space, then any element in the weak closure of B is a limit of a weakly convergent sequence of B.

THEOREM 1.7. *Let $f\colon X \to Y$ be a* seq. *closed multifunction. If* *the product space $X \times Y$ is closure sequential, then f is closed on X.*

Proof. Let $\{(x_\alpha, y_\alpha)\}$ be a net in $\mathrm{Gr}(f)$ and $(x_\alpha, y_\alpha) \to (x_0, y_0)$. Since $X \times Y$ is closure sequential and $(x_0, y_0) \in \mathrm{cl}\,\mathrm{Gr}(f)$, we can find a sequence $\{(x'_n, y'_n)\} \in \mathrm{Gr}(f)$ such that $(x'_n, y'_n) \to (x_0, y_0)$. It follows then from the seq. closedness of f that $(x_0, y_0) \in \mathrm{Gr}(f)$. This shows that f is closed. $\square$

2. Upper semicontinuity.

DEFINITION 2.1. Let X be a topological space and E a topological linear Hausdorff space. A multifunction $f: X \to E$ is said to be mild upper semicontinuous at a point x_0 of X if, for each neighborhood V of the origin 0 in E, there exists a neighborhood U of x_0 such that $f(U) \subseteqq f(x_0) + V$ (or equivalently for any convergent net $\{x_\alpha\}$ with limit x_0, eventually $f(x_\alpha) \subseteqq f(x_0) + V$). We shall say that f is mild upper semicontinuous if it is mild upper semicontinuous at each point of X.

The above concept of mild u.s.c. is different from the usual u.s.c. in the literature: (Smithson [16], Kuratowski [14]).

DEFINITION 2.2. Let X, Y be topological spaces. A multifunction $f: X \to Y$ is said to be u.s.c. at a point x_0 of X if, for each given open set $G \supseteqq f(x_0)$, there exists a neighborhood U of x_0 such that $f(U) \subseteqq G$ (or equivalently the set $f^+(G)$ is a neighborhood of x_0, or equivalently for any net $\{x_\alpha\}$ in X with $x_\alpha \to x_0$, eventually $f(x_\alpha) \subseteqq G$.

f is u.s.c. if it is u.s.c. at each point of X, (or equivalently $f^+(G)$ is an open set of X for every open set G of Y).

We can characterize u.s.c. by means of f^- as the next proposition indicates.

PROPOSITION 2.3. *A multifunction $f: X \to Y$ is* u.s.c. *if and only if, for each closed set A of Y, $f^-(A)$ is a closed set of X.*

Proof. Suppose f is u.s.c. on X. Let A be a closed set of Y. The relation $X \backslash f^-(A) = f^+(Y \backslash A)$ implies $f^-(A)$ is closed as $Y \backslash A$ and $f^+(Y \backslash A)$ are open. For the converse, let G be an open set in Y. Since $Y \backslash G$ is closed, then $X \backslash f^-(Y \backslash G) = f^+(Y \backslash (Y \backslash G)) = f^+(G)$ is open, hence f is u.s.c. $\square$

Clearly the u.s.c. in Definition 2.2 implies mild u.s.c. since for any open neighborhood V of 0 in E, $f(x_0) + V$ is open and contains $f(x_0)$. The following example shows that u.s.c. is stronger than mild u.s.c.

EXAMPLE 1. Let $X \equiv [0, 1]$ and E be the real line. We define $f: X \to E$ by $f(x) \equiv [0, 1]$ for $0 \leqq x < 1$ and $f(1) \equiv [0, 1)$. f is then mild u.s.c. However, it is not u.s.c. at $x = 1$ since $f^{+}[(-1, 1)] = \{1\}$ which is not a neighborhood of 1.

The two definitions, however, are equivalent if f is compact-valued. This follows from the fact that in topological linear spaces for a compact set K contained in an open set G there exists a neighborhood V of 0 such that $K + V \subseteqq G$ (cf Holmes [13], p. 109).

The next two theorems indicate the relationship between mild u.s.c. (resp. u.s.c.) and closedness of multifunctions.

THEOREM 2.4. *If $f: X \to E$ is a mild* u.s.c., *closed valued multifunction, then f is closed.*

Proof. Let $\{(x_\alpha, y_\alpha)\}$ be a net in $\mathrm{Gr}(f)$ that converges to $(x, y) \in X \times E$. Let W be an open neighborhood of 0 in E and choose an open neighborhood V of 0 such that $V + V \subseteqq W$. Since f is mild u.s.c. at x, $\{y_\alpha\}$ is eventually in $f(x) + V$. Thus

$$y \in \mathrm{cl}\,(f(x) + V) \subseteqq f(x) + V + V \subseteqq f(x) + W,$$

whence $y \in \bigcap_W \{f(x) + W\colon W \text{ is an open neighborhood of } 0\} = f(x)$ as the set $f(x)$ is closed. $\square$

THEOREM 2.5. *Let $f: X \to Y$ be an* u.s.c., *closed valued multifunction with Y a regular topological space. Then f is closed.*

Proof. Let $\{(x_\alpha, y_\alpha)\}$ be a net in $\mathrm{Gr}(f)$ with $(x_\alpha, y_\alpha) \to (x, y) \in X \times Y$. Suppose $y \notin f(x)$. Since Y is regular, there are disjoint open sets G_1, G_2 in Y with $y \in G_1$ and $f(x) \subseteqq G_2$. Then by the u.s.c. of f, $f^{+}(G_2)$ is a neighborhood of x. The net $\{x_\alpha\}$ is eventually in $f^{+}(G_2)$ and so $\{y_\alpha\}$ is eventually in G_2. Hence $y \in \mathrm{cl}\, G_2 \subseteqq Y \backslash G_1$, contradicting $y \in G_1$. Therefore $y \in f(x)$ implying f is closed. $\square$

The following example shows that a closed multifunction is not necessarily mild u.s.c. and hence not u.s.c.

EXAPLE 2. Let $X = [0, 1]$ and $E = R^2$, the 2-dimensional Euclidean space. Define a multifunction $f: X \to E$ by

$$f(x) = \{(u, v) \in R^2\colon -\infty \leqq u < +\infty,\ v \leqq xu\}.$$

Then f is closed since its graph

$$\mathrm{Gr}(f) = \{(x, u, v) \in R^3 \colon 0 \leqq x \leqq 1, -\infty \leqq u < +\infty, v \leqq xu\}$$

is closed. But f is not mild u.s.c. at every x of X.

3. Upper semicontinuous multifunctions with compact values. In 1965, Whyburn [18] gave a characterization theorem for compact valued, u.s.c. multifunctions. He employed the concept of "directed family", i.e., a filterbase (see, e.g., Dugundj [9]). His theorem essentially reads as follows:

A multifunction $f\colon X \to Y$ is a compact valued, u.s.c. multifunction of two topological spaces X and Y if and only if it preserves directedness of families (i.e., if and only if for each filterbase $\mathscr{M}$ in X converging to a point $x \in X$, any filterbase subordinated to the image filterbase $\{f(m)\colon m \in \mathscr{M}\}$ in Y has a cluster point in $f(x)$).[1]

We state below a characterization theorem of compact valued, u.s.c. multifunctions in terms of nets rather than the concept of filterbase. This theorem improves the previous result of Whyburn. It reads:

THEOREM 3.1. *A multifunction $f\colon X \to Y$ is compact valued and* u.s.c. *at x if and only if, for every net $\{(x_\alpha, y_\alpha)\}$ in* Gr(f) *with $x_\alpha \to x$, $\{y_\alpha\}$ has a cluster point in $f(x)$.*

Proof. Let f be compact valued and u.s.c. at x and let $\{x_\alpha\}$ be a net in X with $x_\alpha \to x \in X$ and $y_\alpha \in f(x_\alpha)$. Suppose $\{y_\alpha\}$ did not have a cluster point in $f(x)$. Then for each point $y \in f(x)$ there is an open neighborhood U_y of y such that $\{y_\alpha\}$ is eventually not in U_y. The family $\{U_y\colon y \in f(x)\}$ forms an open covering of $f(x)$. It follows from the compactness of $f(x)$ that there are finitely many points $\{y_1, \cdots, y_n\} \subset f(x)$ such that $f(x) \subset \bigcup_{i=1}^n U_{y_i} \equiv U$. Clearly $\{y_\alpha\}$ is also eventually not in U. Now by the u.s.c. of f, we can find an open neighborhood V of x so that $f(V) \subseteqq U$. But as $x_\alpha \to x$, $\{x_\alpha\}$ is eventually in V implying that $\{y_\alpha\}$ is eventually in U. These contradictory facts establish that $\{y_\alpha\}$ must have a cluster point in $f(x)$.

As for the converse, we first show that $f(x)$ is compact. For this, let $\{y_\alpha\}$ be any net in $f(x)$. Set $x_\alpha = x$, then $x_\alpha \to x$. Thus by hypothesis $\{y_\alpha\}$ has a cluster point in $f(x)$, proving $f(x)$ compact. It remains to prove that f is u.s.c. at x. Suppose the assertion were false. Then there exists an open set G containing $f(x)$ such that $U \cap X \backslash f^+(G) \neq \varnothing$ for every neighborhood U of x. The neighborhood base $\{U_\alpha\}$ at x is directed by set inclusion. Since

[1] A point p is said to be a cluster point of a net $\{x_\alpha\}$, if for each neighborhood U of the point p the net $\{x_\alpha\}$ is frequently in U.

$U_\alpha \cap f^-(Y\backslash G) = U_\alpha \cap X\backslash f^+(G) \neq \varnothing$, we can choose (x_α, y_α) in $X \times Y$ such that $x_\alpha \in U_\alpha$ and $y_\alpha \in f(x_\alpha) \cap Y\backslash G$. Clearly $\{x_\alpha\}$ is a net with limit x. Thus $\{y_\alpha\}$ has a cluster point y in $f(x)$. We have also $y \in Y\backslash G$ since $Y\backslash G$ is closed. This implies that $f(x) \cap Y\backslash G \neq \varnothing$, contradicting $f(x) \subset G$. Hence f is u.s.c. at x. □

REMARK. If y is a cluster point of $\{y_\alpha\}$, then $\{y_\alpha\}$ has a subnet converging to y. Using this fact, we may restate Theorem 3.1 as below:

A multifunction $f: X \to Y$ is compact valued and u.s.c at x if and only if, for every net $\{(x_\alpha, y_\alpha)\}$ in $\mathrm{Cr}(f)$ with $x_\alpha \to x$, $\{y_\alpha\}$ has a subnet converging to a point of $f(x)$.

COROLLARY 3.2. *A single valued function f: $X \to Y$ is continuous if only if for any convergent net $\{x_\alpha\}$ in X with limit x, $f(x)$ is a cluster point of $\{f(x_\alpha)\}$.*

Proof. This follows from the fact that, for single valued functions u.s.c. is equivalent to continuity. □

With the help of characterization Theorem 3.1 we can derive some useful facts about compact valued, u.s.c. multifunctions (cf. Berge [1]).

PROPOSITION 3.3. *If $f: X \to Y$ is a compact valued,* u.s.c. *multifunction and K is a compact set of X, then $f(K)$ is compact.*

Proof. Let $\{y_\alpha\}$ be an arbitrary net in $f(K)$. Choose $\{x_\alpha\}$ in K such that $y_\alpha \in f(x_\alpha)$. Since K is compact, we may assume that $\{x_\alpha\}$ converges to some $x \in K$. Therefore by Theorem 3.1 $\{y_\alpha\}$ has a cluster point in $f(x) \subseteqq f(K)$. Hence $f(K)$ is compact. □

PROPOSITION 3.4. *Let f: $X \to Y$ be a multifunction with X Hausdorff. Then* $\mathrm{Gr}(f)$ *is compact in $X \times Y$ if and only if X is compact and f is* u.s.c. *with compact values.*

Proof. If $\mathrm{Gr}(f)$ is compact, then X is obviously compact. To show that f is compact valued and u.s.c., let $x_\alpha \to x$, $y_\alpha \in f(x_\alpha)$. Then $(x_\alpha, y_\alpha) \in \mathrm{Gr}(f)$ and we may extract a subnet $(x_\alpha, y_\alpha) \to (x', y) \in \mathrm{Gr}(f)$. Since $x_\alpha \to x$, we have $x' = x$ whence $y \in f(x)$. It follows from Theorem 3.1 that f is u.s.c. with compact values.

Conversely, suppose that f is u.s.c. with compact values. Let $\{(x_\alpha, y_\alpha)\}$ be a net in $\mathrm{Gr}(f)$. Since X is compact, we may extract a subnet of $\{x_\alpha\}$ that converges to some point x_0. By Theorem 3.1

$\{y_\alpha\}$ has a cluster point $y_0 \in f(x_0)$. Therefore (x_0, y_0) is a cluster point of $\{(x_\alpha, y_\alpha)\}$, proving $\mathrm{Gr}(f)$ compact. □

DEFINITION 3.5. Let X and Y be topological spaces. A multifunction $f: X \to Y$ is said to be locally compact on X if given any point x in X, there exists a neighborhood V of x such that $f(V)$ is relatively compact.

Contrary to the Example 2 of § 2, we have the following result.

THEOREM 3.6. *Let $f: X \to Y$ be a closed multifunction. If f is locally compact on X, then f is a compact valued,* u.s.c. *multifunction.*

Proof. Let $\{x_\alpha, y_\alpha)\}$ be a net in $\mathrm{Gr}(f)$ with $x_\alpha \to x$. We show $\{y_\alpha\}$ has a cluster point in $f(x)$. By hypothesis we can choose a neighborhood V of x such that $f(V)$ is relatively compact. $\{x_\alpha\}$ is eventually in V and hence $\{y_\alpha\}$ eventually belongs to $f(V)$. Thus $\{y_\alpha\}$ has a subnet converging to some $y \in Y$ which is then a cluster point of $\{y_\alpha\}$. As f is closed, we have $y \in f(x)$. Now an application of Theorem 3.1 finishes the proof. □

COROLLARY 3.7. *If $f: X \to Y$ is a closed multifunction with its range $f(X)$ lying in a compact set of Y, then f is a compact valued,* u.s.c. *multifunction.*

In contrast to Theorem 2.5, a compact valued u.s.c. multifunction $f: X \to Y$ is automatically closed as long as Y is Hausdorff, regardless whether Y is regular or not. This is the next theorem.

Theorem 3.8. *Let $f: X \to Y$ be a compact valued,* u.s.c. *multifunction with Y Hausdorff. Then f is closed.*

Proof. Let $\{(x_\alpha, y_\alpha)\}$ be a net in $\mathrm{Gr}(f)$ with limit (x, y). Since Y is Hausdorff, y is the only cluster point of $\{y_\alpha\}$. By Theorem 3.1, $\{y_\alpha\}$ has a cluster point z in $f(x)$, hence $y = z \in f(x)$. Therefore f is closed. □

4. **Squential upper semicontinuity.** If the properties stated in Definitions 2.1 and 2.2 are only true for sequences, then we have the concepts of sequential u.s.c. and sequential mild u.s.c.. More precisely, we have:

DEFINITION 4.1. Let $f: X \to Y$ be a multifunction of two topological spaces X and Y. f is said to be sequential u.s.c. at a point x of X if, for any open set G containing $f(x)$ and any sequence

$\{x_n\}$ of X with $x_n \to x$, there is an integer n_0 such that $f(x_n) \subseteq G$ for all $n \geqq n_0$. Also f is said to be sequential u.s.c. if it is sequential u.s.c. at every point of X.

Similarly we say a multifunction f from a topological space X to a topological linear space E is sequential mild u.s.c. at a point x of X if, for each neighborhood V of the origin in E and any convergent sequence $x_n \to x$, there is an integer n_0 such that $f(x_n) \subseteq f(x) + V$ for all $n \geqq n_0$. Also f is said to be sequentially mild u.s.c. if it is sequential mild u.s.c. at every point of X.

To characterize sequential u.s.c., we need the following definition (cf. Wilansky [19]):

DEFINITION 4.2. Let X be a topological space and let A be a subset of X. We say that

(i) A is sequentially open if given any sequence $\{x_n\}$ in X which converges to a point in A, eventually $\{x_n\}$ is in A; and

(ii) A is sequentially closed if given any sequence $\{x_n\}$ in A and x in X with $x_n \to x$, then $x \in A$.

Moreover, we say that the space X itself is sequential if every sequentially closed subset of X is closed in X.

It is easy to verify the following proposition.

PROPOSITION 4.3. *Let $f\colon X \to Y$ be a multifunction, X, Y two topological spaces.*

(i) *f is sequential* u.s.c. *if and only if $f^+(G)$ is sequentially open for any open set G in Y.*

(ii) *f is sequential* u.s.c. *if and only if $f^-(A)$ is sequentially closed for any closed set A in Y.*

We now examine the relationship between u.s.c. and sequential u.s.c.

THEOREM 4.4. *An* u.s.c. *multifunction $f\colon X \to Y$ is sequential* u.s.c. *The converse is true if the space X is sequential.*

Proof. Let A be a closed subset of Y. If f is u.s.c., then $f^-(A)$ is closed and hence sequentially closed. Conversely, if X is sequential and f is sequentially u.s.c., then sequential closedness of $f^-(A)$ implies that $f^-(A)$ is closed, and the desired conclusion follows from Proposition 2.3. $\square$

DEFINITION 4.5. A topological space Y is said to have property *SC* (resp. *CS*) if every sequentially compact (resp. compact) set of Y is relatively compact (resp. sequentially compact).

We know that for metric spaces the two properties are equivalent. The same occurs for normed linear spaces with weak topology.

The next result is immediate from Theorem 4.4.

THEOREM 4.6. *Let $f: X \to Y$ be a multifunction with closed values.*

(1) *If f is compact valued and* u.s.c., *then it is sequentially compact valued and sequentially* u.s.c., *providing Y has property CS.*

(2) *If f is sequentially compact valued and sequential* u.s.c., *then it is* u.s.c. *with compact values, providing X is sequential and Y has property SC.*

THEOREM 4.7. *A multifunction $f: X \to Y$ is sequentially compact valued and sequential* u.s.c. *at x_0 if, for any sequence $\{(x_n, y_n)\} \subset \mathrm{Gr}(f)$ with $x_n \to x_0$, $\{y_n\}$ has a convergent subsequence with limit in $f(x_0)$.*

Proof. Let $\{y_n\}$ be any sequence in $f(x_0)$. Then by hypothesis $\{y_n\}$ has a convergent subsequence with limit in $f(x_0)$. Hence $f(x_0)$ is sequentially compact. For the sequential u.s.c. of f at x_0, let G be an open set containing $f(x_0)$ and $\{x_n\}$ a sequence in X converging to x_0. We must show that $\{x_n\}$ is eventually in $f^+(G)$. Indeed, suppose the assertion were false. Then $\{x_n\}$ is frequently in $X \backslash f^+(G) = f^-(Y \backslash G)$. We may select from $\{x_n\}$ a subsequence $\{x_m\}$ lying in $f^-(Y \backslash G)$. Choose a point y_m in $f(x_m) \cap Y \backslash G$. Thus by hypothesis $\{y_m\}$ has a subsequence converging to a point y_0 in $f(x_0)$. Since $Y \backslash G$ is closed, we have also $y_0 \in Y \backslash G$, whence $y_0 \in f(x_0) \cap Y \backslash G$. This contradicts that $f(x_0) \subseteqq G$. Hence $\{x_n\}$ is eventually in $f^+(G)$, proving that f is sequential u.s.c. at x_0. □

THEOREM 4.8. *Let $f: X \to Y$ be a compact valued,* u.s.c. *multifunction with Hausdorff space Y possessing property CS. Then for any sequence $\{(x_n, y_n)\} \subset \mathrm{Gr}(f)$ with $x_n \to x$, $\{y_n\}$ has a convergent subsequence with limit in $f(x)$.*

Proof. Suppose $\{(x_n, y_n)\}$ is a sequence in $\mathrm{Gr}(f)$ with $x_n \to x$. Then the set $B \equiv \cup \{x_n\} \cup \{x\}$ is compact in X. Since f is u.s.c. with compact values, it follows from Proposition 3.3 that $f(B)$ is compact, whence sequentially compact by property CS. Being a sequence in $f(B)$, $\{y_n\}$ has a convergent subsequence with limit y. Since f is also closed (Theorem 3.8) we have $y \in f(x)$ and the assertion follows. □

We are now in a position to state the following theorem.

THEOREM 4.9. *Let X be a sequential space and Y a Hausdorff space with properties CS and SC. Let $f: X \to Y$ be a multifunction with closed values. Then f is compact valued and* u.s.c. (*or sequentially compact valued and sequentially* u.s.c.) *if and only if for any sequence $\{(x_n, y_n)\}$ of* $\mathrm{Gr}(f)$ *with $x_n \to x$, $\{y_n\}$ has a convergent subsequence with limit in $f(x)$.*

Proof. We note that by Theorem 4.6 f is compact valued and u.s.c. if and only if it is sequentially compact valued and sequentially u.s.c. An application of Theorems 4.7 and 4.8 finishes the proof. □

5. **Property** (Q). We shall discuss in this section another property of multifunctions, called property (Q). Property (Q) was introduced by Cesari in [5], and used in a number of variants by many authors (see, e.g., Olech [15], Castaing and Valadier [4], Cesari and Suryanarayana [8]). In the present generality it is convenient to choose the following precise definition:

DEFINITION 5.1. Let $f: X \to E$ be a multifunction from a topological space X to a topological linear space E.

(i) f is said to have property (Q) at a point x_0 of X if, for any net $\{x_\alpha: \alpha \in J\}$ in X with $x_\alpha \to x_0$, the inclusion

$$\bigcap_{\alpha \in J} \operatorname{clco} \bigcup_{\beta \geqq \alpha} f(x_\beta) \subseteqq f(x_0)$$

holds.

(ii) f is said to have the sequential property (Q) at $x_0 \in X$ if the inclusion

$$\bigcap_{n=1}^{\infty} \operatorname{clco} \bigcup_{k=n}^{\infty} f(x_k) \subseteqq f(x_0)$$

holds for any convergent sequence $x_n \to x_0$. Also we say that f has property (Q) (resp. sequential property (Q)) if it has property (Q) (resp. sequential property (Q)) at every point in X.

REMARK. We note that if f has either property (Q) or sequential property (Q) at x_0, then the set $f(x_0)$ is closed and convex. To see this, simply take $x_\alpha = x_0$ for all $\alpha \in J$ in Definition 5.1. Also obviously property (Q) implies sequential property (Q).

PROPOSITION 5.2. *If $f: X \to E$ is a multifunction that has pro-*

perty (Q) at x_0, then f is closed at x_0 and f is also closed at x_0 for E with the weak topology $\sigma(E, E^*)$.

Proof. The assertions follow directly from Theorem 1.3 and the relations

$$\bigcap_{\alpha\in J} \mathrm{cl} \bigcup_{\beta\geqq\alpha} f(x_\beta) \subseteqq \bigcap_{\alpha\in J} \mathrm{cl}_w \bigcup_{\beta\geqq\alpha} f(x_\beta) \subset \bigcap_{\alpha\in J} \mathrm{cl}_w\, co \bigcup_{\beta\geqq\alpha} f(x_\beta)$$
$$= \bigcap_{\alpha\in J} \mathrm{clco} \bigcup_{\beta\geqq\alpha} f(x_\beta) \subseteqq f(x_0)$$

for any net $x_\alpha \to x_0$. □

Let G be a finite measure space and E a real Banach space. Denote by $L_1(G, E)$ the space of all Bochner integrable functions on G with values in E.

DEFINITION 5.3. Let X be a topological space. A multifunction $f: G \times X \to E$ is said to have sequential property (Q) with respect to x at a point x_0 of X if the inclusion

$$\bigcap_{n=1}^{\infty} \mathrm{clco} \bigcup_{k=n}^{\infty} f(t, x_k) \subseteqq f(t, x_0)$$

holds for almost every t in G and any sequence $x_n \to x_0$.

The following is a Banach space version of Cesari's closure theorem (cf. Cesari [5]).

THEOREM 5.4. *Let $f: G \times X \to E$ be a multifunction that satisfies sequential property (Q) with respect to x in X. Let ξ, ξ_n, $n = 1, 2, \cdots$, be integrable functions in $L_1(G, E)$, and z, z_n, $n = 1, 2 \cdots$, functions from G to X, satisfying the relation $\xi_n(t) \in f(t, z_n(t))$ almost everywhere in G, and such that $\xi_n \to \xi$ weakly in $L_1(G, E)$, $z_n(t) \to z(t)$ in X for almost every t in G as $n \to \infty$. Then $\xi(t) \in f(t, z(t))$ almost everywhere in G.*

Proof. Since $\xi_n \to \xi$ weakly, by Banach-Saks-Mazur's lemma (Ekeland and Temam [11]), there is a sequence of finite convex combinations $\{v_n\}$ such that

$$v_n = \sum_i \lambda_{in}\xi_{n+i}\,, \quad \sum_i \lambda_{in} = 1\,, \quad \lambda_{in} \geqq 0\,,$$

and $v_n \to \xi$ strongly in $L_1(G, E)$. Thus we may assume, by passing to subsequence if necessary, that $v_n(t) \to \xi(t)$ in E for almost every t in G.

Fix a $t \in G$ for which $v_n(t)$ converges to $\xi(t)$ in E and at which $f(t, x)$ has sequential property (Q) with respect to x in X, $z_n(t) \to$

$z(t)$, and satisfy the relation $\xi_m(t) \in f(t, z_m(t))$ for all m. We note that these are true for every point of G modulo a subset of measure zero. Clearly $v_n(t) \in \operatorname{clco} \bigcup_{k=n}^{\infty} f(t, z_k(t))$ because $\xi_m(t) \in f(t, z_m(t))$ and $v_n(t)$ is a finite convex combination of $\xi_m(t)$ $m \geqq n$. We observe that if we denote $A_n \equiv \operatorname{clco} \bigcup_{k=n}^{\infty} f(t, z(t))$, then $\{A_n\}$ is decreasing, that is, $A_n \supseteqq A_m$ if $m > n$.

Since $v_n(t) \to \xi(t)$, we claim: $\xi(t) \in \bigcap_{n=1}^{\infty} A_n$. Indeed, if $\xi(t) \notin \bigcap_{n=1}^{\infty} A_n$, then $\xi(t) \notin A_N$ for some N. It follows from Hahn-Banach theorem that we may find a continuous linear functional $h \in E^*$ and α such that $h[\xi(t)] < \alpha < h(z)$ for all $z \in A_N$. Thus $h[v_n(t)] > \alpha$ as $v_n(t) \in A_n \subseteqq A_N$ for all $n \geqq N$. Since $v_n(t) \to \xi(t)$, we see that $h[\xi(t)] \geqq \alpha$. However, this is in contradiction with the fact that $h[\xi(t)] < \alpha$, proving that $\xi(t) \in \bigcap_{n=1}^{\infty} A_n$, and whence $\xi(t) \in f(t, z(t))$ by sequential property (Q). Therefore $\xi(t) \in f(t, z(t))$ almost everywhere in G. □

The rest of this section states conditions under which a multifunction has property (Q).

THEOREM 5.5. *Let E be a locally convex space and X be a topological space. If $f: X \to E$ is mild* u.s.c. *with closed convex values, then f has property* (Q).

Proof. Let $x \in X$ and $\{x_i: i \in D\}$ be a net in X that converges to x. Let U be an open convex neighborhood of the origin 0 in E and choose an open convex neighborhood of 0 such that $V + V \subseteqq U$. Since f is mild u.s.c. at x, there exists a neighborhood N of x such that $f(N) \subseteqq f(x) + V$. As $x_i \to x$, there is an $i_0 \in D$ such that, for all $i \geqq i_0$, $f(x_i) \subseteqq f(x) + V$ and hence $\bigcup_{i \geqq i_0} f(x_i) \subseteqq f(x) + V$. The convexity of $f(x) + V$ implies

$$\bigcap_{j \in D} \operatorname{clco} \bigcup_{i \geqq j} f(x_i) \subseteqq \operatorname{cl}(f(x) + V) \subseteqq f(x) + V + V \subseteqq f(x) + U .$$

It follows that $\bigcap_{j \in D} \operatorname{clco} \bigcup_{i \geqq j} f(x_i) \subseteqq f(x)$ as U is arbitrary and $f(x)$ is closed. □

COROLLARY 5.6. *Let $f: X \to E$ be a multifunction with closed convex values. If either of the following holds*:

(i) *f is* u.s.c.

(ii) *f is locally compact and has closed graph,*

then f has property (Q).

Proof. If (i) holds, f is mild u.s.c. The assertion follows from Theorem 5.5. If (ii) holds, f is u.s.c. by Theorem 3.6. This reduces to (i). □

Analogous to Theorem 5.5, we have

THEOREM 5.7. *If f: $X \to E$ is a sequentially mild* u.s.c. *multifunction with closed convex values, then f has sequential property (Q).*

Proof. The proof is similar to that of Theorem 5.5. with nets replaced by sequences. □

THEOREM 5.8. *Let X be a metric space, E a Banach space and $f: X \to E$ a multifunction which closed convex values. Suppose that for any $x \in X$ and any sequence $\{(x_n, y_n)\} \subset \mathrm{Gr}(f)$ with $x_n \to x$, $\{y_n\}$ has a weakly convergent subsequence with limit in $f(x)$. Then f has sequential property (Q).*

Proof. By Eberlein-Smulian theorem (Dunford & Schwartz [1]), a set is weakly compact if and only if it is sequentially weakly compact. Therefore E_w, the space E endowed with weak topology $\sigma(E, E^*)$, possesses properties *SC* and *CS*. It follows now from Theorem 4.9 that f is weakly compact valued and u.s.c. from X to E_w. Therefore, by Theorem 5.6, f has sequential property (Q), i.e., for any $x_n \to x$,

$$\bigcap_{n=1}^{\infty} \mathrm{cl}_w \operatorname{co} \bigcup_{k=n}^{\infty} f(x_k) \subseteq f(x) .$$

But $\mathrm{cl}_w \operatorname{co} Z = \operatorname{clco} Z$ for any set Z in E. Hence

$$\bigcap_{n=1}^{\infty} \operatorname{clco} \bigcup_{k=n}^{\infty} f(x_k) \subseteq f(x) .$$ □

COROLLARY 5.9. *Let V, U, Z be Banach spaces and f be a function from $V \times U$ to Z satisfying: $f(v_n, u_n) \to f(v, u)$ for any sequence $\{v_n\} \subset V$, $\{u_n\} \subset U$ such that $v_n \to v$ and $u_n \rightharpoonup u$. If $\Gamma: V \to U_w$ is a weakly compact valued,* u.s.c. *multifunction and $f(v, \Gamma(v))$ is convex for each $v \in V$, then the multifunction g: $V \to Z$, defined by $g(v) \equiv f(v, \Gamma(v))$, has sequential property (Q).*

Proof. It suffices to show that g satisfies the hypothesis of Theorem 5.8. Let $\{(v_n, z_n)\} \subset \mathrm{Gr}(g)$ with $v_n \to v$. We need only to prove that $\{z_n\}$ has a weakly convergent subsequence with limit in $g(v)$. By definition of g, there are $u_n \in \Gamma(v_n)$ such that $z_n = f(v_n, u_n)$. Since Γ is weakly compact valued and u.s.c., we can extract, by

Theorem 4.9, a subsequence $u_{n_k} \rightharpoonup u \in \Gamma(v)$. Thus, by the hypothesis on f, $z_{n_k} = f(v_{n_k}, u_{n_k}) \rightharpoonup f(v, u) \in g(v)$. $\square$

6. Maximal monotone multifunctions. We consider in this section multifunctions in a locally convex topological vector space with monotonicity property.

Let E be a Hausdorff locally convex space and E^* its dual with duality pairing $\langle \ , \ \rangle$. We shall denote by E' the locally convex space obtained from E^* by endowing it with the weak-* topology $\sigma(E^*E)$.

DEFINITION 6.1. Let D be a subset of E. A multifunction T: $D \to E^*$ is said to be monotone if $\langle x^* - y^*, x - y\rangle \geqq 0$ whenever $x^* \in T(x)$ and $y^* \in T(y)$. It is called maximal monotone if, in addition, its graph $\mathrm{Gr}\,(T)$, is not properly contained in the graph of a monotone multifunction on E. The following two lemmas are well known.

LEMMA 6.2. *Let $T: D \to E^*$ be a maximal monotone multifunction and $(x_0, x_0^*) \in E \times E^*$. If*

$$\langle x_0^* - y^*, x_0 - y\rangle \geqq 0$$

for any $(y, y^*) \in \mathrm{Gr}\,(T)$, then $x_0^* \in T(x_0)$.

LEMMA 6.3. *Let $T: D \to E^*$ be a maximal monotone multifunction. Then $T(x)$ is a weak-* closed convex set of E^* for every x of D.*

DEFINITION 6.4. A multifunction $T: D \to E^*$ is said to be locally bounded at a point x of D if there exists a neighborhood U of x such that the set $T(U)$ is an equicontinuous subset of E^*.

THEOREM 6.5. *Let $T: D \to E^*$ be a maximal monotone multifunction which is locally bounded at each point of D. Then T has a closed graph in $D \times E'$.*

Proof. Let $\{(x_\alpha, x_\alpha^*)\}$ be a net in $\mathrm{Gr}\,(T)$ with limit (x_0, x_0^*) in $D \times E'$. It is sufficient to show that $y_0 \in T(x_0)$. By the monotonicity of T, we have $\langle x_\alpha^*, -v^*, x_\alpha - v\rangle \geqq 0$ for every (v, v^*) in $\mathrm{Gr}\,(T)$ and every index α. Since T is locally bounded at x_0, the set $\{x_\alpha^*: \alpha \geqq \alpha_0\}$ is equicontinuous for some index α_0. Thus $x_\alpha \to x_0$, $x_\alpha^* \rightharpoonup x_0^*$ give $\langle x_0^* - v^*, x_0 - v\rangle \geqq 0$ for all (v, v^*) in $\mathrm{Gr}\,(T)$. The maximal monotonicity of T implies that $x_0^* \in T(x_0)$. $\square$

COROLLARY 6.6. *With the hypotheses of Theorem* 6.5,

(1) *T is a weak-* compact valued,* u.s.c. *multifunction from D to E'.*

(2) *T satisfies property* (Q), i.e.,

$$\bigcap_{\alpha} \mathrm{cl}_{w^*} \cdot \mathrm{co} \bigcup_{\beta \leqq \alpha} T(x_\beta) \subseteqq T(x)$$

holds for all $x \in D$ and any convergent net $x_\alpha \to x$.

Proof. For (1), this follows from Theorem 3.6 while (2) follows from Corollary 5.6(ii). □

REMARK. It is clear from Theorem 6.5 and Corollary 6.6 that a maximal monotone multifunction satisfies closed graph property, u.s.c., mild u.s.c. and property (Q) if it is locally bounded.

If E is a Fréchet space, then by a theorem of Fitzpatrick-Hess-Kato [12] a monotone multifunction from D to E^* is locally bounded at each interior point of D. A consequence of this result and of Corollary 6.6 is the theorem below.

THEOREM 6.7. *Let E be a Fréchet space, and $T: D \to E^*$ a maximal monotone multifunction with D open. Then T is a weak-* compact valued,* u.s.c. *and satisfies property* (Q) *when T is identified as a multifunction from D into E'.*

COROLLARY 6.8. *Let E be a reflexive Banach space and $T: E \to E^*$ a maximal monotone multifunction. Then T has property* (Q).

Proof. This follows from the fact that in E^* the weak and weak-star topologies coincide. Thus $\mathrm{cl}_{w^*} \cdot \mathrm{co}\, Z = \mathrm{cl}_w \cdot \mathrm{co}\, Z = \mathrm{cl}\,\mathrm{co}\, Z$ for any set Z in E^*. Hence, by Theorem 6.7, $\bigcap_\alpha \mathrm{cl}\,\mathrm{co} \bigcup_{\beta \geqq \alpha} T(x_\beta) \subseteqq T(x)$ for any convergent net $x_\alpha \to x$. □

REMARK. Corollary 6.8 was also obtained by Suryanarayana [17] by a different argument.

REFERENCES

1. C. Berge, *Espaces topologiques et functions multivoques*, Dunod, Paris, 1959.
2. H. Brézis, *Operateurs Maximaux Monotones et Semigroupes de Contractions Dans les Espaces de Hilbert*, North Holland, 1973.
3. H. Brézis, L. Nirenberg and G. Stampacchia, *A remark on Ky Fan's minimax principle*, Boll. Unione Mat. Ital., (4) **6**, (1972), 293–300.
4. C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Lecture notes in Math. No. 580, Springer Verlag, 1977.
5. L. Cesari, *Existence theorems for weak and usual optimal solutions in Lagrange*

problems with unilateral constraint, I and II, Trans. Amer. Math. Soc., **124** (1966), 396–412, 413–430.
6. L. Cesari, *Convexity and property (Q) in optimal control theory*, SIAM J. Control, **12** (1974), 705–720.
7. ———, *Closure theorems for orientor fields and weak convergence*, Archive Rat. Mech. Anal., **55** (1974), 332–356.
8. L. Cesari and M. B. Suryanarayana, *Upper semicontinuity properties of set valued functions*, to appear.
9. J. Dugundji, *Topology*, Allyn and Bacon, Inc., 1970.
10. N. Dunford and J. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1958.
11. I. Ekeland and R. Temam, *Analyse Convexe et Problemes Variationnels*, Dunod, 1974.
12. P. M. Fitzpatrick, P. Hess, and T. Kato, *Local boundedness of monotone type operators*, Proc. Japan. Acad., **48** (1972), 275–277.
13. R. B. Holmes, *Geometric Functional Analysis and its Applications*, Springer Verlag, New York, 1975.
14. K. Kuratowski, *Les fonctions semi-continues dans l'espace des ensembles fermes*, Fund. Math., **18** (1932), 148–166.
15. C. Olech, *A note concerning set valued measurable functions*, Bull. Acad. Polon. Sci., **13** (1965), 317–321.
16. R. E. Smithson, *Multifunctions*, Nieuw Archiefvoor Wiskunde 3, 20, (1972), 31–53.
17. M. B. Suryanarayana, *Monotonicity and upper semicontinuity*, Bull. Amer. Math. Soc., **82** (1976), 936–938.
18. G. T. Whybur, *Continuity of multifunctions*, Proc. Nat. Acad. Sci. U.S.A., **54** (1965), 1494–1501.
19. A. Wilansky, *Topology for Analysis*, Ginn, 1970.
20. Zangwill, *Nonlinear Programming*, Prentice Hall.

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