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ON THE ZEROS OF COMPOSITE POLYNOMIALS

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Let $P(z) = \sum_{j=0}^{n} C(n, j)A_j z^j$ and $Q(z) = \sum_{j=0}^{n} C(n, j)B_j z^j$, $A_n B_n \neq 0$, be two polynomials of the same degree n. If $P(z)$ and $Q(z)$ are apolar and if one of them has all its zeros in a circular region C, then according to a famous result known as Grace's theorem, the other will have at least one zero in C. In this paper we propose to relax the condition that $P(z)$ and $Q(z)$ are of the same degree. Instead, we will assume $P(z)$ and $Q(z)$ to be the polynomials of arbitrary degree n and m respectively, $m \leqq n$, with their coefficients satisfying an apolar type relation and obtain certain generalizations of Grace's theorem for the case when the circular region C is a circle $|z| = r$. As an application of these results, we also generalize some results of Szegö, Cohn and Egerváry.

Two polynomials

$$P(z) = \sum_{j=0}^{n} C(n, j)A_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{n} C(n, j)B_j z^j , \qquad A_n B_n \neq 0 ,$$

of the same degree n are said to be apolar if their coefficients satisfy the relation

$$(1) \qquad A_0 B_n - C(n, 1)A_1 B_{n-1} + C(n, 2)A_2 B_{n-2} + \cdots + (-1)^n A_n B_0 = 0 .$$

As to the relative location of the zeros of $P(z)$ and $Q(z)$, we have the following fundamental result due to Grace [1, p. 61].

THEOREM A. *If $P(z)$ and $Q(z)$ are apolar polynomials and if one of them has all its zeros in a circlar region C, then the other will have at least one zero in C.*

Here we propose to relax the condition that the polynomials $P(z)$ and $Q(z)$ are of the same degree and prove

THEOREM 1. *If $P(z) = \sum_{j=0}^{n} C(n, j)A_j z^j$ and $Q(z) = \sum_{j=0}^{m} C(m, j)B_j z^j$ are two polynomials of degree n and m respectively, $m \leqq n$, such that*

$$(2) \qquad C(m, 0)A_0 B_m - C(m, 1)A_1 B_{m-1} + \cdots + (-1)^m C(m, m)A_m B_0 = 0 ,$$

then the following holds.

(i) *If $Q(z)$ has all its zeros in the circle $|z| \leqq r$, then $P(z)$ has at least one zero in $|z| \leqq r$.*

(ii) *If $P(z)$ has all its zeros in the region $|z| \geqq r$, then $Q(z)$ has at least one zero in $|z| \geqq r$.*

REMARK 1. If in Theorem 1, the polynomial $P(z)$ has all its zeros in a circle $|z| \leqq r$ and $m < n$, then the polynomial $Q(z)$ need not have any in zero in $|z| \leqq r$. For consider the polynomials

$$P(z) = 1 + z + z^2 + \cdots + z^n \equiv \sum_{j=0}^{n} C(n, j) A_j z^j , \qquad n > 1$$

and

$$Q(z) = n + z ,$$

then $m = 1 < n$ and the relation (2) is satisfied. But $P(z)$ has all its zeros in the circle $|z| \leqq 1$, whereas the only zero of $Q(z)$ lies in $|z| > 1$.

In case $m = n$, Theorem 1 reduces to Theorem A when the circular region C is the circle $|z| = r$.

For the proof of Theorem 1, we need the following lemmas.

LEMMA 1. *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| > r$, $r > 0$ and $|w| \leqq r$, then the polynomial*

$$P_1(z) = nP(z) + (w - z)P'(z)$$

has all its zeros in $|z| > r$.

This result is, essentially, due to Szegö [2]. For the sake of completeness we shall present an independent proof of this lemma.

Proof of Lemma 1. The polynomial $P(z)$ has all its zeros in $|z| > r > 0$, therefore, if $z_1, z_2, \cdots, z_n$ are the zeros of $P(z)$, then $|z_j| > r$ for $j = 1, 2, \cdots, n$ and

$$\frac{zP'(z)}{P(z)} = \sum_{j=1}^{n} \frac{z}{z - z_j} .$$

Now if $z = re^{i\theta}$, $0 \leqq \theta < 2\pi$, then we have

$$\operatorname{Re} \frac{re^{i\theta}P'(re^{i\theta})}{P(re^{i\theta})} = \sum_{j=1}^{n} \operatorname{Re} \frac{re^{i\theta}}{re^{i\theta} - z_j} < \sum_{j=1}^{n} \frac{1}{2} = \frac{n}{2} .$$

This implies

$$\operatorname{Re} \frac{zP'(z)}{nP(z)} < \frac{1}{2} \qquad \text{for} \quad |z| = r .$$

Equivalently

$$\left|\frac{zP'(z)}{nP(z)}\right| < \left|1 - \frac{zP'(z)}{nP(z)}\right| \qquad \text{for} \quad |z| = r .$$

Since $P(z) \neq 0$ for $|z| = r$, it follows that

$$(3) \qquad |zP'(z)| < |nP(z) - zP'(z)| \qquad \text{for} \quad |z| = r .$$

Applying Rouche's theorem and noting that $P(z) \neq 0$ for $|z| \leqq r$, we conclude that the polynomial $nP(z) - zP'(z)$ has no zero in $|z| < r$. If now w is any complex number such that $|w| \leqq r$, then from (3) we have

$$|wP'(z)| \leqq r|P'(z)| = |zP'(z)| < |nP(z) - zP'(z)| \qquad \text{for} \quad |z| = r .$$

This implies according to Rouche's theorem again, that the polynomials $nP(z) - zP'(z)$ and $nP(z) + (w - z)P'(z)$ have the same number of zeros in $|z| < r$. Consequently, the polynomial $nP(z) + (w - z)P'(z)$ has no zero in $|z| < r$. This polynomial does not vanish for $|z| = r$ either. Because, if for some $z = z_0$, with $|z_0| = r$

$$nP(z_0) + (w - z_0)P'(z_0) = 0 ,$$

then

$$|nP(z_0) - z_0P'(z_0)| = |wP'(z_0)| \leqq r|P'(z_0)| = |z_0P'(z_0)| .$$

But this is a contradiction to (3). Hence we conclude that the polynomial $nP(z) + (w - z)P'(z)$ has no zero in $|z| \leqq r$ and this proves the lemma.

An immediate consequence of Lemma 1 is the following

LEMMA 2. *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \geqq r$, $r > 0$ and $|w| < r$, then the polynomial*

$$P_1(z) = nP(z) + (w - z)P'(z)$$

has all its zeros in $|z| \geqq r$.

We also need

LEMMA 3 [1, p. 52, Eq. (13, 9)]. *If $P(z) = \sum_{j=0}^{n} C(n, j)A_j z^j$ is a polynomial of degree n and $w_1, w_2, \cdots, w_m$ are m, $m \leqq n$, arbitrary real or complex numbers, then the kth polar derivative*

$$P_k(z) = (n - k + 1)P_{k-1}(z) + (w_k - z)P'_{k-1}(z) , \qquad k = 1, 2, \cdots, m$$

of $P(z)$ with $P_0(z) = P(z)$, can be written in the form

$$P_k(z) = \sum_{j=0}^{n-k} C(n-k, j) A_j^{(k)} z^j ,$$

where

$$A_j^{(k)} = n(n-1) \cdots (n-k+1) \sum_{i=0}^{k} S(k, i) A_{i+j} ,$$

and $S(k, i)$ being the symmetric function consisting of the sum of all possible products of $w_1, w_2, \cdots, w_k$ taken i at a time.

Proof of Theorem 1. Let $w_1, w_2, \cdots, w_m$ be the zeros of $Q(z)$, so that we have

$$(4) \qquad \sum_{j=0}^{m} C(m, j) B_j z^j = B_m (z - w_1)(z - w_2) \cdots (z - w_m) .$$

Equating the coefficients of the like powers of z on the two sides of (4), we get

$$(5) \qquad C(m, j) B_{m-j} = C(m, m-j) B_{m-j} = (-1)^j S(m, j) B_m$$

where $S(m, j)$ is the symmetric function consisting of the sum of all possible products of $w_1, w_2, \cdots, w_m$ taken j at a time.

Now suppose that all the zeros of $Q(z)$ lie in $|z| \leqq r$. We have to show that at least one zero of $P(z)$ lies in $|z| \leqq r$. Assume the contrary. That is, assume that the polynomial $P(z)$ has all its zeros in $|z| > r$. Since $|w_i| \leqq r$, $i = 1, 2, \cdots, m$ it follows by the repeated applications of Lemma 1 that all the zeros of each polar derivative

$$(6) \qquad P_k(z) = (n-k+1) P_{k-1}(z) + (w_k - z) P'_{k-1}(z) , \quad k = 1, 2, \cdots, m ,$$

also lie in $|z| > r$. Hence in particular all the zeros of $P_m(z)$ lie in $|z| > r$. But by Lemma 3, $P_m(z)$ can be written as

$$(7) \qquad P_m(z) = \sum_{j=0}^{n-m} C(n-m, j) A_j^{(m)} z^j ,$$

were

$$\begin{aligned} A_j^{(m)} &= n(n-1) \cdots (n-m+1) \sum_{i=0}^{m} S(m, i) A_{i+j} \\ &= \frac{n(n-1) \cdots (n-m+1)}{B_m} \sum_{i=0}^{m} (-1)^i C(m, i) B_{m-i} A_{i+j} . \end{aligned}$$

Since by hypothesis

$$A_0^{(m)} = \frac{n(n-1) \cdots (n-m+1)}{B_m} \sum_{i=0}^{m} (-1)^i C(m, i) B_{m-i} A_i = 0 ,$$

therefore, if $n > m$, we get from (7) $P_m(0) = 0$. This shows that $z = 0$ is a zero of $P_m(z)$, which is a contradiction to (6). In case $n = m$, from (7) we have

$$P_m(z) \equiv A_0^{(m)} = 0 .$$

Since

$$P_m(z) = P_{m-1}(z) + (w_m - z)P'_{m-1}(z) ,$$

it follows that

$$P_{m-1}(w_m) = 0 .$$

But $|w_m| \leqq r$, this contradicts (6) again. Hence in any case we conclude that $P(z)$ must have a zero in $|z| \leqq r$. This completes the proof of the first part of the theorem.

To establish part (ii) of Theorem 1, we suppose that all the zeros of $P(z)$ lie in $|z| \geqq r$. We have to show that at least one zero of $Q(z)$ lies in $|z| \geqq r$. Assume that all the zeros of $Q(z)$ lie in $|z| < r$, so that $|w_i| < r$, $i = 1, 2, \cdots, m$. Then it follows by the repeated applications of Lemma 2 that all the zeros of each polar derivative

$$P_k(z) = (n - k + 1)P_{k-1}(z) + (w_k - z)P'_{k-1}(z) , \qquad k = 1, 2, \cdots, m ,$$

lie in $|z| \geqq r$. We shall now proceed similarly as before and complete the proof of the 2nd part of the theorem.

We may apply Theorem 1 to the polynomials $z^n\ P(1/z)$ and $z^m Q(1/z)$ to get the following

Corollary 1. *If* $P(z) = \sum_{j=0}^{n} C(n, j)A_j z^j$, $A_0 A_n \neq 0$ *and* $Q(z) = \sum_{j=0}^{m} C(m, j)B_j z^j$, $B_0 B_m \neq 0$ *are two polynomials of degree n and m respectively,* $m \leqq n$, *such that*

$$(8) \qquad C(m, 0)B_0 A_n - C(m, 1)B_1 A_{n-1} + \cdots + (-1)^m C(m, m)B_m A_{n-m} = 0 ,$$

then the following holds.

(i) *If $Q(z)$ has all its zeros in $|z| \geqq r$, then $P(z)$ has at least one zero in $|z| \geqq r$.*

(ii) *If $P(z)$ has all its zeros in $|z| \leqq r$, then $Q(z)$ has at least one zero in $|z| \leqq r$.*

The next corollary is obtained by applying Theorem 1 to the polynomials $P(z)$ and $z^m\ Q(1/z)$ with $r = 1$.

Corollary 2. *If* $P(z) = \sum_{j=0}^{n} C(n, j)A_j z^j$, $A_n \neq 0$ *and* $Q(z) = \sum_{j=0}^{m} C(m, j)B_j z^j$, $B_0 B_m \neq 0$ *are two polynomials of degree n and m*

respectively, $m \leqq n$, such that

$$C(m, 0)A_0B_0 - C(m, 1)A_1B_1 + \cdots + (-1)^m C(m, m)A_mB_m = 0 \,, \tag{9}$$

then the following holds.

(i) *If $Q(z)$ has all its zeros in $|z| \geqq 1$, then $P(z)$ has at least one zero in $|z| \leqq 1$.*

(ii) *If $P(z)$ has all its zeros in $|z| \geqq 1$, then $Q(z)$ has at least one zero in $|z| \leqq 1$.*

If we apply Theorem 1 to the polynomials $z^n P(1/z)$ and $Q(z)$ with $r = 1$, we get the following

COROLLARY 3. *If $P(z) = \sum_{j=0}^{n} C(n, j)A_jz^j$, $A_0A_n \neq 0$ and $Q(z) = \sum_{j=0}^{m} C(m, j)B_jz^j$, $B_m \neq 0$ are two polynomials of degree n and m respectively, $m \leqq n$, such that*

$$C(m, 0)A_nB_m - C(m, 1)A_{n-1}B_{m-1} + \cdots + (-1)^m C(m, m)A_{n-m}B_0 = 0 \,, \tag{10}$$

then we have the following:

(i) *If $Q(z)$ has all its zeros in $|z| \leqq 1$, then $P(z)$ has at least one zero in $|z| \geqq 1$.*

(ii) *If $P(z)$ has all its zeros in $|z| \leqq 1$, then $Q(z)$ has at least one zero in $|z| \geqq 1$.*

As an application of Theorem 1, we shall deduce the following partial generalization of a result due to Szegö [1, p. 65].

THEOREM 2. *If all the zeros of a polynomial $P(z) = \sum_{j=0}^{n} C(n, j)A_jz^j$ of degree n lie in $|z| \geqq r$ and if β is a zero of the polynomial $Q(z) = \sum_{j=0}^{m} C(m, j)B_jz^j$, $B_0B_m \neq 0$ of degree m, $m \leqq n$, then every zero w of the polynomial $R(z) = \sum_{j=0}^{m} C(m, j)A_jB_jz^j$ of degree m, has the form $w = -\alpha\beta$ where α is a suitably chosen point in $|z| \geqq r$.*

Proof of Theorem 2. If w is a zero of $R(z)$, then

$$R(w) = \sum_{j=0}^{m} C(m, j)A_jB_jw^j = 0 \,. \tag{11}$$

Equation (11) shows that the polynomials

$$P(z) = C(n, 0)A_0 + C(n, 1)A_1z + \cdots + C(n, n)A_nz^n$$

and

$$z^mQ(-w/z) = C(m, 0)(-1)^m B_mw^m + \cdots - C(m, m-1)B_1wz^{m-1} + C(m, m)B_0z^m$$

satisfy the condition of Theorem 1. Since all the zeros of $P(z)$ lie in $|z| \geqq r$, it follows from the 2nd part of Theorem 1 that $z^m Q(-w/z)$ has at least one zero in $|z| \geqq r$. If $\beta_1, \beta_2, \cdots, \beta_m$ are the zeros of $Q(z)$, then the zeros of $z^m Q(-w/z)$ are $-w/\beta_1, -w/\beta_2, \cdots, -w/\beta_m$. One of these zeros must be α where $|\alpha| \geqq r$. Therefore, we must have $w = -\alpha\beta_j$ for some $j = 1, 2, \cdots, m$. This complete the proof.

Exactly in the same way as Theorem 2, we may deduce the following result from the 2nd part of Corollary 1.

THEOREM 3. *If all the zeros of the polynomial* $P(z) = \sum_{j=0}^{n} C(n, j) A_j z^j$ *of degree* n *lie in* $|z| \leqq r$ *and if* β *is a zero of the polynomial* $Q(z) = \sum_{j=0}^{m} C(m, j) B_j z^j$, $B_0 B_m \neq 0$, *then every zero* w *of the polynomial*

$$R(z) = \sum_{j=0}^{m} C(m, j) A_{n-m+j} B_j z^j \,, \qquad m \leqq n \,,$$

has the form $w = -\alpha\beta$ *where* α *is a suitably chosen point in* $|z| \leqq r$.

From Theorem 3, we immediately deduce the following corollary which presents a generalization of a result due Cohn and Egerváry [1, p. 66, Cor. (16, 1a)].

COROLLARY 4. *If all the zeros of* $P(z) = \sum_{j=0}^{n} C(n, j) A_j z^j$ *of degree* n *lie in* $|z| \leqq r$ *and if all the zeros of* $Q(z) = \sum_{j=0}^{m} C(m, j) B_j z^j$ *of degree* m *lie in* $|z| < s$, $m \leqq n$, *then all the zeros of the polynomial*

$$R(z) = \sum_{j=0}^{m} C(m, j) A_{n-m+j} B_j z^j$$

of degree m *lie in* $|z| < rs$.

This follows from the fact that $|\alpha| \leqq r$ and $|\beta| < s$ implies $|w| < rs$.

REMARK 2. In very much the same way as above, we can deduce from Theorem 1 and from Corollaries 1–3 many other interesting results.

REFERENCES

1. M. Marden, *Geometry of Polynomials*, 2nd ed., Mathematical Surveys, no. 3, Amer, Math. Soc., Providence, R. I., 1966.

2. G. Szegö, *Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen*, Math. Z., **13** (1922), 28–55.

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