PACIFIC JOURNAL OF MATHEMATICS
Vol. 103, No. 2, 1982

HOMEOMORPHIC CLASSIFICATION OF CERTAIN INVERSE LIMIT SPACES WITH OPEN BONDING MAPS

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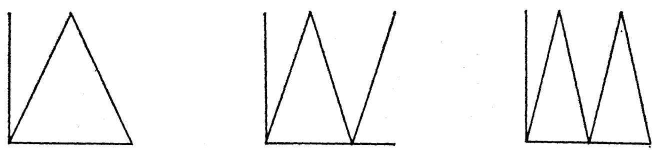
Let $I = [0, 1]$. Let ${}^N f$ be the Nth degree hat function from I to I. For example, ${}^2 f$, ${}^3 f$, and ${}^4 f$ are pictured below:

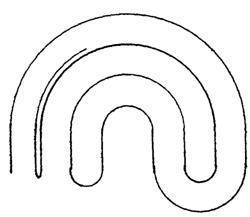


^2f ^3f ^4f

We are interested in classifying the spaces which are inverse limits of the unit interval using these bonding maps. In particular, for a fixed integer $N \geqq 2$, we are interested in classifying (up to homeomorphism) the space D_N, which is $\varprojlim \{I, {}^N f\}$. The main result of this paper is:

THEOREM: D_N is homeomorphic to D_M if and only if M and N have the same prime factors.

Overview. Let $D_N = \varprojlim \{I, {}^N f\}$. These spaces are often called Knaster continua since D_2 is, in fact, the Knaster Bucket Handle:



$\cong D_2$.

Bellamy [1] and latter Oversteegen-Rogers [2] used D_8 to construct examples of tree-like continua without the fixed point property. It appears improbable that their techniques can be modified to construct a similar example from D_2. This resurrects a question raised in a paper by J. W. Rogers, Jr.—Are there three topologically different D_N's?

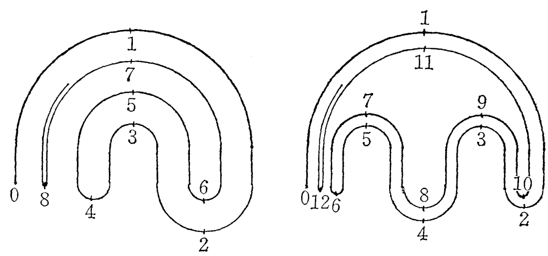
The answer, as previously stated is:

THEOREM. *D_N is homeomorphic to D_M if and only if M and N have the same prime factors. (Allowing different bonding maps we will show there are precisely c topologically different Knaster type continua.)*

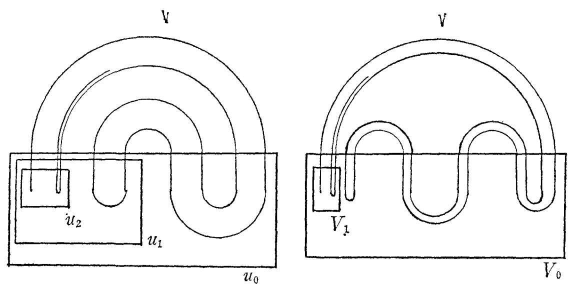
Assuming they have the same prime factors we will demonstrate an inverse limit homeomorphism between the two.

The objective of this section will be to outline, without proofs, the steps in showing that D_2 and D_6 are not homeomorphic. Subsequent sections will provide the details of the proofs.

Consider the composant of D_2 and D_6 containing the end-point as parameterized below.

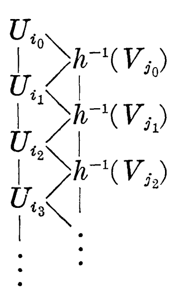


Consider the special basis about the point 0 in D_2 and D_6.



Observe that the integer points in U_i is exactly the collection $\{2n2^i\colon n \text{ is a nonnegative integer}\}$. The integer points in V_i is exactly the collection $\{2n6^i\colon n \text{ is a nonnegative integer}\}$.

If there were a homeomorphism $h\colon D_2 \to D_6$ it would take the end-point-composant of D_2 onto the end-point-composant of D_6 in an order preserving manner. Furthermore we could construct the following infinite lattice of open sets.



Where $h(U_{i_0}) \subset V_0$.

We need two definitions. Suppose $A = \{a_i\}_{i=0}^{\infty}$ and $B = \{b_i\}_{i=0}^{\infty}$ are two increasing sequences of nonnegative integers, then:

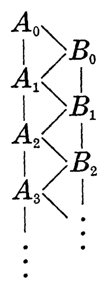
$A \supset B$ if and only if $a_0 = b_0 = 0$ and $b_i \in A$ for every i.

$A\ k\ B$ if and only if $A \supset B$ and $b_i = a_{ki}$ for every i.

Now we construct, from our lattice of open sets, a lattice of sequences. First we get the chain:

$$\begin{array}{c} A_0 \\ | \\ A_1 \\ | \\ A_2 \\ | \\ \vdots \end{array}$$

where A_n is the collection of integers in U_{i_n}, and for every i there is an integer r_i so that $A_i\ 2^{r_i}\ A_{i+1}$. Each A_i is an arithmetic sequence. Unfortunately, the integers in V_{j_0} may not be mapped to integers in U_{i_0} under h^{-1}. However, each integer in V_{j_0} will be mapped into some arc component of U_{i_0} and at most one integer is mapped into any arc component. Thus, using a "nearest integer function", this allows us to construct a lattice:



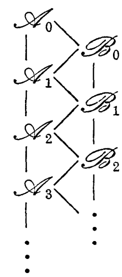
B_n is the subsequence of A_n obtained by picking those integers in U_{i_n} that are on the same arc component of U_{i_n} containing some $h^{-1}(m)$ where m is some integer point in V_{j_n}.

This lattice has the properties that for every i there exist r_i and s_i so that $A_i\ 2^{r_i}\ A_{i+1}$, $B_i\ 6^{s_i}\ B_{i+1}$ and $A_i \supset B_i$ and $B_i \supset A_{i+1}$.

At one time I had hoped to show that no such lattice exists. I have been unable to do this.

By picking some special but very natural chainings of D_2 and D_6 we can establish one more useful fact. We pick a nested sequence of chainings of $D_2 - \mathscr{A}_0 \succ \mathscr{A}_1 \succ \mathscr{A}_2 \succ \mathscr{A}_3 \succ \cdots$ where $\mathscr{A}_i \succ \mathscr{A}_{i+1}$ means $\mathscr{A}_{i+1}$ refines $\mathscr{A}_i$. Further, U_i will be the first link in $\mathscr{A}_i$. Similarly pick a nested sequence of chainings of $D_6 - \mathscr{B}_0 \succ \mathscr{B}_1 \succ \mathscr{B}_2 \succ \cdots$.

We could then get an infinite lattice of chaining:



Use the first link in each chaining to construct the same lattice of sequences we had earlier. Knowledge about the chainings helps establish that B_1 must be an arithmetic sequence. (I don't know any way of insuring that B_0 is arithmetic.)

For any arithmetic sequence C let δC be the difference between two consecutive elements. Since B_1 is arithmetic and $B_1\ 6^{s_1}\ B_2$ we see $\delta B_2 = 6^{s_1}\delta B_1$. Since $B_2 \supset A_3$ there is some constant k so that $\delta A_3 = k\delta B_2 = k6^{s_1}\delta B_1$. We know that δA_3 is some power of 2 and this is a contradiction.

NOTATION. We begin with a very particular and convenient description of the continua under consideration. Suppose N is a fixed positive integer. Let ${}^N f_i\colon [0, N^{i+1}] \to [0, N^i]$ be the "hat-function" such that ${}^N f_i(mN^i) = 0$ whenever m is even and ${}^N f_i(mN^i) = N^i$ whenever m is odd ($m = 0, 1, 2, \cdots, N$) and it is linear in between.

DEFINITION. $D_N = \varprojlim \{[0, N^i], {}^N f_i\}$. That is, D_N is the inverse limit of the following sequence:

$$[0, 1] \xleftarrow{N_{f_0}} [0, N] \xleftarrow{N_{f_1}} [0, N^2] \xleftarrow{N_{f_2}} [0, N^3] \xleftarrow{N_{f_3}} \cdots .$$

With this "parameterization" of D_N we shall have a very nice correspondence between the nonnegative real line and the composant of D_N containing the end-point $\langle 0, 0, \cdots \rangle$. We shall call this composant the $\bar{0}$-composant of D_N. This composant has a particularly simple form-namely, it is

$$\{\bar{x} = \langle x_n \rangle \in D_N \,|\, \text{for some } k,\ x_k = x_{k+1} = x_{k+2} = x_{k+3} = \cdots\}$$

that is the eventually constant elements of D_N. Given any nonnegative real number x denote $\bar{x}$ as the unique point on $\bar{0}$-composant whose coordinates are eventually all x. An integer point $\bar{n}$ is a point whose coordinates are eventually the integer n.

Define $\mu\colon \bar{0}\text{-composant} \times \bar{0}\text{-composant} \to R$, where R is the set of real numbers, by $\mu(\bar{x}, \bar{y}) = |x - y|$. In a sense this measures the arc length distance between $\bar{x}$ and $\bar{y}$.

Define η: $\bar{0}$-composant $\rightarrow N$, where N is the set of nonnegative integers, by $\eta(\bar{x}) = [x + 1/2]$, where [ ]: $R \rightarrow N$ is the greatest integer function. η is, in a sense, a nearest integer function. The effect of η is to find the integer point $\bar{n}$ "closest" to $\bar{x}$ as measured by μ and then $\eta(\bar{x}) = n$.

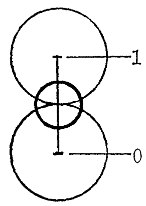
We wish to investigate the possibility of a homeomorphism $h: D_M \rightarrow D_N$. Some very specific chainings of D_M and D_N will be helpful in this investigation. Let $N_n = [0, N^n]$ and $M_n = [0, M^n]$ be the nth coordinate of D_N and D_M respectively. Then $\pi_n: D_M \rightarrow M_n$ is the projection of D_M onto M_n. (π_n will also be used to denote the projection $\pi_n: D_N \rightarrow N_n$. There should be no confusion when read in context, however.) Now define the special chainings of D_N and D_M:

$$\begin{aligned}\mathscr{N}_n^m = \pi_n^{-1}\{&[0, 2^{1-m}), (2^{-m}, 3\cdot 2^{-m}), \cdots,\\ &(N^n - 3\cdot 2^{-m}, N^n - 2^{-m}), (N^n - 2\cdot 2^{-m}, N^n]\}\\ \mathscr{M}_n^m = \pi_n^{-1}\{&[0, 2^{1-m}), (2^{-m}, 3\cdot 2^{-m}), \cdots,\\ &(M^n - 3\cdot 2^{-m}, M^n - 2^{-m}), (M^n - 2\cdot 2^{-m}, M^n]\}\,.\end{aligned}$$

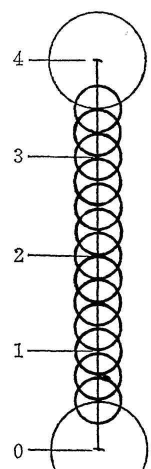
There are four significant properties of these chainings:

1. $\mathscr{N}_i^j$ is refined by $\mathscr{N}_m^n$ (denoted $\mathscr{N}_i^j \succ \mathscr{N}_m^n$) if and only if $j \leqq n$ and $i \leqq m$. Similarly for $\mathscr{M}_n^m$.

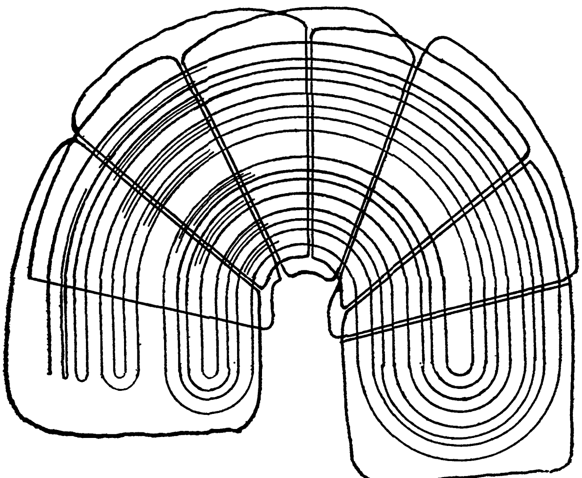
2. The first link of $\mathscr{N}_m^{n+1}$ intersects only the first link of $\mathscr{N}_m^n$. In general, if $\mathscr{N}_i^j \succ \mathscr{N}_m^n$, $j \leqq n$, then the first link of $\mathscr{N}_m^n$ intersects only the first link of $\mathscr{N}_i^j$. Similarly for $\mathscr{M}_m^n$.



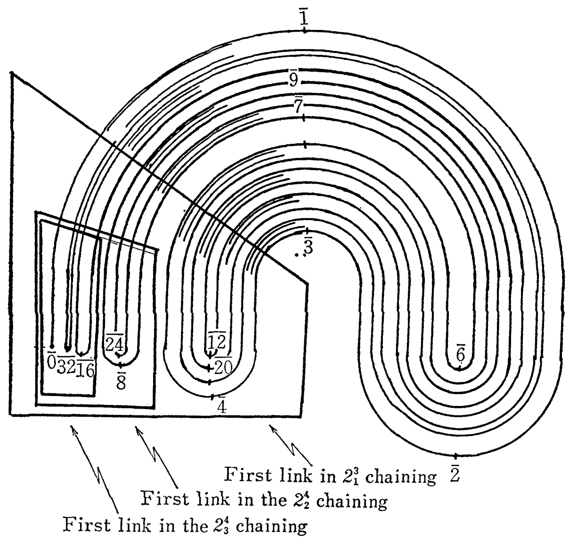
The gates in the 0th coordinate of D_2 corresponding to the super-script 2.

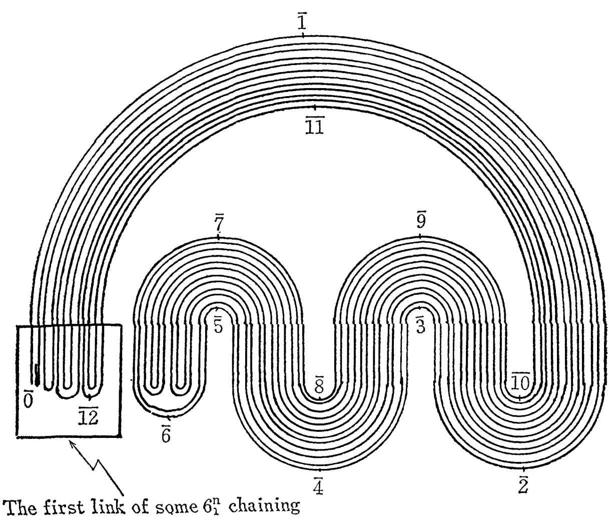


The gates in the 2th coordinate of D_2 corresponding to the super-script 2.



The 2_1^3 chaining of D_2





3. For all $i \geqq 1$ the set of integer points in the first link of $\mathscr{N}_i^j$ (resp. $\mathscr{M}_i^j$) is exactly $\{\bar{n} \mid n = k2N^i$ (resp. $n = k2M^i$) $k \in N\}$.

4. $\mu(\bar{x}, \bar{y}) < 1/4$ for any $\bar{x}$ and $\bar{y}$ on the same component in the first link of $\mathscr{N}_i^4$ or $\mathscr{M}_i^4$.

Non-Homeomorphic D_N's.

THEOREM 1. *If there exists a homeomorphism $h: D_M \to D_N$, then there exists an infinite sequence of chainings*:

$$\mathscr{M}_{i_0}^{j_0} \succ h^{-1}(\mathscr{N}_{n_0}^{m_0}) \succ \mathscr{M}_{i_0}^{j_0} \succ h^{-1}(\mathscr{N}_{n_1}^{m_1}) \succ \cdots .$$

If we denote $\mathscr{V}_\alpha$ as the first link of $\mathscr{N}_{n_\alpha}^{m_\alpha}$ and $\mathscr{U}_\alpha$ as the first link of $\mathscr{M}_{i_\alpha}^{j_\alpha}$, then the only link of $\mathscr{M}_{i_n}^{j_n}$ that intersects $h^{-1}(\mathscr{V}_n)$ is $\mathscr{U}_n$ and the only link of $h^{-1}(\mathscr{N}_{n_i}^{m_i})$ that intersects $\mathscr{U}_{i+1}$ is $h^{-1}(\mathscr{V}_i)$.

Further, $\mathscr{U}_0$ and $h(\mathscr{U}_0)$ are both so small that for any $\bar{x}$ and $\bar{y}$, both on the same $\bar{0}$-composant component of $\mathscr{U}_0$, $\mu(\bar{x}, \bar{y}) < 1/4$ and $\mu(h(\bar{x}), h(\bar{y})) < 1/4$.

Proof of Theorem 1. Pick $\mathscr{M}_{i_0}^{j_0}$ so that $j_0 > 4$ and $h(\mathscr{U}_0)$ is a subset of the first link of $\mathscr{N}_1^4$. Then $\mu(\bar{x}, \bar{y}) < 1/4$ and $\mu(h(\bar{x}), h(\bar{y})) < 1/4$ for any $\bar{x}$ and $\bar{y}$ in the same component of $\mathscr{U}_0$.

Pick $\mathscr{N}_{n_0}^{m_0}$ so that $h^{-1}(\mathscr{V}_0)$ is contained in the first link of $\mathscr{M}_{i_0}^{j+1}$. Then $\mathscr{U}_0$ is the only link of $\mathscr{M}_{i_0}^{j_0}$ that intersects $h^{-1}(\mathscr{V}_0)$.

Pick $\mathscr{M}_{i_1}^{j_1}$ so that $h(\mathscr{U}_1)$ is contained in the first link of $\mathscr{N}_{n_0}^{m_0+1}$. Then $h^{-1}(\mathscr{V}_0)$ is the only link of $h^{-1}(\mathscr{N}_{n_0}^{m_0})$ that intersects $\mathscr{U}_1$.

Continue this process indefinitely.

Define α_n to be the set of integer points in $\mathscr{U}_n$ and β_i to be the set of integer points in $\mathscr{V}_i$. Then as noted earlier:

$$\alpha_n = \{\bar{m} \mid m = k2M^{i_n}, \ k \text{ is a nonnegative integer}\}$$
$$\beta_i = \{\bar{m} \mid m = k2N^{n_i}, \ k \text{ is a nonnegative integer}\} .$$

DEFINITION. Given two increasing sequences of nonnegative integers $A = \{a_i\}_{i=0}^{\infty}$ and $B = \{b_i\}_{i=0}^{\infty}$ we say $A \supset B$ if and only if conditions (a) and (b) hold.

(a) for every $b_i \in B$, b_i is also an element of A.

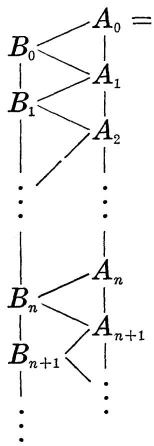
(b) $a_0 = b_0$.

DEFINITION. $A \; k \; B$ if and only if $b_n = a_{kn}$ for every n.

Observe that for every $n \leqq m$ there exists an r so that $\eta(\alpha_n) \; M^r \; \eta(\alpha_m)$ and for every $i \leqq j$ there exists an s so that $\eta(\beta_i) \; N^s \; \eta(\beta_j)$.

DEFINITION. $A_n = \eta(\alpha_n)$; $B_n = \eta(h^{-1}(\beta_n))$.

THEOREM 2. *If there exists a homeomorphism* $h\colon D_M \to D_N$, *then there exists an infinite lattice of increasing sequences of nonnegative integers so that for every* i *and* j $b_{1,i+1} - b_{1,i} = b_{1,j+1} - b_{1,j}$ *and for every* n *there exists nonnegative integers* r_n *and* s_n *so that* $A_n\ M^{r_n}\ A_{n+1}$ *and* $B_n\ N^{s_n}\ B_{n+1}$ *and* $B_n \supset A_{n+1}$ *and* $A_n \supset B_n$. *The first element in each sequence is* 0.



$A_0 = \{$nonnegative integer multiples of some power of 2$\}$

$A_n = \{a_{n,i}\}_{i=0}^{\infty}$
$B_n = \{b_{n,i}\}_{i=0}^{\infty}$

Proof of Theorem 2. Let $A_n = \eta(\alpha_n)$. Since α_n is the integer points in $\mathscr{U}_n \in \mathscr{M}_{i_n}^{j_n}$ and α_{n+1} is the integer points in $\mathscr{U}_{n+1} \in \mathscr{M}_{i_{n+1}}^{j_{n+1}}$ where $j_{n+1} \geqq j_n$, and $i_{n+1} \geqq i_n$ there exists r_n so that $A_n\ M^{r_n}\ A_{n+1}$.

Step 1. $\eta \circ h^{-1}\colon \beta_n \to A_n$ is an injection. Hence $A_n \supset B_n = \eta(h^{-1}(\beta_n))$.

Proof of Step 1. Suppose $\bar{x} \in \beta_n$, $\bar{x} \neq \bar{y}$ and $\bar{y} \in \beta_n$ then $\mu(\bar{x}, \bar{y}) \geqq 1$. The only way $\eta(h^{-1}(\bar{x})) = \eta(h^{-1}(\bar{y}))$ is for $h^{-1}(\bar{x})$ and $h^{-1}(\bar{y})$ to be on the same component of $\mathscr{U}_0$ which implies $\bar{x}$ and $\bar{y}$ are in the same component of $\mathscr{V}_n$ (hence on the same component of $\mathscr{V}_0$) but any two points on the same component of $\mathscr{V}_0$ have $(\bar{x}, \bar{y}) \leqq 1/4$, a contradiction.

Step 2. $\eta(h^{-1}(\beta_n)) \supset A_{n+1}$.

Proof of Step 2. $h^{-1}(\beta_n) \subset h^{-1}(\mathscr{V}_n)$. More specifically, each $\bar{0}$-composant component of $h^{-1}(\mathscr{V}_n)$ contains a unique element of $h^{-1}(\beta_n)$. $\mathscr{U}_{n+1} \subset h^{-1}(\mathscr{V}_n)$, so every $\bar{0}$-composant component of $\mathscr{U}_{n+1}$ is a subset of some $\bar{0}$-composant component of $h^{-1}(\mathscr{V}_n)$. Hence, for every element $\bar{y} \in \alpha_{n+1}$ there is a unique $\bar{x} \in \beta_n$ so that $\bar{y} = \eta(h^{-1}(\bar{x}))$.

Step 3. $\beta_n\ N^{s_n}\ B_{n+1}$.

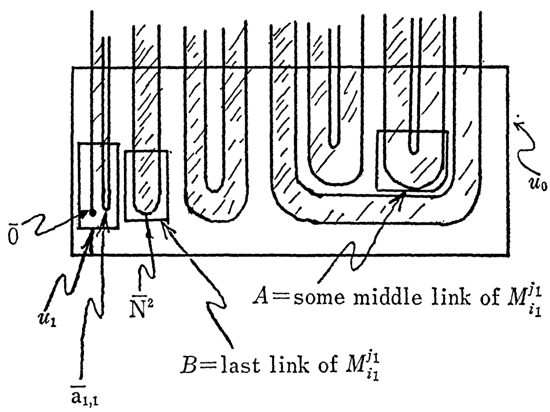
Proof of Step 3. We observed earlier that for every n there is some s_n so that $\eta(\beta_n)$ N^{s_n} $\eta(\beta_{n+1})$. Since $\eta \circ h^{-1}\colon \beta_n \to A_n$ is a one to one order preserving map B_n N^{s_n} B_{n+1}.

Step 4. Let p and q be two consecutive elements of α_1 and J be the arc from p to q. If $\mathscr{A}$ is a link of $\mathscr{M}_{i_1}^{j_1}$ other than the last link and $\mathscr{B}$ is the last link of $\mathscr{M}_{i_1}^{j_1}$ then $J \cap \mathscr{B}$ has exactly one component and $J \cap \mathscr{A}$ has exactly two components. In particular $J \cap \mathscr{U}_1$ has exactly two components.

Proof of Step 4. This is simply the observation that the $\bar{0}$-composant goes from the first link, through each link in order to the last. It then turns around and goes from the last link to the first in reverse order.

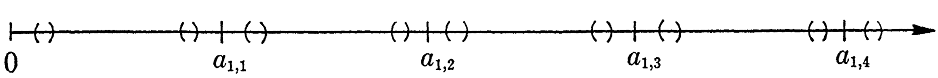
The following diagram may be helpful.

$\mathscr{B}$ is the last link of $\mathscr{M}_{i_1}^{j_1}$
$\mathscr{U}_1$ is the first link of $\mathscr{M}_{i_1}^{j_1}$
$\mathscr{U}_0$ is the first link of $\mathscr{M}_{i_1}^{j_1}$.

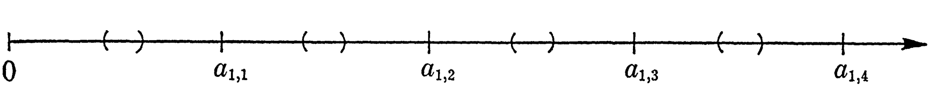


Consider the continuous one to one, onto function f: nonnegative reals $\to \bar{0}$-composant, so that $f(x) = \bar{x}$. Then

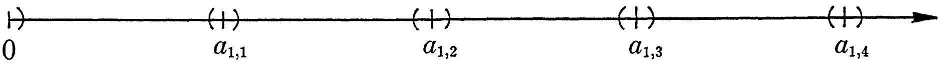
$f^{-1}(\mathscr{A})$ looks like:



$f^{-1}(\mathscr{B})$ looks like:



$f^{-1}(\mathscr{U}_1)$ looks like:



Step 5. The number of B_0 elements between any two consecutive A_1 elements is constant.

Proof of Step 5. Note that any maximal subchain of $\mathscr{M}_{i_1}^{j_1}$ contained in $\mathscr{U}_0$ has only one link containing integer points. Call this link $\mathscr{U}$. Step 2 implies $A_1 \subset B_0$, so that if there exists some $\bar{x} \in \beta_0$ and $\bar{y} \in \mathscr{U}_0 \cap \mathscr{U}$ so that $\eta(h^{-1}(\bar{x})) = \eta(\bar{y})$, then there is some $\bar{x}_1 \in \beta_0$ for every $\bar{y}_1 \in \mathscr{U}_0 \cap \mathscr{U}$ so that $\eta(h^{-1}(\bar{x}_1)) = \eta(\bar{y}_1)$.

We know how the arc connecting two consecutive elements of α_1 passes through the links of $\mathscr{M}_{i_1}^{j_1}$. If $\mathscr{U} \in \mathscr{M}_{i_1}^{j_1}$ and $\mathscr{U} \cap \beta_0 \neq \varnothing$ and $\mathscr{U}$ is not an end link then any arc connecting two consecutive elements of α_1 passes through $\mathscr{U}$ exactly twice. That is there are exactly two components of $\mathscr{U}$ that are subsets of the arc.

If $\mathscr{U}$ is an end link then there is only one component of $\mathscr{U}$ that is a subset of the arc.

B_0 either contains all or none of $\eta(\beta_0 \cap \mathscr{U})$. Since this hold for every $\mathscr{U} \in \mathscr{M}_{i_1}^{j_1}$ where $\mathscr{U} \cap \beta_0 \neq \varnothing$ the number of B_0 elements between any two consecutive A_1 elements is constant.

Step 6. $b_{1,i+1} - b_{1,i}$ is a constant for all i.

Proof of Step 6. Suppose $b_{1,1} = b_{0,L} = a_{1,K}$. Then $b_{1,2} = b_{0,2L}$, and since the number of B_0 elements between two consecutive A_1 elements is constant, $b_{1,2} = a_{1,2K}$, and in general $b_{1,n} = a_{1,nK}$. Since $a_{1,(i+1)K} - a_{1,iK}$ is a constant, $b_{1,i+1} - b_{1,i}$ is also a constant.

The steps have established Theorem 2.

THEOREM 3. *If there exists a prime p so that $p \mid N$ and $p \nmid M$ then there does not exist a homeomorphism $h: D_M \to D_N$.*

Proof of Theorem 3. If 2 divides N and 2 does not divide M then D_N has one end point and D_M has two end points, hence D_M is not homeomorphic to D_N.

Consider the case when $p \neq 2$, p divides N and p does not divide M. Theorem 2 says that B_1 is arithmetic. $B_1\ N^{s_1}\ B_2$ then implies that B_2 is arithmetic and in particular $N^{s_1}(b_{1,i+1} - b_{1,i}) = b_{2,j+1} - b_{2,j}$ for every i and j. Further, since A_3 is arithmetic, $B_2 \supset A_3$ implies there exists a fixed integer c so that $cN^{s_1}(b_{1,i+1} - b_{1,i}) = a_{3,j+1} - a_{3,j}$ for all i and j. This is impossible since the left hand side is divisible by p and the right hand side is two times some power of M and hence not divisible by p.

Homeomorphic D_N's.

THEOREM 4. *If M and N have the same prime factors then D_M is homeomorphic to D_N.*

It will be more convenient at this point to consider $D_M = \varprojlim \{I, {}^M f\}$ where ${}^M f: [0, 1] \to [0, 1]$ so that for $n = 0, 1, 2, \cdots, N$ ${}^M f(n/M) = 0$ whenever n is even and ${}^M f(n/M) = 1$ whenever n is odd.

These open functions satisfy the property:

$$ {}^{MN} f = {}^M f \circ {}^N f = {}^N f \circ {}^M f \quad \text{and hence} \quad {}^N f^n = {}^{N^n} f . $$

For notational convenience we will denote ${}^M f$ by M in this section.

Proof of Theorem 4. It will be sufficient to show that if $M = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p^{a_n}$ is a prime factorization of M and $R = p_1 p_2 p_3 \cdots p_n$ that D_M is homeomorphic to D_R.

Step 1. Suppose there is a prime p dividing R and $M = pR$, then D_M is homeomorphic to D_R.

Proof of Step 1. Construct the inverse limit map $h: D_M \to D_R$

$$ \begin{array}{ccccccccccc} I & \xleftarrow{M} & I & \xleftarrow{M} & I & \xleftarrow{M} & I & \xleftarrow{M} & I & \xleftarrow{M} & \cdots \\ \downarrow i & & \downarrow p & & \downarrow p^2 & & \downarrow p^3 & & \downarrow p^4 & & \\ I & \xleftarrow{R} & I & \xleftarrow{R} & I & \xleftarrow{R} & I & \xleftarrow{R} & I & \xleftarrow{R} & \cdots \end{array} $$

h is an open, continuous, onto map. It is left to show that h is one to one. Under the specified conditions p^2 divides M so M/p^2 is an integer.

Suppose $h(\bar{x}) = h(\bar{y})$, then

$$ \begin{aligned} \pi_i(\bar{x}) &= M^i(\pi_{2i}(\bar{x})) = \frac{M^i}{p^{2i}} \circ p^{2i}(\pi_{2i}(\bar{x})) = \frac{M^i}{p^{2i}} \circ p^{2i}(\pi_{2i}(\bar{y})) \\ &= M^i(\pi_{2i}(\bar{y})) = \pi_i(\bar{y}) . \end{aligned} $$

This establishes the step. A simple repetition of the step proves the theorem.

Counting the homeomorphism classes. We now have the theorem:

THEOREM 5. *D_M is homeomorphic to D_N if and only if M and N have the same prime factors.*

This yields countably many distinct homeomorphism classes. By considering a slightly larger collection of spaces we can show that there are uncountably many distinct homeomorphism classes.

Consider the collection of inverse limit spaces with open bonding maps as before but not necessarily with fixed bonding maps. That is consider any sequence of primes $\{\mathfrak{p}_i\}_{i=0}^{\infty}$ where $\mathfrak{p}_0 = 1$. Define the functions $\mathscr{P}_{n+1}\colon [0, \prod_{i=0}^{n+1} \mathfrak{p}_i] \to [0, \prod^n \mathfrak{p}_i]$, where $\mathscr{P}_{n+1}(m \prod^n \mathfrak{p}_i) = 0$ whenever m is even and $\mathscr{P}_{n+1}(m \prod^n \mathfrak{p}_i) = \prod^n \mathfrak{p}_i$ whenever m is odd ($m = 0, 1, 2, \cdots, \mathfrak{p}_{n+1}$) and linear in between. Then define $D_{\{\mathfrak{p}_i\}} = \varprojlim \{[0, \prod^n \mathfrak{p}_i], \mathscr{P}_n\}$ to be the following inverse limit space:

$$[0, 1] \xleftarrow{\mathscr{P}_1} [0, \mathfrak{p}_1] \xrightarrow{\mathscr{P}_2} [0, \mathfrak{p}_1\mathfrak{p}_2] \xleftarrow{\mathscr{P}_3} [0, \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3] \xleftarrow{\mathscr{P}_4} \cdots .$$

This generalizes the previous notion since D_N is $D_{\{\mathfrak{p}_i\}}$ where $\mathfrak{p}_i = N$ for all $i \neq 0$.

Let P represent the set of all primes and 2^P represent the set of all nonempty subsets of P. The cardinality of 2^P is uncountable. For each set in 2^P we will now construct an inverse limit space that is not homeomorphic to any of the others.

For each K in 2^P construct a sequence that repeats each element of K infinitely often. For example let $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \cdots$ be the elements of K. (To insure a unique construction we might assume $\mathfrak{p}_i \leqq \mathfrak{p}_{i+1}$ for all i.) Construct the sequence $1, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4, \mathfrak{p}_5, \cdots$ call this sequence $\tilde{K}$. Let $\tilde{L}$ be the sequence $1, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_1, \mathfrak{q}_2, \cdots$.

THEOREM 3a. *If K and L are distinct subsets of 2^P then $D_{\tilde{K}}$ and $D_{\tilde{L}}$ are not homeomorphic.*

The line of proof follows the same reasoning as the previous section. That is we first investigate conditions imposed by a homeomorphism $h\colon D_{\tilde{K}} \to D_{\tilde{L}}$ and then show that these conditions cannot be met when K and L are distinct subsets of 2^P.

Without loss of generality we can assume that $h(\bar{0}) = \bar{0}$, so that the $\bar{0}$-composant is mapped onto the $\bar{0}$-composant. We consider special chaining defined exactly as before and note that they have nearly the same properties. The only change is in the third property. Now the integer points in the first link of the chain $\mathscr{K}_i^j$ are exactly:

$$\{\bar{n} \mid n = k2 \prod^{i} \mathfrak{p}_m \text{ where } k \text{ is a nonnegative integer}\} .$$

The following theorems have almost word for word the same proofs as their counterparts in previous sections.

THEOREM 1a. *If there exists a homeomorphism* $h\colon D_{\tilde{K}} \to D_{\tilde{L}}$, *then there exists an infinite sequence of chainings*:

$$\mathscr{K}_{i_0}^{j_0} \succ h^{-1}(\mathscr{L}_{n_0}^{m_0}) \succ \mathscr{K}_{i_1}^{j_1} \succ h^{-1}(\mathscr{L}_{n_1}^{m_1}) \succ \cdots .$$

If we denote $\mathscr{V}_\alpha$ as the first link of $\mathscr{L}_{n_\alpha}^{m_\alpha}$ and $\mathscr{U}_\alpha$ as the first link of $\mathscr{K}_{i_\alpha}^{j_\alpha}$, then the only link of $\mathscr{K}_{i_n}^{j_n}$ that intersects $h^{-1}(\mathscr{V}_n)$ is $\mathscr{U}_n$ and the only link in $h^{-1}(\mathscr{L}_{n_i}^{m_i})$ that intersects $\mathscr{U}_{i+1}$ is $h^{-1}(\mathscr{V}_i)$.

Further, $\mathscr{U}_0$ and $h(\mathscr{U}_0)$ are both so small that for any $\bar{x}$ and $\bar{y}$, both on the same $\bar{0}$-composant component of $\mathscr{U}_0$, $\mu(x, y) < 1/4$ and $\mu(h(x), h(y)) < 1/4$.

THEOREM 2a. *If there exists a homeomorphism* $h\colon D_{\tilde{K}} \to D_{\tilde{L}}$, *then there exists an infinite lattice of increasing sequences of nonnegative integers so that for every* i *and* j $b_{1,i+1} - b_{1,i} = b_{1,j+1} - b_{1,j}$ *and for every* n *there exists nonnegative integers* r_n *and* s_n *so that*

$$A_n \prod_{r_n-1}^{r_n} \mathfrak{p}_i \, A_{n+1}\,, \quad \textit{and} \quad B_n \prod_{s_n-1}^{s_n} \mathfrak{q}_i \, B_{n+1}$$

and $B_n \supset A_{n+1}$ *and* $A_n \supset B_n$. *(It is easy to see that if* $\mathscr{K}$ *and* $\mathscr{L}$ *are distinct subsets of* 2^P, *that is, there exists* $\mathfrak{q} \in \mathscr{L} \backslash \mathscr{K}$, *then* s_n *can be chosen so that* $\mathfrak{q}$ *divides* $\prod_{s_n-1}^{s_n} \mathfrak{q}_i$ *for every* n *but* $\mathfrak{q}$ *does not divide* $\prod_{r_n-1}^{r_n} \mathfrak{p}_i$ *for any* n.)

THEOREM 3a. *If there exists a prime* $\mathfrak{q} \in \mathscr{L} \backslash \mathscr{K}$ *then* $D_{\tilde{K}}$ *is not homeomorphic to* $D_{\tilde{L}}$.

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Received September 2, 1980 and in revised form January 21, 1981. The author expresses thanks to the Mathematics faculty at the University of Wyoming and especially to Ira Rosenholtz and Joe Martin for their interest.

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