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A FOURIER TRANSFORM THEOREM ON NILMANIFOLDS AND NIL-THETA FUNCTIONS

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A theorem describing the image of the Euclidean Fourier transform on certain nilmanifolds is proven. As an application, we compute the Fourier transform of a certain class of distributions on a manifold which is analogous with the class of classical theta functions on C. It is shown that our theta functions satisfy a functional equation which generalizes the functional equation satisfied by the Jacobi theta functions.

In this paper we present what we feel is a fascinating theorem concerning the Euclidian Fourier transform of certain functions on certain nilmanifolds. As an application of our Fourier transform theorem, we compute the Fourier transform of a certain class of distributions on a nilmanifold which is analogous to the class of theta functions on C. It is shown that our theta distributions satisfy a functional equation which generalizes the functional equation satisfied by the Jacobi theta functions.

To describe our Fourier transform theorem, let $\mathscr{B}$ be an associative, finite dimensional, nilpotent algebra over R. $\mathscr{B}$ is said to be left commutative if $xyw = yxw$ for all $x, y, w \in \mathscr{B}$. $\mathscr{B}$ is said to be HAT if $\mathscr{B}$ is left commutative and $\dim \mathscr{Z}(\mathscr{B}) = 1$ ($\mathscr{Z}(\mathscr{B}) =$ center of $\mathscr{B}$). In this case $\mathscr{Z}(\mathscr{B})$ annihilates $\mathscr{B}$ because the two sided annihilator of $\mathscr{B}$ is nontrivial and central. $\mathscr{L}(\mathscr{B})$ is, in fact, precisely the two sided annihilator of $\mathscr{B}$. $\mathscr{B}$ defines a nilpotent Lie algebra by setting $[x, y] = xy - yx$. The corresponding Lie group can be realized as the space $\mathscr{B}$ along with the product $x*y = x + y + xy$. The exponential map $\exp_{\mathscr{B}}$ is given by $\exp_{\mathscr{B}} x = \sum_{n=1}^{\infty} x^n/n!$. The inverse of $x \in \mathscr{B}$ is $\bar{x} = \sum_{n=1}^{\infty} (-1)^n x^n$. (We choose the notation $\bar{x}$ in preference to x^{-1} to avoid confusion with inverses in $\mathscr{B}$, which of course don't exist). When we wish to think of $\mathscr{B}$ as a group we will use the notation $(\mathscr{B}, *)$.

Now suppose that $\mathscr{B}$ is a HAT algebra which contains a closed subgroup Γ such that the component of the identity of Γ is $\mathscr{Z}(\mathscr{B})$ and $\Gamma\backslash(\mathscr{B}, *)$ is compact. Let μ be an abelian character of Γ which is nontrivial on $\mathscr{Z}(\mathscr{B})$. A function ψ on $\mathscr{B}$ is said to be μ-primary if ψ is C^∞ on $\mathscr{B}$ and ψ satisfies $\psi(\gamma * x) = \mu(\gamma)\psi(x)$ for all $\gamma \in \Gamma$ and $x \in \mathscr{B}$. Our Fourier transform theorem describes the Fourier transform of the μ-primary functions.

The sense in which we mean the Fourier transform needs some explanation. Let $d\mathscr{B}$ denote an additive Haar measure on the vector space $\mathscr{B}$. Let $\mathscr{S}(\mathscr{B})$ denote the Schwartz space of rapidly decreasing functions on $\mathscr{B}$. If ψ is any locally integrable function on $\mathscr{B}$ that grows at most polynomially, we may use ψ to define a Schwartz distribution (also denoted ψ) on $\mathscr{S}(\mathscr{B})$ by setting

$$\langle f, \psi \rangle = \int_{\mathscr{B}} f(x)\psi(x)d\mathscr{B}(x) .$$

Schwartz distributions on $\mathscr{B}$ have Fourier transforms which are Schwartz distributions on $\mathscr{B}^*$. It is in this distributional sense that we take the word "Fourier transform". The specific conventions we adopt for the Fourier transform are described below for $f \in \mathscr{S}(\mathscr{B}^*)$. The Haar measure $d\mathscr{B}^*$ on $\mathscr{B}^*$ is normalized by requiring that $\mathscr{F}^{-1}\mathscr{F} = I$.

$$\mathscr{F}f(\lambda) = \int_{\mathscr{B}} f(x) \exp -2\pi i \langle x, \lambda \rangle \, d\mathscr{B}(x) \qquad (\lambda \in \mathscr{B}^*)$$

$$\mathscr{F}^{-1}f(x) - \int_{\mathscr{B}^*} g(\lambda) \exp 2\pi i \langle x, \lambda \rangle \, d\mathscr{B}^*(\lambda) \qquad (x \in \mathscr{B}) .$$

Now $\mathscr{B}$ anti-acts on $\mathscr{B}^*$ on both the left and the right according to the formulas

$$x \cdot \lambda(y) = \lambda(xy)$$
$$\lambda \cdot x(y) = \lambda(yx) .$$

$(\mathscr{B}, *)$ also anti-acts on $\mathscr{B}^*$ by setting $x*\lambda = \lambda + x\lambda$ and $\lambda*x = \lambda + \lambda x$. Each coset of $\mathscr{B}^*/(\mathscr{B}, *)$ and $(\mathscr{B}, *)\backslash\mathscr{B}^*$ is an affine subspace of $\mathscr{B}^*$. Hence each such coset carries an essentially unique Haar measure. Our first theorem on the Fourier transform of μ-primary functions is the following.

THEOREM A. *Let ψ be μ-primary, then $\mathscr{F}\psi$ is supported in a countable union of $\mathscr{B}^*/(\mathscr{B}, *)$ cosets. On each of these cosets, $\mathscr{F}\psi$ is a C^∞ function times the Haar measure of the given coset.*

This theorem, however, is just the beginning of the story. We show that $\mathscr{F}\psi$ is supported in a finite union of $\Gamma\backslash\mathscr{B}^*/(\mathscr{B}, *)$ double cosets. Let $\lambda_1, \cdots, \lambda_n$ be a complete set of representatives for these double cosets. Let $\mathscr{M}(\lambda_i) = \lambda_i*\mathscr{B}$. From the transformation properties of ψ under Γ, $\mathscr{F}\psi$ is determined by its restrictions ψ_i to $\mathscr{M}(\lambda_i)$. On $\mathscr{M}(\lambda_i)$, ψ_i is a C^∞ function. For $x \in \mathscr{B}$, let

$$\tilde{\psi}_i(x) = \exp 2\pi i \langle \bar{x}, \lambda_i \rangle \psi_i(\lambda_i * \bar{x}) .$$

We prove the following

THEOREM B. *For each i there is a closed subgroup Δ_i and a unique character χ_i of Δ_i such that*

(1) $\tilde{\psi}_i(\delta * x) = \chi_i(\delta)\tilde{\psi}_i(x)$ *for all* $\delta \in \Delta$.

(2) $\tilde{\psi}_i$ *is* C^∞ *on the connected hull* $\mathscr{M}_i$ *of* Δ_i *and* ψ_i *is Schwartz transverse to* $\mathscr{M}_i$.

(3) *Let* $\mathscr{C}_i$ *denote the space of functions satisfying* (2) *above relative to* Δ_i. *Then the mapping* $\psi \to (\tilde{\psi}_1, \cdots, \tilde{\psi}_n)$ *of into* $\sum_i \oplus \mathscr{C}_i$ *is an isomorphism of vector spaces.*

(4) *Let* $R(x)$ *denote right translation of functions by* $x \in \mathscr{B}$ $(R(x)f(y) = f(y*x))$. *Then* $R(x)(\tilde{\psi}_i) = (R(x)\tilde{\psi})\tilde{}$.

It is also true that the spaces in question (i.e., the spaces $\mathscr{C}_i$ above and the space of μ-primary functions) can all be given unitary structures in a natural way and that, upon suitable normalization, the mapping $\psi \to (\psi_1, \cdots, \psi_n)$ becomes a unitary isomorphism. Theorems A and B taken together amount to an explicit computation of the Fourier transform of an arbitrary μ-primary function ψ, especially in the case that the induced representation ind $(\Gamma, (\mathscr{B}, *), \mu)$ is irreducible. In this case there is only one space $\mathscr{C}_1$ and the mapping $\psi \to \tilde{\psi}_i$ is an intertwining operator for two irreducible representations ind $(\Gamma, (\mathscr{B}, *), \mu)$ and ind $(\Delta_1, (\mathscr{B}, *), \chi_1)$. The science of constructing such operators is well developed (see [4] e.g.). Typically ψ_1 will be expressible in the form

$$\psi_1(x) = \int_{\Gamma_0 \cap K \backslash K} \psi(kx)\overline{\chi}_i(k)dk$$

where K is an abelian normal subgroup of Δ_i and Δ_i is trivial on the co-compact subgroup $\Gamma_0 \cap K$ of K. A remarkable aspect of such formulas in that $\mathscr{F}\psi(\lambda)$ for a given λ depends only on the values of ψ along certain cosets of certain relatively small subgroups of $(\mathscr{B}, *)$.

As an application we consider the following situation. Let $\mathscr{S}$ be an abelian, nilpotent, finite dimensional algebra over R. Let B be a symmetric, bi-linear form on $\mathscr{S}$ which satisfies $B(xy, w) = B(y, xw)$. We define the scalar log function l on $\mathscr{S}$ by

$$l(x) = \sum_{n=2}^{\infty} (-1)^{n+1} B(x, x^{n-1})/n \ .$$

Note that the sum is really finite so l is a polynomial.

We suppose that there is a vector lattice $\Lambda \subset \mathscr{S}$ such that $\Lambda^2 \subset \Lambda$ and $B(\Lambda, \Lambda) \subset \boldsymbol{Z}$. Let σ be a complex number. We let $\theta_\sigma(x)$ denote the following (usually divergent) series on $\mathscr{S}$.

$$\theta_\sigma(x) = \sum_{n \in \Lambda} \exp 2\pi i(\sigma l(n) + B(n, x)) .$$

This series is called the theta series of period σ associated with $\mathscr{S}$. If σ is real each term of θ_σ is bounded so θ_σ may be considered as a sequence of Schwartz distributions on $\mathscr{S}$. We shall show that in this case θ_σ converges in the sense of distribution theory. One other case in which convergence occurs is where $\mathscr{S}$ is $\boldsymbol{R}^n$ with trivial multiplication and B is the usual scalar product. In this case $l(x) = -\|x\|^2/2$. Let $\Lambda = \boldsymbol{Z}^n$. Then

$$\theta_\sigma(x) = \sum \exp 2\pi i(-\sigma \|n\|^2/2 + n \cdot x) .$$

If σ has real imaginary part, this sum converges uniformly in x and y. The limit is a slight variant of the classical Jacobi theta on $\boldsymbol{C}^n$.

The θ_σ series behave in many ways like classical theta functions. For example, if σ is real we prove the following theorem (which does not use the Fourier transform result).

THEOREM C. *If σ is real, then as a distribution*

$$\theta_\sigma(x) = K(\sigma) \sum_{n \in \Lambda} \exp 2\pi i \sigma l(\sigma^{-1}(x + n))$$

where $K(\sigma)$ is some constant.

In the case that $\mathscr{S} = \boldsymbol{R}^2$ with trivial multiplication and B is the usual scalar product, this result is equivalent with the usual transformation identity of the Jacobi theta functions under the modular group. We also outline a technique for computing $K(\sigma)$.

To apply the Fourier transform theorem we consider a slightly larger class of functions—the mollified theta functions. A linear mapping $\tau_0 \colon \mathscr{S} \to \mathscr{S}$ is said to be a mollifier if τ_0 satisfies $\tau_0(xy) = x\tau_0 y = 0$ for all $x, y \in \mathscr{S}$ and $B(x, \tau_0 y) = B(\tau_0 x, y)$. The 0 map is a mollifier, although $\mathscr{S}$ always has nontrivial mollifiers too. For example, if $e \in \mathscr{S}$ and $e\mathscr{S}^2 = 0$ then the mapping $\tau_0(x) = ex$ is a mollifier. The two variable series

$$\theta_{\sigma,\tau_0}(x, y) = [\exp 2\pi i(\sigma l(x) + B(\tau_0 x, y))]\theta_\sigma(y - \sigma x)$$

is called the mollified theta series with period σ and mollifier τ_0. Using our Fourier transform theorem we may compute the Fourier transfer on $\mathscr{S} \times \mathscr{S}$ of any mollified theta functions. See § III.

II. General theory. We begin with a proof of Theorem A above. Throughout this work we shall write distributions in functional notation. For example if ψ is a Schwartz distribution on a

vector space V and A is a topological isomorphism of V into V which preserves the Haar measure dV, and preserves $\mathcal{S}(V)$, then by $x \to \psi(A(x))$ we shall mean the distribution ψ_A defined by $\langle f, \psi_A \rangle = \langle f \circ A^{-1}, \psi \rangle$. Similarly, if $g(x)$ is a C^∞ function which has bounded derivatives of all orders, by $x \to g(x)\psi(x)$ we shall mean the distribution $f \to \langle fg, \psi \rangle$.

We consider μ-primary functions as distributions on $\mathcal{B}$ as usual and let $\mathcal{F}$ denote the Fourier transform as described above. First we shall prove the statements about the support of $\mathcal{F}\psi$.

LEMMA 1. *Let ψ be a Schwartz distribution on $\mathcal{B}$. Let $L(g)\psi$ be the distribution $x \to \psi(\bar{g} * x)$ and let $R(g)\psi$ be $x \to \psi(x * g)$. (See above comments for notation). Then*

(a) $$\mathcal{F}(L(g)\psi)(\lambda) = \exp -2\pi i \langle g, \lambda \rangle \mathcal{F}\psi(g * \lambda)$$

(b) $$\mathcal{F}(R(g)\psi)(\lambda) = \exp -2\pi i \langle \bar{g}, \lambda \rangle \mathcal{F}\psi(\lambda * \bar{g}) \,.$$ □

Proof. Straightforward.

Now let $\mathcal{L}(\mathcal{B})$ be the left annihilator of $\mathcal{B}$,

$$\mathcal{L}(\mathcal{B}) = \{x \in \mathcal{B} \mid x\mathcal{B} = \{0\}\} \,.$$

Let $\mathcal{M}_\mu \subset \mathcal{B}^*$ be the set of functionals λ such that $\exp 2\pi i\lambda$ equals μ on $\Gamma \cap \mathcal{L}(\mathcal{B}) = \Gamma_{\mathcal{L}}$. It can easily be seen that $\mathcal{M}_\mu$ is nonempty, although the nontriviality also follows from the corollary below.

LEMMA 2. *Let ψ be μ-primary. Then $\mathcal{F}\psi$ has a restriction to $\mathcal{M}_\mu$ in the sense that if $f = g$ on $\mathcal{M}_\mu$, then $\mathcal{F}\psi(f) = \mathcal{F}\psi(g)$ for $f, g \in \mathcal{S}(\mathcal{B}^*)$.*

Proof. There is a character $\tilde{\mu}$ of the abelian group $\mathcal{L}(\mathcal{B})$ which equals μ on $\Gamma_{\mathcal{L}}$. There is a functional $\lambda_0 \in \mathcal{M}_\mu$ such that $\exp 2\pi i\lambda_0 | \mathcal{L}(\mathcal{B}) = \tilde{\mu}$. Let ψ be μ-primary and let

$$\mathcal{N}(x) = \exp -2\pi i \langle x, \lambda_0 \rangle \psi(x) \,,$$

Then $\mathcal{N}(\gamma x) = \mathcal{N}(x)$ for $\gamma \in \Gamma_{\mathcal{L}}$ and $x \in \mathcal{B}$. Also

$$\mathcal{F}\psi(\lambda) = \mathcal{F}\mathcal{N}(\lambda - \lambda_0)$$

so it suffices to show that if $f = 0$ on $\mathcal{M}_\mu - \lambda_0$, then $\mathcal{N}(f) = 0$. To see this, let $g = \mathcal{F}^{-1}f$, $g \in \mathcal{S}(\mathcal{B})$. Let

$$g_0(x) = \sum_{\mathcal{L}} g(x + \gamma) \,.$$

This converges to a C^∞ function because g is Schwartz. It follows from reasoning similar to that found in the proof of Corollary 19 below that $g_0 = 0$ implies that $\mathscr{N}(g) = 0$. But, from the Poisson summation formula

$$g_0(x) = \int_{(\Gamma_{\mathscr{L}})^*} f(\lambda) \exp -2\pi i \langle x, \lambda \rangle d\lambda$$

where $(\Gamma_{\mathscr{L}})^* = \{\lambda \in \mathscr{B}^* \mid \lambda(\Gamma_{\mathscr{L}}) \subset Z\}$. But then $(\Gamma_{\mathscr{L}})^* = \mathscr{M}_\mu - \lambda_0$ so indeed $g_0 = 0$. Our lemma follows from this since $\mathscr{N}(g) = \hat{\mathscr{N}}(f)$. □

Now let $\lambda \in \mathscr{M}_\mu$. Let $\mathscr{M}(\lambda) = \lambda + \mathscr{L}(\mathscr{B})^\perp$. Then $\mathscr{M}_\mu$ is a countable union of sets of the form $\mathscr{M}(\lambda)$. The next lemma shows, among other things that each $\mathscr{M}(\lambda)$ is a coset of $\mathscr{B}^*/(\mathscr{B}, *)$.

LEMMA 3. Let $\lambda \in \mathscr{B}^*$ *and suppose* λ *is nonzero on* $\mathscr{Z}(\mathscr{B})$. *Then* $x \cdot \lambda = 0$ *iff* $x \in \mathscr{L}(\mathscr{B})$. *Also* $\mathscr{L}(\mathscr{B})^\perp = \{\lambda \cdot x \mid x \in \mathscr{B}\}$ *and* $x \cdot \lambda = \lambda \cdot x$ *iff* $x \in \mathscr{Z}(\mathscr{B})$.

Proof. Suppose $x \cdot \lambda = 0$ and $x \notin \mathscr{L}(\mathscr{B})$. Then there is an n such that $x \cdot \mathscr{B}^n = \{0\}$ and $x\mathscr{B}^{n-1} \neq 0$. Then $x \cdot \mathscr{B}^{n-1} \subset \mathscr{Z}(\mathscr{B})$ so $\lambda(x\mathscr{B}^{n-1}) \neq 0$. This is a contradiction to $\dim \mathscr{Z}(\mathscr{B}) = 1$. The second statement is equivalent to saying $\mathscr{L}(\mathscr{B}) = \bigcap_x \ker \lambda \cdot x$. This follows trivially from the first statement.

To prove the third statement, if x is not in $\mathscr{Z}(\mathscr{B})$ there is an n such that $[x, \mathscr{B}^n] = 0$ and $[x, \mathscr{B}^{n-1}] \neq 0$. Then $[x, \mathscr{B}^{n-1}] \subset \mathscr{Z}(\mathscr{B})$ so there is a $y \in \mathscr{B}^{n-1}$ such that $\lambda([x, y]) \neq 0$. But then $(x \cdot \lambda - \lambda \cdot x)(y) \neq 0$, which is a contradiction. □

Next we show the "finiteness" claim in Theorem B. Let $\mathscr{V}_\mu = \{\beta \in \mathscr{B}^* \mid \beta \mid \mathscr{Z}(\mathscr{B}) = \lambda \mid \mathscr{Z}(\mathscr{B})$ for all $\lambda \in \mathscr{M}_\mu\}$.

LEMMA 4. $\mathscr{V}_\mu = \mathscr{B} * \lambda * \mathscr{B}$.

Proof. Consider the mapping $x \to x \cdot \lambda - \lambda \cdot x$ of $\mathscr{B}$ into $\mathscr{Z}(\mathscr{B})^\perp$. By the lemma this mapping is surjective. Let $y = \bar{x}$. Then $x * \lambda * y = \lambda + x\lambda + \lambda x + x\lambda y$. By left commutativity $x\lambda y = \lambda xy = -\lambda(x + y)$ since $x + y + xy = 0$. Hence $x * \lambda * y = \lambda + x \cdot \lambda - \lambda \cdot x$. □

COROLLARY 5. $\mathscr{M}_\mu$ *is a finite union of sets of the form* $\Gamma * \mathscr{M}(\lambda)$, $\lambda \in \mathscr{M}_\mu$.

Proof. Let $\lambda \in \mathscr{M}_\mu$. Let $\tilde{\Gamma} = \{x \in \mathscr{B} \mid x * \lambda \in \mathscr{M}_\mu\}$. Then $x \in \tilde{\Gamma}$ iff $x * (\lambda * \mathscr{B}) = x * \mathscr{M}(\lambda)$ intersects $\mathscr{M}_\mu$. Hence $\tilde{\Gamma} * \mathscr{M}(\lambda) = \mathscr{M}_\mu$ by Lemma 4. $\tilde{\Gamma}$ not be a subgroup of $(\mathscr{L}, *)$. However, on $\mathscr{L}(\mathscr{B})$,

$x*\lambda(y) = \lambda(x*y*\bar{x})$. Hence $\tilde{\Gamma}$ is a union of right Γ cosets. The component of 0 in $\tilde{\Gamma}$ is $\tilde{\Gamma}_0 = \{x \mid x\cdot\lambda \mid \mathscr{L}(\mathscr{B}) = 0\}$. This is a two sided ideal of $\mathscr{B}$ and hence a normal subgroup of $(\mathscr{B}, *)$. $\Gamma*\tilde{\Gamma}_0$ is a closed subgroup of $(\mathscr{B}, *)$ and the quotient $M_0 = (\mathscr{B}, *)/\Gamma*\tilde{\Gamma}_0$ is compact. The image of $\tilde{\Gamma}$ in M_0 is a discrete, closed subset. Hence $\tilde{\Gamma}/\Gamma*\tilde{\Gamma}_0$ is compact. The image of $\tilde{\Gamma}$ in M_0 is a discrete, closed subset. Hence $\tilde{\Gamma}/\tilde{\Gamma}*\tilde{\Gamma}_0$ is finite. Let x_i be a collection of coset representatives for $\tilde{\Gamma}$. Then

$$\mathscr{M}_\mu = \cup\, (x_i*(\Gamma*\Gamma_0))*\mathscr{M}(\lambda)\ . \qquad \square$$

The spaces $\Gamma\backslash\Gamma*\mathscr{M}(\lambda)$ are conveniently describable as homogeneous spaces. Let $\Gamma_0 = \{\gamma \in \Gamma \mid \gamma*\lambda \in \mathscr{M}(\lambda)\} = \{\gamma \in \Gamma \mid \gamma\cdot\lambda \mid \mathscr{L}(\mathscr{B}) = 0\} = \{\gamma \in \Gamma \mid \gamma*\mathscr{M}(\gamma) = \mathscr{M}(\lambda)\}$. Then $\Gamma\backslash\Gamma*\mathscr{M}(\lambda) = \Gamma_0\backslash\mathscr{M}(\lambda)$. From Lemma 3, for each $\gamma \in \Gamma_0$ there is a $\delta \in \mathscr{B}$ such that $\gamma*\lambda = \lambda*\delta$. Let $\Delta_\lambda = \{\delta \in \mathscr{B} \mid \gamma*\lambda = \lambda*\delta$ for some $\gamma \in \Gamma\}$. Note that $\Delta_\lambda \supset \{x \in \mathscr{B} \mid \lambda*x = \lambda\}$. There is a one-to-one correspondence between elements of $\Gamma_0\backslash\mathscr{M}(\lambda)$ and $(\mathscr{B}, *)/\Delta_\lambda$. In fact for $\beta \in \mathscr{M}(\lambda)$ there is an x such that $\beta = \lambda*x$. $\Gamma_0*\beta = \lambda*(x*\Delta_\lambda)$. The mapping $\Gamma_0*\beta \to x*\Delta_\lambda$ defines the correspondence. Hence we have proven the following

PROPOSITION 6. *$\Gamma\backslash\mathscr{M}_\mu$ is a finite union of sets of the form $\Gamma\backslash\Gamma*\mathscr{M}(\lambda)$. Each of these sets is a $(\mathscr{B}, *)$ homogeneous space.*

Next we prove the fact that $\mathscr{F}\psi$ is C^∞ on $\mathscr{M}_\mu$. This the hardest part of the proof of Theorems A and B. The main technique is a pair of theorems on C^∞ vectors for representations of Lie groups due to Poulsen. Specifically, let U be a continuous unitary representation of a Lie group G in a Hilbert space $\mathscr{H}(U)$.

Let $C^\infty(U)$ be the space of differentiable vectors for U — i.e., $C^\infty(U)$ is the set of vectors v for which $g \to U(g)v$ is a C^∞ mapping of G into the representation space $\mathscr{H}(U)$ of U.

We give $C^\infty(U)$ its usual topology (see [9]). Let $C^{-\infty}(U)$ be the *conjugate* dual space to $C^\infty(U)$. The scalar product on $\mathscr{H}(U)$ allows us to imbed $\mathscr{H}(U)$ in $C^{-\infty}(U)$. $C^\infty(U)$ is U invariant. We set $U^\infty = U \mid C^\infty(U)$. We let $U^{-\infty}$ denote the contragredient representation. The theorem which allows us to conclude that $\mathscr{F}\psi$ is a C^∞ function is the following beautiful result due to Poulsen ([9], Corollary 2.1).

THEOREM 7. *Let U_1 and U_2 be unitary representations of the Lie group G. Suppose $T: C^\infty(U_1) \to C^{-\infty}(U_2)$ is a continuous operator which intertwines U_1^∞ and $U_2^{-\infty}$. Then the image of T is contained in $C^\infty(U_2)$ and T is continuous from $C^\infty(U_1)$ to $C^\infty(U_2)$.*

Proof. Let $\beta: C^\infty(U_1) \times C^\infty(U_2) \to \boldsymbol{C}$ be defined by $\beta(x, y) = \langle y, Tx \rangle$. Then apply Corollary 2.1 of [9] and note the comments following 2.1 in [9]. □

The following result, which also is a corollary of results of [9], enables us to explicitly compute $\mathscr{S}\psi$.

THEOREM 8. *If either U_1 or U_2 is a finite direct sum of irreducible representations then T has a unique bounded extension to an intertwining operator from $\mathscr{H}(U_1)$ to $\mathscr{H}(U_2)$.*

Proof. It is known that $C^\infty(U_i)$ is reflexive. Hence $T^*: C^\infty(U_2) \to C^{-\infty}(U_1)$. T has a bounded extension iff T^* does. Hence it suffices to assume that U_1 is a finite sum of irreducible representations. Therefore, it suffices to assume that U_1 is irreducible. By Theorem 7, $T^*: C^\infty(U_2) \to C^\infty(U_1)$. T^*T is then a self intertwining operator for U_1^∞. By Corollary 3.5 of [9], $T^*T = \lambda I$. Hence $\|T\| = |\lambda|^{1/2}$. □

Next we describe the representations we intend to use as U_1 and U_2. First some generalities on induced representations. Let K be a closed subgroup of a nilpotent Lie group N and let V be a unitary representation of K in a Hilbert space $\mathscr{H}(V)$. Let $\mathscr{H}(U^V)$ denote the set of $\mathscr{H}(V)$ valued measurable maps f on N which satisfy

(1) $f(kx) = V(k)f(x)$ for $k \in K$ and $x \in N$.

(2) $\|f\| \in L^2(K\backslash N)$ relative to the canonical measure.

Let U^V be the unitary representation of N in $\mathscr{H}(U^V)$ defined by right translation. We shall also write $U^V = \mathrm{ind}\,(K, N, V)$. Whenever we discuss induced representations we always will be using the above realization unless otherwise stated.

Now let $U_1 = \mathrm{ind}\,(\Gamma, (\mathscr{B}, *), \mathscr{H})$. It follows from theorem of [9] and the compactness of $\Gamma\backslash(\mathscr{B}, *)$ that $C^\infty(U_1)$ is precisely the space of μ-primary functions of $(\mathscr{B}, *)$.

LEMMA 9. *U_1 is a finite sum of irreducible representations.*

Proof. Since $\Gamma\backslash(\mathscr{B}, *)$ is compact it is known that U_1 is an utmost countable sum of irreducible representations and any given irreducible representation occurs in U_1 with utmost finite multiplicity. We claim that U_1 is in fact primary. This will prove our lemma. The primarity of U_1 follows from results of Moore-Wolf and the following sublemma.

Sublemma 10. Every irreducible unitary representation U of

$(\mathscr{B}, *)$, which is nontrivial on $Z(\mathscr{B})$ is square integrable in the sense of Moore-Wolf [8].

This sublemma implies the primarity of U_1 for from Moore-Wolf [8] two square integrable representations are equivalent on $(\mathscr{B}, *)$ iff they have the same character on $Z(\mathscr{B})$. Clearly each irreducible subrepresentation of U_1 is equivalent to every other irreducible subrepresentation by this criterion. To prove the sublemma it suffices to show that for any $\lambda \in \mathscr{B}^*$ such that $\lambda | Z(\mathscr{B}) \neq 0$ we have $\lambda([x, y]) = 0$ for all y iff $x \in Z(B)$. (See Moore-Wolf, loc. cit.) This was shown in Lemma 6 above. □

To describe U_2, let $\mathscr{M}_\mu$ be as before. By the previous results $\Gamma\backslash\mathscr{M}_\mu$ is a finite union of sets of the form $\Gamma\backslash\Gamma * \mathscr{M}(\lambda)$ and each of these sets is a $(\mathscr{B}, *)$ homogeneous space. Homogeneous spaces carry invariant measures so $\Gamma\backslash\mathscr{M}_\mu$ has a $(\mathscr{B}, *)$ invariant measure. This measure is far from unique since $\Gamma\backslash\mathscr{M}_\mu$ is not necessarily connected. However, we may normalize the measure on $\Gamma\backslash\mathscr{M}_\mu$ as follows. $\mathscr{M}_\mu$ is an additive coset of the additive subgroup $\mathscr{M}_0$ of $\mathscr{B}^*$ given by $\mathscr{M}_0 = \{\lambda \in \mathscr{B}^* | \lambda | \Gamma_{\mathscr{L}} = 0\}$. (Recall $\Gamma_{\mathscr{L}} = \Gamma \cap \mathscr{L}(\mathscr{B})$). As such, $\mathscr{M}_\mu$ carries an essentially unique measure $d\mathscr{M}$ which is invariant under the transformation $x \to x + y - z$ for all $x, y, z \in \mathscr{M}_\mu$. Let f be a continuous function on $\mathscr{M}_\mu$ with compact support. Then $\sum_{\Gamma_{\mathscr{L}}\backslash\Gamma} f(\gamma * \lambda) = \tilde{f}(\lambda)$ defines an element of $C_c(\Gamma\backslash\mathscr{M}_\mu)$ and there is a unique choice of invariant measures v on $\Gamma\backslash\mathscr{M}_\mu$ such that $\int_{\mathscr{M}_\mu} f d\mathscr{M} = \sum_{\Gamma\backslash\mathscr{M}_\mu} \tilde{f} dv$. We assume that v has been so chosen.

Let $\mathscr{H}_2(\mu)$ be set of measurable functions on $\mathscr{M}_\mu$ for which

(1) $\mu(\gamma) f(\lambda) = \exp -2\pi i \langle \gamma, \lambda \rangle f(\gamma * \lambda)$ $\gamma \in \Gamma$.

(2) $|f| \in L^2(\Gamma\backslash\mathscr{M}_\mu)$.

Then $\mathscr{H}_2(\mu)$ is a Hilbert space with the obvious innerproduct. We define a representation U_2 of $(\mathscr{B}, *)$ in $\mathscr{H}_2(\mu)$ by setting $U_2(g) f(\lambda) = \exp -2\pi i \langle \bar{g}, \lambda \rangle f(\lambda * \bar{g})$. The point of these definitions is that they mimic the invariance properties of the Fourier transform of primary functions as described in Lemma 1. In fact, if ψ is a primary function, $\mathscr{T}\psi$ meets the *algebraic* condition necessary to define a functional on $C^\infty(U_2)$: $\mu(\gamma)\mathscr{T}\psi(\lambda) = \exp -2\pi i \langle \gamma, \lambda \rangle \mathscr{T}\psi(\gamma * \lambda)$, for all $\gamma \in \Gamma$. The question is, does $\mathscr{T}\psi$ have the correct rate of decay at ∞ to define an element of $C^{-\infty}(U_2)$. In fact, if we can show that $\mathscr{T}\psi \in C^{-\infty}(U_2)$ then $\psi \to \mathscr{T}\psi$ will define a continuous intertwining operator between $C^\infty(U_1)$ and $C^{-\infty}(U_2)$. By Poulsen's theorem (7 above) $\psi \to \mathscr{T}\psi$ will then be a continuous mapping of $C^\infty(U_1)$ into $C^\infty(U_2)$. It will be shown that $C^\infty(U_2)$ consists of C^∞ functions on $\Gamma\backslash\mathscr{M}_\mu$ so we will have proven that $\mathscr{T}\psi$ is a C^∞ function. Actually $C^\infty(U_2)$ is not a subset of $\mathscr{S}(\mathscr{B})$, in part because

the elements of $C^\infty(U_2)$ are supported in $\mathscr{M}_\mu$. This would not cause problems if each element f of $C^\infty(U_2)$ could be extended to an element $\tilde{f}$ of $\mathscr{S}(\mathscr{B}^*)$ for then we could define $\mathscr{T}\psi(f) = \mathscr{T}\psi(\tilde{f})$. Unfortunately, the elements of $C^\infty(U_2)$ cannot be so extended as the elements of $C^\infty(U_2)$ do not tend to zero at infinity. What we will show is the following:

Let $f \in \mathscr{S}(\mathscr{B}^*)$ and let $y \in \Gamma$. Set

$$V(y)f(\lambda) = \mu(y) \exp -2\pi i\langle y, \lambda\rangle f(y*\lambda) .$$

Note that for $y \in \Gamma_{\mathscr{L}} = \Gamma \cap \mathscr{L}(\mathscr{B})$ and $x \in \mathscr{B}$ we have $y*x = y + x$, $y*\lambda = \lambda$ and $\mu(y) = \exp -2\pi i\langle y, \lambda\rangle$. It follows that as a function of y, $V(y)$ is constant on cosets of $\Gamma_{\mathscr{L}}$ in Γ. Of course $\Gamma_{\mathscr{L}} \supset Z(\mathscr{B})$ so that $\Gamma_{\mathscr{L}}\backslash\Gamma$ is discrete. Let

$$\tau_\mu f = \sum_{\Gamma_{\mathscr{L}}\backslash\Gamma} V(\gamma)f \,|\, \mathscr{M}_\mu .$$

We will show that τ_μ is a surjective mapping of $\mathscr{L}(\mathscr{B})$ onto $C^\infty(U_2)$. Furthermore we shall show that $\ker \tau_\mu \subset \ker \mathscr{T}\psi$. It will follow that we may define an element $\tilde{\psi}$ of $C^{-\infty}(U_2)$ by setting $\tilde{\psi}(\tau f) = \mathscr{T}\psi(\bar{f})$. $\psi \to \tilde{\psi}$ will then be an intertwining operator from $C^\infty(U_1)$ to $C^{-\infty}(U_2)$.

It will then be an easy matter to show that $\mathscr{T}\psi$ is a C^∞ function on $\mathscr{M}_\mu$.

To describe $C^\infty(U_2)$ let $\mathscr{M}_\mu = \bigcup_{i=1}^n \Gamma*\mathscr{M}(\lambda_i)$ as in Lemma 5. Let $\mathscr{K}(\lambda_i)$ be the set of functions in $\mathscr{K}_2(\mu)$ which are supported in $\Gamma*\mathscr{M}(\lambda_i)$. The $\mathscr{K}(\lambda_i)$ are mutually orthogonal, invariant subspaces of $\mathscr{K}_2(\mu)$ and $\mathscr{K}_2(\mu) = \sum_{i=1}^n \oplus \mathscr{K}(\lambda_i)$. Let $U(\lambda_i)$ be the restriction of U_2 to $\mathscr{K}(\lambda_i)$. We shall describe $C^\infty(U(\lambda_i))$. Let i be fixed and set $\lambda = \lambda_i$.

As commented above, $\Gamma\backslash\Gamma*\mathscr{M}(\lambda) \approx (\mathscr{B}, *)/\Delta_\lambda$ where $= \{\delta \in \mathscr{B} \mid \lambda*\delta = \gamma*\lambda$ for some $\gamma \in \Gamma_0\}$. This suggests that $U(\lambda)$ should be equivalent with a representation induced from a character χ_λ of Δ_λ. This is the case as we now show. First we define χ_λ.

LEMMA 11. *For $\delta \in \Delta_\lambda$ let*

$$\chi_\lambda(\delta) = \bar{\mu}(\lambda(\delta)) \exp -2\pi i\lambda(\delta - \gamma(\delta))$$

where $\gamma(\delta)$ is any element of Γ such that $\gamma(\delta)\cdot\lambda = \lambda\cdot\delta$. Then $\chi_\lambda(\delta)$ is a well defined character of Δ_λ.

Proof. First we show that χ_λ is well defined—i.e., $\bar{\mu}(\gamma(\delta))$ $\exp -2\pi i\langle\gamma(\delta), \lambda\rangle$ does not depend on the choice of $\gamma(\delta)$ in Γ. Let $\gamma*\lambda = \gamma'*\lambda$ for γ and γ' in Γ. Then $(\gamma'*\bar{\gamma})*\lambda = \lambda$ so $\gamma'*\bar{\gamma} \in \Gamma_{\mathscr{L}}$ by Lemma 3. Let $\sigma = \gamma'*\bar{\gamma}$. Then $\gamma' = \sigma*\gamma = \sigma + \gamma$. By definition,

$\mu(\sigma) = \exp -2\pi i\langle\sigma, \lambda\rangle$ so

$$\mu(\gamma)\exp -2\pi i\langle\gamma, \lambda\rangle = \mu(\gamma')\exp -2\pi i\langle\gamma', \lambda\rangle$$

as claimed.

To show that χ_λ is a character, let δ_1, $\delta_2 \in \Delta_\lambda$. We may take $\gamma(\delta_1*\delta_2) = \gamma(\delta_1)*\gamma(\delta_2)$. Then $\delta_1*\delta_2 - \gamma(\delta_1*\delta_2) = \delta_1 - \gamma(\delta_1) + \delta_2 - \gamma(\delta_2) + \delta_1\delta_2 - \gamma(\delta_1)\gamma(\delta_2)$. Since μ is a character on Γ, it suffices to show that $\langle\delta_1\cdot\delta_2, \gamma\rangle = \langle\gamma(\delta_1)\gamma(\delta_2), \lambda\rangle$. But $\langle\delta_1\cdot\delta_2, \lambda\rangle = \langle\delta_1, \lambda\cdot\delta_2\rangle = \langle\delta_1, \gamma(\delta_2)\cdot\lambda\rangle = \langle\gamma(\delta_2)\delta_1, \lambda\rangle = \langle\gamma(\delta_2), \lambda\cdot\delta_1\rangle = \langle\gamma(\delta_1)\gamma(\delta_2), \lambda\rangle$. □

THEOREM 12. *$U(\lambda)$ is equivalent with the induced representation* $\operatorname{ind}(\Delta_\lambda, \chi_\lambda, (\mathscr{B}, *))$.

Proof. For $f \in \mathscr{K}(\lambda)$ let

$$Tf(x) = \exp 2\pi i\langle\bar{x}, \lambda\rangle f(\lambda*\bar{x}) .$$

Tf is a map of $\mathscr{B}$ into C. We claim Tf belongs to $\mathscr{K}(\operatorname{ind}\Delta_\lambda, \chi_\lambda, (\mathscr{B}, *))$. Let $\delta \in \Delta_\lambda$. Then

$$Tf(\delta*x) = \exp 2\pi i\langle\bar{x}*\bar{\delta}, \lambda\rangle f((\lambda*\delta)*\bar{x}) .$$

Now

$$f((\lambda*\bar{\delta})*\bar{x}) = f(\gamma(\bar{\delta})*(\lambda*\bar{x})) = \bar{\mu}(\gamma(\bar{\delta}))\exp -2\pi i\langle\gamma(\delta), \lambda*\bar{x}\rangle f(\lambda*\bar{x}) .$$

Also

$$\langle\bar{x}*\bar{\delta}, \lambda\rangle = \langle\bar{x} + \bar{\delta} + \bar{x}\bar{\delta}, \lambda\rangle = \langle\bar{x} + \bar{\delta} + \lambda(\bar{\delta})\bar{x}, y\rangle = \langle\gamma(\bar{\delta}), \lambda*\bar{x}\rangle + \langle\bar{\delta} - \lambda(\delta), \lambda\rangle + \langle\bar{x}, \lambda\rangle .$$

Combining these equalities we see

$$Tf(\delta*x) = \chi_\lambda(\delta)Tf(x) .$$

Since the measure on $\Gamma\backslash\Gamma*\mathscr{M}(\lambda)$ is defined by the identification of $\Gamma\backslash\Gamma*\mathscr{M}(\lambda)$ with $\Delta_\lambda\backslash(\mathscr{B}, *)$, T is unitary. We need to show that T intertwines the given representations. This, however, is a straightforward computation similar to the above.

Next we must describe the C^∞ vectors for $\operatorname{ind}(\Delta_\lambda, \chi_\lambda, (\mathscr{B}, *)) = U(\lambda, \chi)$. They are describable in terms of the Schwartz functions on $\mathscr{B}$. Actually our description will be valid for any connected, simply connected nilpotent Lie group. Let N be any such group and let $\mathscr{N}$ be the Lie algebra of N. Let $\mathscr{T}$ be a vector subspace of the vector space $\mathscr{N}$. Let $\mathscr{S}(\mathscr{T})$ denote the Schwartz space of rapidly decreasing C^∞ functions on $\mathscr{T}$. Let $T \subset N$ be the set $T = \exp_N\mathscr{T}$. By $\mathscr{S}(T)$ we shall mean the set of complex functions f

on T such that $f \circ \exp_N$ belongs to $\mathscr{S}(\mathscr{T})$. If $\mathscr{V}$ is a topological vector space, by $\mathscr{S}(T, \mathscr{V})$ we shall mean the set of functions $f: T \to \mathscr{V}$ such that $t \to \langle f(t), w \rangle$ belongs to $\mathscr{S}(T)$ for all $w \in \mathscr{V}^*$ ($\mathscr{V}^*$ is the continuous dual of $\mathscr{V}$).

We shall require an intrinsic description of $\mathscr{S}(N)$. A polynomial function on N is a function p on N such that $p \circ \exp_N$ is a polynomial on N. A right, polynomial, differential operator on N is an operator D expressible in the form $D = \sum p_i X_i$ where X_i are left invariant differential operators on N and p_i are polynomial functions on N.

LEMMA 13. *$\exp_N$ maps the set of polynomial differential operators on $\mathscr{N}$ surjectively onto the set of right polynomial differential operators on N.*

Proof. Let $Y \in \mathscr{N}$. Let $D_Y f(x) = d/dt|_{t=0} f(x + tY)$. We begin by showing that there is a polynomial operator F_Y on N such that $(F_Y f) \circ \exp_N = D_Y(f \circ \exp_N)$. Let $\{X_1, \cdots, X_n\}$ be a Jordan-Hölder basis for $\mathscr{N}$ — i.e., $[X_i, X_j] \in \operatorname{span}\{X_1, \cdots, X_{j-1}\}$ for all i, j. It suffices to show F_{X_k} exists for all j. We proceed by induction on j. For $Y \in \mathscr{N}$ let Y denote also the left invariant differential operator defined by Y. If $i = 1$ we claim that $F_{X_1} = X_1$. In fact

$$X_1 f \circ \exp_N x = \frac{d}{dt}\Big|_{t=0} f(\exp_N x \exp_N tX_1)$$
$$= \frac{d}{dt}\Big|_{t} f(\exp_N (x + tX_1)) = (D_{X_1} f) \circ \exp_N .$$

Now suppose that the theorem is true for all $j_0 < j$. There is a polynomial mapping p of $\mathscr{N} \times \boldsymbol{R}$ into $\mathscr{N}$ such that $\exp_N \times \exp_N tX_j = \exp_N p(x, t)$. Furthermore $p(x, t) = x + tX_j \bmod \operatorname{span}\{\chi_1, \cdots, \chi_{j-1}\}$. Hence

$$X_j f \circ \exp_N(x) = \frac{d}{dt}\Big|_{t=0} f \circ \exp_N p(x, t)$$
$$= D_{X_j}(f \circ \exp_N x) + \sum_{k=1}^{j-1} p_k(x) D_{X_k} f \circ \exp_N x$$
$$= D_{X_j}(f \circ \exp_N x) + \sum_{k=1}^{j-1} p_k(x)(F_{X_k} f) \circ \exp_N x .$$

We get the desired result upon solving the above for D_{X_j}.

It is now clear that $\exp_N$ maps polynomial differential operators on $\mathscr{N}$ to right polynomial differential operators on N as the D_{X_j} generate the algebra of polynomial operations as a module over the polynomial functions. The surjectivity of the mapping is obvious

since it is clear that for any right polynomial operator D, the operator D' given by $f \to (D(f \circ \log_N) \circ \exp_N)$ is polynomial on $\mathcal{N}$. □

COROLLARY 14. *A C^∞ function f belongs to $\mathcal{S}(N)$ iff Df is uniformly bounded for all right polynomial differential operators D.*

Now let K be a closed, but not necessarily connected, subgroup of N. Let χ be a character of K. Let $U^\chi = \text{ind}\,(K, N, \chi)$ realized as usual (see the above comments). Let τ be the map of $\mathcal{S}(N)$ into $\mathcal{H}(U^\chi)$ given by:

$$\tau f(x) = \int_K \chi(k^{-1}) f(kx) dk \,.$$

Our main result on C^∞ vectors is the following

THEOREM 15. *Suppose U^χ is a finite direct sum of irreducible representations. Then τ is a surjective mapping of $\mathcal{S}(N)$ into $C^\infty(U^\chi)$.*

Proof. According to theorem of [9], $C^\infty(U^\chi)$ is the largest space $\mathcal{C}$ of C^∞ functions which satisfies the following

(1) $f \in \mathcal{C}$ implies that $f(kx) = \chi(k) f(x)$ for all $k \in K$ and $x \in N$.

(2) $|f| \in L^2(K/N)$.

(3) $\mathcal{C}$ is closed under all left invariant differential operators.

From this characterization, it is clear that $\mathcal{S}(N) \subset C^\infty(U^\chi)$. The only problem is to prove the surjectivity. Let $L \supset K$ be the smallest closed connected subgroup of N containing K. Let $\mathcal{L}$ be the Lie algebra of L and let $\mathcal{T}$ be a vector compliment to $\mathcal{L}$ in $\mathcal{N}$. Let $T = \exp \mathcal{T}$. Then $N = LT$ and the mapping of $L \times T \to N$ given by $(l, t) \to lt$ is a C^∞ isomorphism.

LEMMA 16. *There is a C^∞ function ϕ on N such that* (i) *ϕ is independent of the t variable and is compactly supported in the l-variable.*

(ii) $\int_K \phi(kx) dk = 1$ *for all* $x \in N$.

(iii) *ϕ is positive.*

Proof. It suffices to assume $L = N$ for we may extend a ϕ which is defined on L to N by setting $\phi(lt) = \phi(l)$. Hence $K\backslash N$ may be assumed compact. (See [1]). Let us also assume for the moment that the identity component K_0 of K is trivial so K is discrete. Let $\pi: N \to K\backslash N$ be the projection map. There is a finite family of open sets $U_1, \cdots, U_n$ of N such that $\pi | U_i$ is a homeomorphism onto

an open subset of $K\backslash N$ and $\pi(U_i)$ cover $K\backslash N$. By passing to a subordinate covering of $K\backslash N$ if necessary, we may assume that there is a partition of unity of $K\backslash N$ relative to the $\pi(U_i)$ — i.e., there are C^∞ functions h_i on $K\backslash N$ such that each h_i is supported in $\pi(U_i)$ and $\sum_{i=1}^n h_i = 1$. Let ϕ_i be the function which is equal to $h_i \circ \pi$ on U_i and is 0 otherwise. Let $\phi = \sum_{i=1}^n \phi_i$.

We claim that $\sum_k \phi(kx) = 1$ for all x. To see this note that $\sum_K \phi_i(kx) = h_i \circ \pi(x)$ since $k_1 U_1 \cap k_2 U_2 = \phi \Rightarrow k_1 = k_2$. Hence $\sum_k \phi(kx) = \sum_K h_i \circ \pi = 1$.

If $K_0 \neq \{e\}$, then K_0 is normal. We form the ϕ corresponding to $K_0\backslash K$ on $K_0\backslash N$. Let $\pi_0 : N \to K_0\backslash N$ be the projection and let $\phi_0 = \phi \circ \pi$. There is a C^∞ function ψ with compact support on cosets of K_0 such that $\int_{K_0} \psi(kx)dk = 1$ for all x. Then $\phi_0\psi$ is the desired ϕ on N. □

We shall use ϕ to prove the surjectivity of τ as follows. Let $f \in C^\infty(U^\chi)$. Then f is a C^∞ function on N. We claim that $f = \tau\phi f$. In fact

$$\begin{aligned}\tau\phi f(x) &= \int_K \chi(k^{-1})\phi(kx) f(kx)dk \\ &= \int_K \phi(kx)dk f(x) \\ &= f(x) .\end{aligned}$$

Hence we need only show that $\phi f \in \mathscr{S}(N)$. Let L and T be as in the above proof. Let p be a polynomial function on T. We shall show that $\sup_{t\in T} |p(t)| \sup_{l\in L} |f(lt)| < \infty$ for all $f \in C^\infty(U^\chi)$. This will imply that $\phi f \in \mathscr{S}(N)$ for let $g : \mathscr{L} \times \mathscr{T} \to C$ be given by $g(l, t) = \phi f(\exp_N l \exp_N t)$. Let $f_0(l, t) = f(\exp_N l \exp_N t)$ and let $\phi_0(l, t) = \phi(\exp_N l \exp_N t)$. The mapping $(l, t) \to \log_N(\exp_N l \exp_N t)$ of $\mathscr{L} \times \mathscr{T}$ into N is a polynomial diffeomorphism with polynomial inverse. Hence $\phi f \in \mathscr{S}(N)$ iff $g \in \mathscr{S}(\mathscr{L} \times \mathscr{T})$. Let D be a polynomial differential operator on $\mathscr{L} \times \mathscr{T}$. Then from the product rule of differentiation $Dg = \sum D_i\phi_0 F_i f_0$ where D_i and F_i are polynomial differential operators on $\mathscr{L} \times \mathscr{T}$. We may assume that the D_i are constant coefficient operators for we may absorb the coefficient of D_i into F_i. Then $D_i\phi_0$ is independent of the $\mathscr{T}$ variable and is compactly supported in the l variable. Let $\phi_i(\exp_N l \exp_N t) = D_i\phi_0(l, t)$ and let $f_i(\exp_N l \exp_N t) = F_i f_0(l, t)$. There is a right, polynomial, differential operator $\tilde{F}_i$ such that $f_i = \tilde{F}_i f$. To show that ϕf is in $\mathscr{S}(N)$, it suffices to show that $\phi_i \tilde{F}_i f$ is bounded. We may write $\tilde{F}_i = \sum p_j X_i$ where p_j are polynomial functions on N and X_j are left invariant differential operators. $X_j f \in C^\infty(U^\chi)$ for all j so it suffices

to show that $\phi_i pf$ is bounded for all polynomial functions p and all $f \in C^\infty(U^\chi)$. To this end, note that p may be written $p(l, t) = \sum_{j=1}^n a_j(l)b_j(t)$ where a_j and b_j are polynomial functions.

If we replace ϕ_i by $\phi_i a_j$ and replace p by b_j, we see that it suffices to show that $\tilde{\phi} pf$ is bounded whenever $\tilde{\phi}$ is a C^∞ function on $N = LT$ which is compactly supported in l and independent of t and p is any polynomial in t and independent of l.

Thus we are reduced to proving the following

LEMMA 17. *If $f \in C^\infty(U^\chi)$, then*

$$\sup_{t \in T} |p(t)| \sup_{l \in L} |f(lt)| < \infty$$

for all polynomial functions p on T.

Proof. For $t \in T$, let f_t denote the function $l \to f(lt)$ on L. If $f \in C^\infty(U^\chi)$ then f_t is C^∞ on L and $|f_t|$ belongs to $L^2(K\backslash L)$ since $K\backslash L$ is compact. Hence f_t belongs to the representation space of $V^\chi = \text{ind}\,(K, L, \chi)$. f_t in fact belongs to $C^\infty(V^\chi)$ for $C^\infty(V^\chi)$ consists of all C^∞ functions h on L which satisfy $h(kl) = \chi(k)h(l)$ since such functions are automatically square integrable modulo K due to compactness. $C^\infty(V^\chi)$ has two natural topologies—the usual topology as a space of C^∞ vectors (see [9]) and the topology of uniform convergence of functions and their derivatives. By the closed graph theorem, these topologies agree on $C^\infty(V^\chi)$. In particular $p(t)f_t$ will be uniformly bounded on L in t if $p(t)f_t$ is bounded as a mapping of T into $C^\infty(V^\chi)$. The mapping $t \to f_t$ has a representation theoretic interpretation. Note that $U^\chi \approx \text{ind}\,(L, N, V^\chi)$ due to the theorem on inducing in stages. We may realize $\text{ind}\,(L, N, V^\chi)$ as a space of $\mathscr{H}(V^\chi)$ valued functions on T. Then $t \to f_t \in \mathscr{H}(\text{ind}\,(L, N, V^\chi))$ and the mapping $f \to f_t$ defines the intertwining operator. To finish the proof of theorem, we need to show:

LEMMA 18. *$C^\infty(\text{ind}\,(L, N, V^\chi))$ is $\mathscr{S}(T, C^\infty(V^\chi))$.*

Proof. By assumption $U^\chi \approx \text{ind}\,(L, N, V^\chi)$ is a finite direct sum of irreducible representations. Hence V^χ is a finite sum of irreducible representations (otherwise U^χ would have an infinite family of mutually orthogonal invariant subspaces). Let $V^\chi = \sum \oplus V_i$ where the V_i are irreducible and let $U_i = \text{ind}\,(L, N, V_i)$. Then $U = \sum \oplus U_i$ and $C^\infty(U) = \sum \oplus C^\infty(U_i)$. It suffices to show that $C^\infty(U_i) = \mathscr{S}(T, C^\infty(V_i))$. Of course, U_i is a finite sum of irreducible representations. However an irreducible representation of a connected subgroup of a nilpotent Lie group N either induces an irreducible

representation of N or induces a representation which decomposes with infinite multiplicity. Hence U_i is irreducible. Now we change realizations of U_i and V_i. There is a *connected* subgroup M of L and a character λ of M such that $V_i \approx \text{ind}(M, L, \lambda)$. Then $U_i \approx \text{ind}(M, N, \lambda)$. Let $\mathcal{M}$ be the Lie algebra of M in $\mathcal{N}$. Let $\mathcal{S}$ be a vector compliment of $\mathcal{M}$ in $\mathcal{L}$. Then $\mathcal{N} = \mathcal{M} \oplus \mathcal{S} \oplus \mathcal{T}$. Let $S = \exp_N \mathcal{S}$. Then $N = MST$. We define the Schwartz class $\mathcal{S}(ST)$ by setting $f \in \mathcal{S}(ST)$ iff the function $g(s, t) = f(\exp_N s \exp_N t)$ belongs to $\mathcal{S}(\mathcal{S} \times \mathcal{T})$. It follows from results of Corwin, Greenleaf and the author that $\text{ind}(M, N, \lambda)$ can be realized in $L^2(ST)$ and $C^\infty(\text{ind}\, M, N, \lambda) = \mathcal{S}(ST)$, [4]. For each $f \in \mathcal{S}(ST)$ and $t \in T$ let $f_t(s) = f(st)$. Then for each t, $f_t \in \mathcal{S}(S)$ and $t \to f_t$ defines a bounded element of $\mathcal{S}(T, \mathcal{S}(S))$. Now $V_i \approx \text{ind}(M, L, \lambda)$ can be realized in $L^2(S)$ and $C^\infty(V_i) \approx \mathcal{S}(S)$ by the results of [4] mentioned above. The representations $\text{ind}(L, N, \text{ind}(M, L, \lambda))$ and $\text{ind}(M, N, \lambda)$ are equivalent and the equivalence is given by mapping $f \to (t \to f_t)$ as defined above. Since $t \to f_t$ belongs to $\mathcal{S}(T, \mathcal{S}(S))$, it follows that $C^\infty(U_i) \approx C^\infty(\text{ind}(L, N, \text{ind}(M, L, \lambda))) \subset \mathcal{S}(T, \mathcal{S}(S))$. But $\mathcal{S}(S) = C^\infty(\text{ind}(M, L, \lambda)) \approx C^\infty(V^i)$. Hence $C^\infty(U_i) = \mathcal{S}(T, C^\infty(V^i))$. □

COROLLARY 19. *Let K be a closed subgroup of N and let χ be a character of K such that $\text{ind}(K, N, \chi) = U^\chi$ is a finite sum of irreducible representations. Then $C^{-\infty}(U^\chi)$ may be canonically identified with the space of distributions ψ on $\mathcal{S}(N)$ which satisfy $\psi(kx) = \chi(k)\psi(x)$ for all $k \in K$. ($\psi(kx)$ denotes left translation of distributions).*

Proof. Let ϕ be a C^∞ function of N such that $\int_K \phi(kx)dk = 1$ as in Lemma 16. Then $f \to \phi f$ defines a continuous mapping of $C^\infty(U^\chi)$ into $\mathcal{S}(N)$. Let ψ be a distribution on $\mathcal{S}(N)$ which satisfies the hypothesis of the theorem. Let $\tilde{\phi}(f) = \chi(\phi \bar{f})$. ($-$denotes complex conjugate. Recall C^∞ is the conjugate dual). Then $\tilde{\phi} \in C^{-\infty}(U^\chi)$. We claim that $\tilde{\psi}$ does not depend on the choice of ϕ. Let ϕ_0 be another such ϕ and let $f_0 = \phi_0 f - \phi f$. Then $\tau f_0 = 0$. We need to show that $\psi(\bar{f}_0) = 0$. To see this let $g \in \mathcal{S}(N)$. Let $L(x)g(y) = g(x^{-1}y)$. Let $\bar{f}_0 * g = \int_N \bar{f}_0(x) g dx$, interpreted as an $\mathcal{S}(N)$ valued integral. We shall show that this integral converges in $\mathcal{S}(N)$. Granting this for the moment we see

$$\psi(\bar{f}_0 * g) = \int_N \bar{f}_0(x)\psi(L(x)g)dx$$

$$= \int_{K\backslash N} \int_K \bar{f}_0(kx)\psi(L(kx)g)dkdx = \int_{K\backslash N} \tau f_0(x)^- \psi(L(x)g)dx = 0 .$$

Here we have used $\psi(L(kx)g) = \chi(k)\psi(L(x)g)$. Hence $\psi(\bar{f}_0 * g) = 0$ for all g. This implies $\psi(\bar{f}_0)=0$ as claimed. Convergence of the $\mathscr{S}(N)$ valued integral means precisely that for all $\psi \in \mathscr{S}'(N)$, $\bar{f}_0(x)\psi(L(x)g) \in L^1(N)$. (Recall that $\mathscr{S}(N)$ is reflexive and the integral is defined by reflexivity). This will follow from the following lemma.

LEMMA 20. *There is a polynomial function p on N depending on ψ and g such that $|\psi(L(x)g)| \leqq p(x)$ for all $x \in N$.*

Proof. As shown above $\mathscr{S}(N)$ can be characterized as the space of functions g such that Dg is bounded for all right polynomial differential operators D. For each such operator D let $\rho_D(g)=\|Dg\|_\infty$. The family $\{\rho_D\}$ defines the topology on $\mathscr{S}(N)$ so there are operators D_i such that

$$|\psi(g)| \leqq \sum \|D_i g\|_\infty$$

for all $g \in \mathscr{S}(N)$. We may take $D_i = p_i X_i$ where X_i is left invariant differential operator on N and p_i is a polynomial function on N. Then $|\psi(L(x)g)|$ is bounded by a sum of terms of the form $\|p_i L(x) X_i g\|_\infty = \|(L(x^{-1})p_i)X_i g\|_\infty$. There are polynomial functions q and r on N such that $p_i(xy) \leqq |q(x)||r(y)|$ for all x, y in N and all i. Hence $\| L(x^{-1})p_i X_i g\|_\infty \leqq |q(x)| \| rX_i g\|_\infty$. Thus, $|\psi(L(x)g)| \leqq \sum (\|rX_i g\|_\infty)|q(x)|$. □

Now we are ready to prove our first main theorem. We refer the reader to the material following Sublemma 10 for notation.

THEOREM 21. *Let ψ be a μ-primary function on N. Then $\mathscr{T}\psi$ is a C^∞ function on $\mathscr{M}_\mu$. In fact there is a function $f \in \mathscr{S}(\mathscr{B}^*)$ such that $\mathscr{T}\psi(\lambda) = \tau_\mu f(\lambda)$ for all $\lambda \in \mathscr{M}_\mu$. The image of $\mathscr{T}$ is precisely the image of τ_μ.*

Proof. Let ψ and $\mathscr{K}_2(\mu)$ and U_2 be as before.

LEMMA 22. *τ_μ is a surjective mapping of $\mathscr{S}(\mathscr{B}^*)$ onto $C^\infty(U_2)$.*

Proof. From Lemma 5, $\mathscr{M}_\mu$ is a finite union of sets of the form $\Gamma * \mathscr{M}(\lambda)$ for $\lambda \in \mathscr{M}_\mu$. $C^\infty(U_2)$ is the direct sum of $C^\infty(U(\lambda))$ where $U(\lambda)$ is (as before) the restriction of U_2 to the space of functions in $\mathscr{K}_2(\mu)$ supported in $\Gamma * \mathscr{M}(\lambda)$. $\Gamma \backslash \Gamma * \mathscr{M}(\lambda)$ is in turn isomorphic with $\Gamma_0 \backslash \mathscr{M}(\lambda)$ where $\Gamma_0 = \{\gamma \in \Gamma \mid \gamma \cdot \lambda | \mathscr{L}(\mathscr{B}) = 0\}$. Let $f \in \mathscr{S}(\mathscr{M}(\lambda))$ and let

$$\tau_0 f(\lambda) = \sum_{\Gamma_{\mathscr{L}} \backslash \Gamma_0} \bar{\mu}(\gamma) \exp -2\pi i \langle \gamma, \lambda \rangle f(\gamma * \lambda) .$$

It suffices, we claim, to show that τ_0 is a surjective mapping of $\mathscr{S}(\mathscr{M}(\lambda))$ onto $C^\infty(U(\lambda))$, considered as a space of functions on $\mathscr{M}(\lambda)$. In fact let $g \in C^\infty(U(\lambda))$, considered as a function on $\Gamma * \mathscr{M}(\lambda)$. Then, assuming that τ_0 is surjective, there is a function f in $\mathscr{S}(\mathscr{M}(\lambda))$ such that $\tau_0 f = g | \mathscr{M}(\lambda)$. There is a neighborhood U of $\mathscr{M}(\lambda)$ such that $U \cap \mathscr{M}(\lambda') = \phi$ if $\lambda' \in \mathscr{M}(\lambda)$. f may be extended to an element $f' \in \mathscr{S}(\mathscr{B}^*)$ which is supported in U. It is easily verified that $\tau_\mu f'$ is zero off of $\Gamma * \mathscr{M}(\lambda)$ in $\mathscr{M}_\mu$ and $\tau_\mu f' | \Gamma * \mathscr{M}(\lambda) = g$, proving our claim.

To show the surjectivity of τ_0 we trace through the identification of $U(\lambda)$ with $\operatorname{ind}(\Delta_\lambda, \chi_\lambda, \mathscr{B}) = U^\chi$ developed above. It will turn out that τ_0 is essentially the τ associated with U^χ. In fact, for $f \in \mathscr{S}(\mathscr{M}(\lambda))$ let $Tf: \mathscr{B} \to C$ be given by $Tf(x) = \exp 2\pi i \langle \bar{x}, \lambda \rangle f(\lambda * \bar{x})$. Let $\{x \in \mathscr{B} \mid \lambda * x = \lambda\} = (\Delta_\lambda)_0$. $(\Delta_\lambda)_0$ is in fact the identity component of Δ_λ. For $x \in (\Delta_\lambda)_0$ and $y \in \mathscr{B}$, $\lambda(y*x) = \lambda(y) + \lambda(x)$. It follows that

$$Tf(x*y) = \exp 2\pi_i \langle \bar{x}, \lambda \rangle Tf(y) = \chi_\lambda(x) Tf(y)$$

for $x \in (\Delta_\lambda)_0$. Since $(\Delta_\lambda)_0$ is connected, there is a Schwartz function g such that

$$Tf(x) = \int_{(\Delta_\lambda)_0} \chi_\lambda(k^{-1}) g(kx) dk \ .$$

We claim that $\tau g = T\tau_0 f$. In fact

$$\begin{aligned} \tau g &= \sum_{(\Delta_\lambda)_0 \backslash \Delta_\lambda} Tf(\delta x) \chi_\lambda(\bar{\delta}) \\ &= \sum \exp 2\pi i \langle \bar{x} * \bar{\delta}, \lambda \rangle \chi_\lambda(\bar{\delta}) f((\lambda * \bar{\delta}) * \bar{x})) \ . \end{aligned}$$

Let $\lambda * \bar{\delta} = \gamma(\bar{\delta}) * \lambda$ where $\gamma(\bar{\delta}) \in \Gamma_0$. $\gamma(\bar{\delta})$ is uniquely determined modulo $\Gamma_\mathscr{L}$ and the mapping $\delta \to \gamma(\bar{\delta})$ gives rise to a one-to-one correspondence between $\Gamma_\mathscr{L} \backslash \Gamma_0$ and $(\Delta_\lambda)_0 \backslash \Delta_\lambda$. Furthermore, in the proof of Theorem 12, it was shown that

$$\exp 2\pi_i \langle \bar{x} * \bar{\delta}, \lambda \rangle = \exp 2\pi i (\langle \gamma(\bar{\delta}), \lambda * \bar{x} \rangle + \langle \bar{x}, \lambda \rangle) \chi_\lambda(\delta) \ .$$

Hence

$$\tau g(x) = \exp 2\pi i \langle \bar{x}, \lambda \rangle \sum_{\Gamma_\mathscr{L} \backslash \Gamma_0} \exp 2\pi i \langle \gamma, \lambda * \bar{x} \rangle f(\gamma * \lambda * \bar{x}) = T\tau_0 f(x) \ .$$

It follows that the image of τ_0 is contained in $C^\infty(U(\lambda))$. Conversely, let g $\mathscr{S}(\mathscr{B})$. Let

$$g_0(x) = \int_{(\Delta_\lambda)_0} \chi_\lambda(k^{-1}) g(kx) dx$$

and let $f(\lambda * \bar{x}) = \exp -2\pi i \langle \bar{x}, \lambda \rangle g_0(\bar{x})$. On $(\Delta_\lambda)_0$, $\chi_\lambda(x) = \exp 2\pi i \langle x, \lambda \rangle$ so f is well defined. It is easily seen that $f \in \mathscr{S}(\mathscr{M}(\lambda))$ and as

above $T_\tau f = \tau g$. Hence τ_0 is onto $C^\infty(U(\chi))$ as claimed.

Now let η be a Schwartz distribution on $\mathscr{B}^*$ which satisfies $\eta(\gamma*\lambda) = \bar{\mu}(\gamma)\exp -2\pi i\langle\gamma, \lambda\rangle\eta(\lambda)$ for all $\gamma \in \Gamma$.

LEMMA 23. *There is an element $\eta' \in C^{-\infty}(U_2)$ such that $\eta(\bar{f}) = \eta'(\tau_\mu f)$ for all $f \in \mathscr{S}(\mathscr{B}^*)$.*

Proof. For all $\lambda \in \mathscr{M}_\mu$ let η_λ be the restriction of η to $\mathscr{M}(\lambda)$. By this we mean the following. For each $f \in \mathscr{S}(\mathscr{M}(\lambda))$ there is an element $\tilde{f} \in \mathscr{S}(\mathscr{B}^*)$ such that $\tilde{f}$ is zero on $\mathscr{M}_\mu \sim \mathscr{M}(\lambda)$. Let $\eta_\lambda(f) = \eta(\tilde{f})$. This is independent of the extension $\tilde{f}$ by Lemma 2 and its proof. We lift η_λ to a distribution $\eta_\mathscr{B}$ on $\mathscr{S}(\mathscr{B})$ by setting

$$\eta_\mathscr{B}(g) = \eta_\lambda(T^{-1}g_0)$$

where T is as above and

$$g_0(x) = \int_{(\Delta_\chi)_0} \chi_\lambda(k^{-1})g(kx)dk\ .$$

Then $\eta_\mathscr{B}(\delta x) = \chi_\lambda(\delta)\eta_\mathscr{B}(x)$ for $x \in \mathscr{B}$. By Corollary 19, there is a functional $\eta'_\mathscr{B}$ in $C^{-\infty}(U^\chi)$ such that $\eta_\mathscr{B}(\bar{g}) = \eta'_\mathscr{B}(\tau g)$. Let $\eta'_\chi(f) = \eta'_\mathscr{B}(Tf)$ for $f \in C^\infty(U(\lambda))$. Then, for $f \in \mathscr{S}(\mathscr{B}^*)$,

$$\begin{aligned}\eta'_\lambda(\tau_0 f) &= \eta'_\mathscr{B}(T\tau_0 f)\\ &= \eta'_\mathscr{B}(\tau g)\\ &= \eta_\mathscr{B}(g)\\ &= \eta_\lambda(f)\end{aligned}$$

where g is any element of $\mathscr{S}(\mathscr{B})$ such that

$$Tf(x) = \int_{(\Delta_\chi)_0} \chi_\lambda(k^{-1})g(kx)dk\ .$$

Now write $\mathscr{M}_\mu = \bigcap_{i=1}^n \Gamma*\mathscr{M}(\lambda_i)$. We use the isomorphism of U_2 with $\sum \oplus U(\lambda_i)$ to define an element η' of $C^{-\infty}(U_2)$ by setting $\eta' = \sum \eta_{\lambda_i}$. It is easily seen that $\eta'(\tau_\mu f) = \eta(\bar{f})$, as desired. □

It follows from Poulsen's theorem above that $(\mathscr{T}\psi)' \in C^\infty(U_2)$ for all μ-primary functions ψ since $\psi \to (\mathscr{T}\psi)'$ is a C^∞ intertwining operator.

LEMMA 24. *$\mathscr{T}\psi \in C^\infty(\mathscr{M}_\mu)$ and $(\mathscr{T}\psi)' = \mathscr{T}\psi|\mathscr{M}_\mu$.*

Proof. Let $(\mathscr{T}\psi)' = g \in C^\infty(U_2)$. Let $h \in C^\infty(\mathscr{M}_\mu)$ and suppose h has compact support. Since $\mathscr{M}_\mu$ is a countable union of affine subspaces, we may define Lebesgue measure on $\mathscr{M}_\mu$. Let dx denote

this measure. Let

$$\langle g, h\rangle = \int_{\mathscr{M}_\mu} \bar{h} g dx \ .$$

Then it can be seen that

$$\langle g, h\rangle = \int_{\Gamma\backslash\mathscr{M}_\mu} \left(\sum_{\Gamma_{\mathscr{L}}\backslash\Gamma} \bar{h}(\gamma * \lambda) g(\gamma * \lambda) \right) dv \ .$$

Using the invariance property of g this is seen to the $(g, \tau_\mu h)$ where $(\ ,\ )$ indicates the $\mathscr{H}_2(\mu)$ innerproduct. But by definition

$$(\tau_\mu h, g) = (\mathscr{T}\psi)'(\tau_\mu h) = \mathscr{T}\psi(\bar{h}) \ .$$

Hence $\mathscr{T}\psi$ is integration against g, as claimed. □

Next we prove surjectivity. Let $g \in C^\infty(U_2)$. Then $g \in C^\infty(\mathscr{M}_\mu)$ and g is bounded on $\mathscr{M}_\mu$ since g is in the image of τ_μ.

We consider g as a distribution on $\mathscr{B}^*$ given by integrating over $\mathscr{M}_\mu$ against gdx where dx is as above. As a distribution, gdx has an inverse Fourier transform, say $\mathscr{T}^{-1}(gdx) = \psi \in \mathscr{S}(\mathscr{B})$. It is easily verified that ψ satisfies $\psi(\gamma * x) = \mu(\gamma)\psi(x)$ for all $\gamma \in \Gamma$. Hence ψ may be considered as an element of $C^{-\infty}(U_1)$. Hence, as above, $\mathscr{T}^{-1}$ maps $C^\infty(U_2)$ into $C^\infty(U_1)$. $\mathscr{T}^{-1}$ is the inverse of $\mathscr{T}$ so $\mathscr{T}$ maps $C^\infty(U_1)$ one-to-one and onto $C^\infty(U_2)$. □

III. Examples.

EXAMPLE 1. In this section we describe a specific class of algebras from which our main examples come. Our class is motivated by Howe's work [6]. Let $\mathscr{I}$ be an abelian, nilpotent algebra which carries a nondegenerate symmetric form B which satisfies $B(xy, w) = B(y, xw)$. Let $\tau_0 : \mathscr{I} \to \mathscr{I}$ be a mollifier (see § I above). Let $\mathscr{A} = \mathscr{I} \times \mathscr{I}$ with the algebra structure $(x_1, x_2)\cdot(y_1, y_2) = (x_1y_1, x_1y_2)$. We shall adopt the notation that if $x \in \mathscr{A}$, then the ith component of x will always be denoted by x_i. On $\mathscr{A}$ let ϕ be the form

$$\phi(x, y) = B(x_1, y_2) - B(y_1, x_2) \ .$$

It is easily seen that ϕ satisfies the "twine" idenities:

$$\phi(xy, w) = \phi(y, xw) = \phi(x, [y, w]) \ .$$

ϕ is a nondegenerate, bi-linear form on $\mathscr{A}$ and $\mathscr{A}$ is a left communtative algebra. Let $\tau : \mathscr{A} \to \mathscr{A}$ be the form

$$\tau x = (\tau_0 x_1, (I - \tau_0)x_2) \ .$$

Let $\tau^\phi = I - \tau$.

Then τ is a left, $\mathscr{A}$-module homomorphism of $\mathscr{A}$ into $\mathscr{L}(\mathscr{A})$ which satisfies

$$\phi(\tau x, y) = \phi(x, \tau^\phi y)$$

for all $x, y \in \mathscr{A}$. Let $\alpha(x, y) = \phi(x, \tau y)$.

LEMMA 25. *Let* $\mathscr{B} = \mathscr{A} \times \boldsymbol{R}$ *with the product* $(x, s)(y, t) = (xy, \alpha(x, y))$. *Then* $\mathscr{B}$ *is a HAT algebra. Furthermore* $[(x, s), (y, t)] = ([x, y], \phi(x, y))$.

Proof. Associativity and left commutativity of $\mathscr{B}$ are equivalent to the identities

$$\alpha(xy, w) = \alpha(yx, w) = \alpha(x, yw) .$$

These follow easily from the twine identities and the properties of τ. The Lie bracket identity is equivalent with $\phi(x, y) = \alpha(x, y) - \alpha(y, x)$ which follows from the fact that $I - \tau$ is the adjoint of τ relative to ϕ. It is now easily seen that $Z(\mathscr{B}) = 0 \times \boldsymbol{R}$. □

We shall use similar direct product notation for elements of $\mathscr{B} = \mathscr{A} \times \boldsymbol{R}$ – e.g., if $x \in \mathscr{B}$, then $x = (x_1, x_2)$ where $x_1 \in \mathscr{A}$ and $x_2 \in \boldsymbol{R}$. Note that under our conventions $x_1 = ((x_1)_1, (x_1)_2)$ where $(x_1)_i \in \mathscr{S}$. We shall seldom use this notation however.

Now, suppose Λ is a vector lattice in $\mathscr{S}$ such that $\Lambda \cdot \Lambda \subset \Lambda$. Let us also suppose that $\tau_0 \colon \Lambda \to \Lambda$. Let $\Lambda^* = \{x \in \mathscr{S} \mid B(\Lambda, x) \subset Z\}$. Let $\Lambda_0 = \Lambda \times \Lambda^* \subset \mathscr{A}$ and let $\Gamma = \Lambda_0 \times \boldsymbol{R} \subset \mathscr{B}$. Then Γ is co-compact subgroup of $(\mathscr{B}, *)$. Let μ be the character of the subgroup Γ defined by $\mu(x) = \exp 2\pi i\, x_2$. μ is a character because $\alpha(\Lambda, \Lambda) \subset Z$.

LEMMA 26. $U^\mu = \operatorname{ind}(\Gamma, (\mathscr{B}, *), \mu)$ *is irreducible.*

Proof. We assume that the reader is familiar both with Kirillov theory [7] and the results of [5] or [10]. Let $\lambda \in \mathscr{B}^*$ be defined by $\lambda(x) = x_2$. Let $\mathscr{M} = (0 \times \mathscr{S}) \times \boldsymbol{R} \subset \mathscr{B}$. $\mathscr{M}$ is an ideal of $\mathscr{B}$ which satisfies $\mathscr{M}^2 = \{0\}$. It is easily seen that as a Lie algebra, $\mathscr{M}$ is an abelian, maximal subordinate subalgebra for λ and λ is integer valued on $\Gamma \cap \mathscr{M}$. The mapping $\exp_{\mathscr{B}}$ from $\mathscr{B}$ to $(\mathscr{B}, *)$ is $x \to x + x^2/2 + \cdots + x^n/n! + \cdots$ so on $\mathscr{M}$, $\exp_{\mathscr{B}}$ is the identity map. Hence the character corresponding to λ on $\mathscr{M}$ is

$$\chi(m) = \exp 2\pi i \langle m, \lambda \rangle .$$

From Kirillov theory $\operatorname{ind}(\mathscr{M}, (\mathscr{B}, *), \chi) = U^\chi$ is irreducible. Since

$(\mathscr{B}, *)$ has square integrable representations (Lemma 10), U^χ is determined by its character on $Z(\mathscr{B}) = 0 \times \boldsymbol{R}$. U^μ has the same character as U^χ on $Z(\mathscr{B})$ so U^μ is a direct sum of copies of U^χ. U^μ may in fact be identified with the U^χ primary summand of $\operatorname{ind}(\Gamma, (\mathscr{B}, *), 1)$. The multiplicity of U^χ in U^μ is given by formulas in [10]. Let $x \in \mathscr{B}$ and let $\chi_x(m) = \chi(x*m*\bar{x})$. Since $\mathscr{M}$ is normal in $(\mathscr{B}, *)$, χ_x is a character of $\mathscr{M}$. Let

$$X = \{x \in \mathscr{B} \mid \chi_x(\Gamma \cap \mathscr{M}) \equiv 1\} .$$

Then X is a finite union of $\Gamma*\mathscr{M}$ cosets and the multiplicity of U^χ in U^μ is the number of such cosets. However, note that $x*y*\bar{x} = x + \bar{x} + x\bar{x} + xy + y\bar{x} + xy\bar{x} = y + xy + y(\bar{x} + x\bar{x}) = y + xy - yx = y + [x, y]$. Hence $\chi_x(y) = \chi(y) \exp 2\pi i\ \phi(x_1, y_1)$. Thus $x \in X$ iff $\boldsymbol{Z} \supset \phi(x_1, \Lambda_0 \cap (0 \times \mathscr{S})) = B((x_1)_1, \Lambda^*)$. Hence $(x_1)_1 \in \Lambda$ so $X = \Gamma^*\mathscr{M}$. Hence the multiplicity is one so U^μ is irreducible.

Now to explicitly describe the Fourier transform, we use ϕ to identify $\mathscr{B}$ with $\mathscr{B}^*$. Specifically, for $y \in \mathscr{B}$ we let y^* denote the functional defined by

$$\langle x, y^* \rangle = \phi(x_1, y_1) + x_2 y_2 .$$

LEMMA 27. $x \cdot y^* = (x_1 y_1 - y_2 \tau^\phi x_1, 0)^*$

$$y^* \cdot x + ([x_1, y_1] + y_2 \tau x_1, 0)^* .$$

Proof.

$$\begin{aligned} \langle xw, y^* \rangle &= \phi(x_1 w_1, y_1) + \alpha(x_1, w_1) y_2 \\ &= \phi(w_1, x_1 y_1) - \phi(w_1, y_2 \tau^\phi x_1) \\ &= \langle w_1, (x_1 y_1 - y_2 \tau^\phi x_1, 0)^* \rangle . \end{aligned}$$

Hence $xy^* = (x_1 y_1 - y_2 \tau^\phi x_1, 0)^*$. Similarly

$$y^* \cdot x = ([x_1, y_1] + y_2 \tau x_1, 0)^* . \qquad \square$$

Now let $\lambda = (O, 1)^*$.

Since U^μ is irreducible, $\Gamma \backslash \mathscr{M}_\mu$ is connected so $\Gamma \backslash \mathscr{M}_\mu = \Gamma_0 \backslash \mathscr{M}(\lambda)$ where $\Gamma_0 = \{\gamma \in \Gamma \mid \gamma \cdot \lambda \mid \mathscr{L}(\mathscr{B}) = 0\}$. From Lemma 3, $\mathscr{M}(\lambda) = \lambda^* \mathscr{B} = \{(\tau x, 1)^* \mid x \in \mathscr{A}\}$. Recall $\Delta_\lambda = \{\delta \in \mathscr{B} \mid \lambda \cdot \delta = \gamma \cdot \lambda \text{ for some } \gamma \in \Gamma\}$. Hence $\Delta_\lambda = \tau^{-1}(\tau^\phi(\Lambda \times \Lambda^*)) \times \boldsymbol{R}$. For $\delta \in \Delta_\lambda$, let $\gamma_1(\delta)$ be any element of $(\tau^\phi)^{-1}(\tau(\delta_1)) \cap (\Lambda \times \Lambda^*)$ and let $\gamma(\delta) = \gamma_1(\delta) \times 0$. The character χ_λ of Δ_λ is described by

$$\begin{aligned} \chi_\lambda(\delta) &= \mu(\gamma(\delta)) \exp 2\pi i \langle (\delta - \gamma(\delta)), \lambda \rangle \\ &= \exp 2\pi i \delta_2 . \end{aligned}$$

Let $\Omega: \mathscr{H}(U^\mu) \to \mathscr{H}(U^{\chi_\lambda})$ be an intertwining operator. The Fourier transform $\mathscr{T}$ of a μ-primary function g is described by

$$\mathscr{T} g(\lambda * x) = C \exp -2\pi i \langle x, \lambda \rangle \Omega f(\bar{x})$$

from Theorem 21. Let $\tilde{\tau}: \tau(\mathscr{A}) \to \mathscr{A}$ be a linear mapping such that $\tau\tilde{\tau}(x) = x$ for all $x \in \tau(\mathscr{A})$. Then $(x, 1)^* = \lambda^*(\tilde{\tau}x, 0)$ so this formula may be written as

$$\mathscr{T} g((x, 1)^*) = C\Omega g((\tilde{\tau}x, 0)^-) .$$

The special case of $\tau_0 = 0$ should be mentioned. In this case Δ_λ is the *connected* subgroup $\mathscr{P}_0 = (\mathscr{S} \times 0) \times \boldsymbol{R}$, and $\chi_\lambda(p) = \exp 2\pi i p_2$. The intertwining operator from U^μ to $\operatorname{ind} \chi^\lambda$ is known by [4]. It is, for $f \in C^\infty(U^\mu)$, $Tf(x) = \int_{\Gamma \subset \mathscr{S} \setminus \mathscr{S}_0} \chi_\lambda(p)^- f(p * x) dp$. This then describes the Fouries transform. Note that in this case $\tau(\mathscr{A}) = 0 \times \mathscr{S}$ and $\tilde{\tau}$ can be chosen to be the identity map. Furthermore $((0, x), 0)^- = ((0, -x), 0)$. Hence

$$\mathscr{T} f(((0, x), 1)^*) = \int_{\mathscr{S}_0 \cap \Gamma \setminus \mathscr{S}_p} f(p^* 0), (((0, -\bar{x}), 0))\chi_\lambda .$$

Note also that if we identify $\mathscr{B}$ and $\mathscr{B}^*$ in the more "conventional" manner by setting

$$\langle (x, t), (y, x) \rangle = B(x_1, t_1) + B(x_2, y_2) + ts .$$

Then $((0, x), 1)^* = ((-x, 0), 1)$ so we would write

$$\mathscr{T} f(((x, 0), 1)) = \int_{\mathscr{S}_0 \cap \Gamma \setminus \mathscr{S}_0} f(p^*((0, x), 1))\chi_\lambda(p)^- dp .$$

The "flip" from $(x, 0)$ to $(0, x)$ seems very interesting.

Now let us discuss theta functions. Let $\sigma \in \boldsymbol{R}$ and let $\mathscr{P}_\sigma = \{((s, \sigma s), t) \mid s \in \mathscr{A}, t \in \boldsymbol{R}\}$. Then $\mathscr{P}_\sigma$ is an abelian subalgebra of $\mathscr{B}$. Let χ_σ be the character of $\mathscr{P}_\sigma$ given by $\chi_\sigma(p) = \exp 2\pi i \lambda(\log_{\mathscr{B}} p)$.

We shall say that a Schwartz distribution θ_σ on $\mathscr{B}$ is a μ-theta-distribution if

(a) $\theta_\sigma(\gamma * x) = \mu(\gamma)\theta_\sigma(x)$ for all $\gamma \in \Gamma$

(b) $\theta_\sigma(x * p) = \theta_\sigma(x)\chi_\sigma(p)$ for all $p \in \mathscr{P}_\sigma$.

The number σ is called the period of θ_σ. Note that (b) can be written

$$\theta_\sigma(x * \exp_{\mathscr{B}} p) = \exp 2\pi i \lambda(p)\theta_\sigma(x) .$$

Differentiating this equality we see that

$$P\theta_\sigma = 2\pi i \lambda(P)\theta_\sigma$$

for all left invariant differential operators P which pass through $\mathscr{P}_\sigma$.

Note that from Lemma 19, θ_σ gives rise to an element ω_σ of

$C^{-\infty}(U^\mu)$ which satisfies $(U^\mu)^{-\infty}(p)\omega_\sigma = \chi_\sigma(p)\omega_\sigma$. Conversely, every such ω_σ gives rise to a unique μ-theta distribution θ_σ.

LEMMA 28. *For each $\sigma \in \boldsymbol{R}$ there is an essentially unique (up to scalar multiples) theta distribution θ_σ.*

Proof. Let χ_σ be as above. $\mathscr{P}_\sigma$ is a maximal subordinate subalgebra of $\mathscr{B}$ relative to λ so the representation $U^\sigma = \operatorname{ind}(\mathscr{P}_\sigma, (\mathscr{B}, *), \chi_\sigma)$ is irreducible and equivalent with U^μ. Now, from Cartier's C^∞-Frobenius reciprocity theorem, the number of times U^μ occurs in U^σ is the dimension of the space of elements $\omega \in C^{-\infty}(U^\mu)$ which satisfy $U^\mu(p)\omega = \chi_\sigma(p)\omega$. ([3], theorem). Since the multiplicity is one, we're done. □

We would like to explicitly describe θ_σ. To this end let $\mathscr{M}$ and χ be as in the proof of Lemma 26 above. Let $U^\chi = \operatorname{ind}(\mathscr{M}, (\mathscr{B}, *), \lambda)$. It is a simple matter to construct an element ω of $C^{-\infty}(U^\chi)$ such that $U^\chi(p)\omega = \chi_\sigma(p)\omega$. Once we have constructed such a ω_σ we shall construct an intertwining operator T from U^χ to U^μ. Then the image of ω under T will be ω_σ. To construct ω, recall that $C^{-\infty}(U^\chi)$ can be cannonically identified with the space of Schwartz distributions ψ which satisfy

$$\psi(m*x) = \chi(m)\psi(x) \quad \text{for all } m \in \mathscr{M} .$$

Let ω be the distribution defined by the function ω defined by

$$\omega(m*p) = \chi(m)\chi_\sigma(p) \qquad m \in \mathscr{M},\ p \in \mathscr{P}_\sigma .$$

Note that $\mathscr{M}*\mathscr{P}_\sigma = \mathscr{B}$ and on $\mathscr{M} \cap \mathscr{P}_\sigma$, χ and χ_σ agree so this does define a function ω.

Next we need to describe T. This is easy due to results [16] and [4]. For $f \in C^\infty(U^\chi)$, the function $\gamma \to f(\gamma * x)$ is constant on cosets of $\Gamma \cap \mathscr{M}$ in Γ for each s. Let

$$Tf = \sum_{\Gamma \cap \mathscr{M} \backslash \Gamma} f(\gamma * x) .$$

Then by [4], this sum converges uniformly on compact subsets to an element of $C^\infty(U^\mu)$. T is the desired intertwining operator. The image of ω under T (i.e., $\omega \circ T^*$) is then the element of $C^{-\infty}(U^\mu)$ defined by the distribution

$$\theta_\sigma(x) = \sum_{\Gamma \cap \mathscr{M} \backslash \Gamma} \omega(\gamma * x) .$$

This sum converges in the sense of distributions.

Next we would like to compute θ_σ explicitly in terms of the

"coordinates" for $\mathscr{B}$. Since $\theta_\sigma((w, t)) = \exp 2\pi i t\theta_\sigma((w, 0))$, it suffices to compute $\theta_\sigma(w, 0)$. First we must compute χ_σ which means computing χ_σ. χ_σ involves $\log_{\mathscr{B}}$ so we compute $\log_{\mathscr{B}}(w, 0)$ first. We begin by noting that $\alpha((x, y), (x, y')) = \phi((x, y), \tau(x', y')) = \phi((x, y), (\tau_0 x'(I - \tau_0)y')) = B(x, (I - \tau_0)y') - B(y, \tau_0 x')$. Now by induction it is seen that for $n \geqq 2$,

$$\begin{aligned}((x, y), 0)^n &= ((x^n, x^{n-1}y), \alpha((x, y), (x^{n-1}, x^{n-2}y))) \\ &= ((x^n, x^{n-1}y), B(x, (I - \tau_0)x^{n-2}y) - B(y, \tau_0 x^{n-1}))\ .\end{aligned}$$

If $n = 2$ this says

$$((x, y), 0)^2 = ((x^2, xy), -2B(\tau_0 x, y) + B(x, y))\ .$$

If $n > 2$, then $\tau_0 x^{n-1} = 0$ since $\tau_0 \mathscr{S}^2 = 0$. Thus $((x, y), 0)^n = ((x^n, x^{n-1}y), B(x, x^{n-2}))$. Now suppose $y = \sigma x$. Let

$$l(x) = \sum_{n=2}^{\infty} (-1)^{n+1} B(x, x^{n-1})/n\ .$$

The following lemma is easily proven from the above comments.

LEMMA 29.

$$\log_{\mathscr{B}}((x, \sigma x), 0) = ((\log_{\mathscr{S}} x,\ 6 \log_{\mathscr{Z}} x),\ \sigma B(\tau_0 x, x) + \sigma l(x))\ .$$

Hence $\chi_\sigma((x, \sigma x), 0) = \exp 2\pi i\sigma(B(\tau_0 x, x) + l(x))$. Now let $((x, y), 0) \in \mathscr{B}$. Then $((x, y), 0) = ((0, y - \sigma x), -\alpha((0, y - \sigma x), (x, \sigma x)))^*((x, \sigma x), 0) = ((0, y - \sigma x), B(y - \sigma x, \tau_0 x))^*((x, \sigma x))$. Hence

$$\begin{aligned}\omega((x, y), 0) &= \exp 2\pi i(B(y - \sigma x, \tau_0 x) + \sigma B(\tau_0 x, x) + \sigma l(x)) \\ &= \exp 2\pi i(B(y, \tau_0 x) + \sigma l(x))\ .\end{aligned}$$

Now the set $\Lambda_1 = \{((n, 0), 0) \mid n \in \Lambda\}$ is a complete set of inequivalent representatives of the $\Gamma \cap \mathscr{M}$ cosets in Γ. Given $w = ((x, y), 0)$ in $\mathscr{B}$ and $\gamma = ((n, 0), 0)$ in Λ_1,

$$\gamma * w = ((n * x, y + ny), B(n, (I - \tau_0)y))\ .$$

Now, note that $\tau_0(n * x) = \tau_0(n + x)$ since $\tau_0 \mathscr{S}^2 = 0$. Hence,

$$\begin{aligned}\mathscr{B}(y + ny, \tau_0(n * x)) &= \mathscr{B}(y + ny, \tau_0(n + x)) \\ &= \mathscr{B}(\tau_0(y + ny),\ n + x) = \mathscr{B}(\tau_0 y, n + x)\ .\end{aligned}$$

Hence

$$\begin{aligned}\omega(\gamma * w) &= \exp 2\pi i((\tau_0 y, x + n) + \sigma l(n * x) + B(n, (I - \tau_0)y)) \\ &= \exp 2\pi i(B(\tau_0 y, x) + B(n, y) + \sigma l(n * x))\ .\end{aligned}$$

Finally, then

$$\theta_\sigma(w) = \exp 2\pi i B(\tau_0 y, x) \sum_{n \in \Lambda} \exp 2\pi i (B(n, y) + \sigma l(n*x)) .$$

We simplify this by means of the following lemma which follows easily from Lemma 29 above with $\tau_0 = 0$.

LEMMA 30. *For all* $x, y \in \mathscr{S}$, $l(x^*y) = l(x) + l(y) - B(x, y)$. It *follows from this lemma that*

$$\theta_\sigma(w) = \exp 2\pi i (B(\tau_0 y, x) + l(x)) \sum \exp 2\pi i (B(n, y - \sigma x) + \sigma l(n)) .$$

This is exactly the mollified theta series of § I.

To prove the theta identity we re-compute θ_σ in the $\tau_0 = 0$ case. We follow the same procedure as above except that we use the subgroup $\mathscr{P}_0$ and the character χ_0 instead of the pair $(\mathscr{M}, \chi)$. Let $\omega_0(p_0*p_1) = \chi_0(p_0)\chi_\sigma(p_1)$ for $p_0 \in \mathscr{P}_0$, $p_1 \in \mathscr{P}_\sigma$. Then from the uniqueness of θ_σ

$$\theta_\sigma(w) = K(\sigma) \sum_{\mathscr{P}_0 \cap I \backslash \Gamma} \omega_0(\gamma * w) .$$

To explicitly compute ω_0, observe that $((x, y), 0) = ((u, 0), 0)*((v, \sigma v), t)$ where $u = x - \sigma^{-1}y$, $v = x*\bar{u}$ and $t = -\sigma B(u, v)$. Then $\omega_0(w) = \exp 2\pi i\sigma(l(v) - B(u, v))$. We simplify using the above lemma.

$$l(v) = l(x) + l(\bar{u}) - B(x, \bar{u}) .$$
$$0 = l(u*\bar{u}) = l(u) + l(\bar{u}) - B(u, \bar{u}) .$$

$B(u, v) = B(u, x + x\bar{u} + \bar{u}) = B(u + u\bar{u}, x) + B(u, \bar{u}) = B(\bar{u}, x) + B(u, \bar{u})$. Hence

$$\begin{aligned} \omega_0(w) &= \exp 2\pi i\sigma(l(x) - l(u)) \\ &= \exp 2\pi i\sigma(l(x) - l(x - \sigma^{-1}y)) . \end{aligned}$$

Hence we see that

$$\theta_\sigma(w, 0) = K(\sigma) \exp 2\pi i\sigma l(x) \sum_{n \in \Lambda} \exp 2\pi i\sigma l(\sigma^{-1}(\sigma x - y + n)) .$$

Equating the above expression of θ_σ with our first expression and replacing $\sigma x - y$ by x we see:

THEOREM 31.

$$\begin{aligned} &\sum \exp 2\pi i(\sigma l(n) + B(x, n)) \\ &\quad = \sum \exp 2\pi i\sigma l(\sigma^{-1}(x + n)) . \end{aligned}$$

This is the desired theta identity.

REMARK. It was originally our hope that one could use the

generalized theta series to compute sums of the form $\sum_{n=0}^{p-1} \exp 2\pi i n^k/p$ where p is some integer. Formulas for such sums are known classically only for $0 \leqq k \leqq 2$. The relevance of the theta series is as follows. Let $\mathscr{J}_Q = \operatorname{span}_Q \Lambda$. Let $x \in \mathscr{J}_Q$ and let $\sigma \in Q$. It is easily verified that there is a sub-lattice $\Lambda_0 \subset \Lambda$ such that the functions $\exp 2\pi i B(n, x) + \sigma l(n)$ and $\exp 2\pi i \sigma l(\sigma^{-1}(x + n))$ are both constant on coset of Λ_0 in Λ as functions of n. Let:

$$\theta_\sigma^Q(x) = [\Lambda_0 : \Lambda]^{-1} \sum_{\Lambda_0 \backslash \Lambda} \exp 2\pi i (B(n, x) + \sigma l(n))$$

$$\psi_\sigma^Q(x) = [\Lambda_0 : \Lambda]^{-1} \sum_{\Lambda_0 \backslash \Lambda} \exp 2\pi i \sigma l(\sigma^{-1}(x + n)) \, .$$

The function θ_σ^Q represents, in some very real sense, the "values" of the distribution θ_σ at the elements of $\mathscr{J}_Q$. It seems reasonable to conjecture that $\theta^Q = K(\sigma)\psi_\sigma^Q$. Granted this, let σ be an integer chosen so that $\sigma l(n) \in \boldsymbol{Z}$ for all $n \in \Lambda$. It is easy to see that such σ exist. Let $x = 0$. Then $\theta_\sigma^Q(x) = 1$ while $\psi_\sigma^Q(x) = \sum \exp 2\pi\sigma l(\sigma^{-1}n)$. The theta identity says $\psi_\sigma^Q(0) = K(\sigma)^{-1}$. Of course $\sigma l(\sigma^{-1}n)$ can have arbitrarily large degree so we can compute some greater than quadratic sums provided we can compute $K(\sigma)$. $K(\sigma)$ should be computable as follows.

For ρ arbitrary, let $U^\rho = \operatorname{ind}(\mathscr{P}_\rho, (\mathscr{B}, *), \chi_\rho)$. Also set $U^\chi = \operatorname{ind}(\mathscr{M}, (\mathscr{B}, *), \chi)$. Let $T^{\rho,0}$ be the operator from $\mathscr{H}(U^\rho)$ to $\mathscr{H}(U^0)$ defined by $T^{\rho,0}f(x) = \int_{\mathscr{P}_\rho \cap \mathscr{P}_0 \backslash \mathscr{P}_0} f(p*x)\bar{\chi}_0(p)dp$. It is easily verified that $T^{\rho,0}$ defines an intertwining operator U^ρ to U^0. Let $T^{\rho,\chi}: U^\rho \to U^\pi$ be defined similarly using $(\mathscr{M}, \chi)$ in place of $(\mathscr{P}_0, \chi_0)$. $T^{\rho,0}$ and $T^{\rho,\chi}$ may be extended to intertwining operators on the $C^{-\infty}$ spaces of the corresponding representations. Let $\delta \in C^{-\infty}(U)$ be defined by $\delta(f) = f(0)$.

Note that $\delta(U^\sigma(p)f) = f(p) = \chi_\sigma(p)f(0) = \chi_\sigma(p)\delta(f)$ for $p \in \mathscr{P}_\sigma$. It is easily verified that $T^{\sigma,0}(\delta) = \omega_0$ and $T^{\sigma,\chi}(\delta) = \omega$. Recall ω_0 is defined by $\omega_0(p_0 * p_\sigma) = \chi_0(p_0)\chi_\sigma(p_\sigma)$ and $\omega(m * p_\sigma) = \chi(m)\chi_\sigma(p_\sigma)$ for $p_0 \in \mathscr{P}_0$, $p_\sigma \in \mathscr{P}_\sigma$ and $m \in \mathscr{M}$. Let $T^{0,\chi}$ and $T^{\chi,\lambda}$ be the respective intertwining operators from U^0 and U^χ to $U^\lambda = \operatorname{ind}(\Gamma Z(\mathscr{B}, *), \lambda)$ used in the computation of θ_σ above. Then $\theta_\sigma = T^{\chi,\lambda}\omega = K(\sigma)T^{0,\chi}\omega_0$. Hence $T^{\chi,\lambda}T^{\sigma,\gamma}\delta = K(\sigma)T^{0,\lambda}T^{\sigma,0}\delta$. From the uniqueness of intertwining operators from this implies $T^{\chi,\lambda}T^{\sigma,\chi} = K(\sigma)T^{0,\lambda}T^{\sigma,0}$. However

$$\begin{aligned} T^{\chi,\lambda}T^{\sigma,\chi} &= T^{\chi,\lambda}T^{0,\chi}(T^{0,\chi})^{-1}T^{\sigma,\lambda} \\ &= CT^{0,\chi}(T^{0,\chi})^{-1}T^{\sigma,\chi} \end{aligned}$$

where C is a constant independent of σ.

It follows that

$$C^{-1}K(\sigma)T^\sigma = (T^{0,\chi})^{-1}T^{\sigma,\chi}$$

i.e., up to constant multiple $K(\sigma)$ is the intertwining "constant" of

$T^{\sigma,0}$, $T^{0,\chi}$ and $T^{\sigma,\chi}$. Note that this constant doesn't depend on Γ at all. It should be expressible as an integral of some sort. Presuming that the intertwining constant can be computed we can evaluate C by setting $\sigma = 1$, $x = 0$ and using the theta identity which says $1 = K(1)\psi_1^0(0) = K(1)\sum \exp 2\pi i l(n)$. This computes $K(1)$ from which C can be found.

We have not been anxious to carry out the above procedure for the following reason. In every example we have tried to compute, so far, we only obtain information concerning the linear or quadratic sums, regardless of the nilpotent degree of $\mathcal{S}$. This is probably due to the tendency of nilpotent analysis to reduce to the two step case. The formulas for $K(\sigma)$ should still be computed, as the expressions for $\psi_\sigma(0)$ in terms of Gauss sums are complicated and a formula for them might express interesting number theoretic information. It is conceivable also that by exploiting a more general class of algebras one could obtain more information. For example, one might try making $\mathcal{S}$ non-abelian or using solvable groups instead of nilpotent groups.

Next we compute the Fourier transform of the mollified theta series. We shall do this in general only in the case that the mollifier τ_0 is trivial. However, we shall demonstrate the effect of the mollifier by considering two special cases. One case is the Heisenberg group and the other is a case that $\mathcal{S}$ is cyclic (generated by one generator). The effect of the mollifier is quite interesting. The effect is to increase the singular support of the Fourier transform. At first glance it may seem undesirable to increase the singular support. However, recall that the Fourier transform of a C^∞ function is a C^∞ function times Haar measure on the singular support, and is zero off of the singular support. Hence the larger the singular support, the *more* the Fourier transform looks like a C^∞ function.

Another curious fact concerning the effect of the mollifier on the Fourier transform is that the mollifier does *not* alter the isomorphism class of the Lie algebra. In fact a mollified algebra gives rise to precisely the *same* Lie algebra as the corresponding unmollified algebra. Despite this the Fourier transforms of the same function can be quite different, depending on the particular algebra chosen to represent the Lie algebra. This occurs even in the three dimensional case.

To describe the Fourier transform of the mollified theta series in the $\tau_0 = 0$ case, let ω_0 be as before:

$$\omega_0(p_0 * p_\sigma) = \chi_0(p_0)\chi_\sigma(p_\sigma) .$$

From the above computations $\mathcal{F}\theta_{\sigma,0}$ may be identified canonically

with an element of $C^{-\infty}(U^{\chi_\lambda})$ where $U^{\chi_\lambda} = \text{ind}\,(\Delta_\lambda, (B, *), \chi_\lambda)$. In the $\tau_0 \equiv 0$ case, $\Delta_\lambda = \mathscr{P}_0$, $\chi_\lambda = \chi_0$. The uniqueness of $\theta_{\sigma,0}$ implies the following.

THEOREM 32. *There is a constant $C(\sigma)$ such that $\mathscr{T}\theta_{\sigma,0} = C(\sigma)\omega_0$.*

Thus, except for computing $C(\sigma)$, the Fourier transform of $\theta_{\sigma,0}$ is automatic. We shall not attempt to compute $C(\sigma)$ as we have no need for this information at the moment.

In the mollified cases, things become more complicated.

EXAMPLE II. Let $\mathscr{S} = \boldsymbol{R}$ with trivial multiplication. Let $\Lambda = \boldsymbol{Z}$ and let $B(x, y) = xy$. Let $\tau_0 : \mathscr{S} \to \mathscr{S}$ be the map $\tau_0(x) = (1/2)x$. τ_0 is a mollifier. The mapping $\tau : \mathscr{S} \times \mathscr{S} \times \mathscr{S}$ is given by $\tau(x, y) = 1/2(x, y)$. Hence $\Delta_\lambda = \tau^{-1}(\tau^\phi(\Lambda \times \Lambda)) \times \boldsymbol{R} = \boldsymbol{Z}^2 \times \boldsymbol{R} = \Gamma$. It is easily checked that $\chi_\lambda = \mu$. Hence, in this case, the Fourier transform is a scalar multiple of the identity map. Note, however that we are using an unconventional identification of $\mathscr{B}$ with $\mathscr{B}^*$. In more conventional terms $\mathscr{T}\psi$ is supported in $(\boldsymbol{R} \times \boldsymbol{R}) \times \boldsymbol{Z}$ and $\mathscr{T}\psi((x, y), n) = C\psi(\Gamma((-y, x), 0)$.

EXAMPLE III. Let $\mathscr{S}$ be generated by a single generator e subject to the relation $e^3 = 0$. Let B be defined by

$$\begin{aligned} B(e^i, e^j) &= 1 \qquad \text{if } i + j = 3 \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

We extend B bi-linearly to $\mathscr{S} \times \mathscr{S}$. It is easily verified that $B(xy, w) = B(x, yw)$ for all x, y, w. Let $\tau_0 : \mathscr{S} \times \mathscr{S}$ be multiplication by e. τ_0 is a mollifier. Let $\Lambda \subset \mathscr{S}$ be the span of $\{e^1, e^2\}$ over $\boldsymbol{Z}$. It is easily computed (from $\tau^{-1}(\tau^\phi(\Lambda \times \Lambda))$) that $\Delta_\lambda = \{((x, ne^2), t) | x \in \mathscr{S}, m \in \boldsymbol{Z}, t \in \boldsymbol{R}\}$ and χ_λ is the map $((x, ne^2), t) \to \exp 2\pi i t$. An intertwining operator from U_0 to U^{χ_λ} is given by

$$\tau f(x) = \sum_n f((0, ne^2), 0^* x) .$$

From the uniqueness of θ_{σ,τ_0}, it follows that $\mathscr{T}\theta_{\sigma,\tau_0}(w) = \omega_0((0, ne^2), 0)^*(w, 0)$. Letting $w = (x, y)$ and computing we see

$$\begin{aligned} \mathscr{T}\theta_{\sigma,\tau_0}(w) &= C(\sigma) \sum \omega_0(w + ne^2) \\ &= c(\sigma) \sum \omega_0(x, y + ne^2) . \end{aligned}$$

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