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ON HEREDITARY RINGS AND NOETHERIAN V-RINGS

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The purpose of this paper is to examine conditions under which (1) a left noetherian left V-ring is left hereditary and (2) a left noetherian left V-ring is a two sided noetherian V-ring. For (1), left noetherian left V-rings which satisfy the restricted left minimum (RLM) condition are examined. The RLM condition is shown to be equivalent to $E(R)/R$ a semisimple left R-module. Consequently, hereditary is equivalent to $E(R)/\mathrm{R}$ semisimple in the two sided case. Two sided noetherian V-rings which are critically nice are also examined. In this case, hereditary is shown to be equivalent to $E(R)/R$ injective and smooth. For (2), a theorem of Faith's concerning left QI-domains is extended to left noetherian left V-rings.

1. Introduction and definitions. A ring R is called a *left V-ring* provided every simple left R-module is injective. The definition of V-ring is due to Villamayor who has shown that a ring is a left V-ring if and only if every left ideal is the intersection of maximal left ideals. Consequently, all left V-rings are semiprime. Kaplansky has shown that a commutative ring is a V-ring if and only if it is regular. It follows that every commutative noetherian V-ring is semisimple artinian. Cozzens [4] showed that this result does not extend to the noncommutative case by producing an example of a nonartinian, two sided hereditary noetherian V-domain over which all cyclic modules are semisimple or free. This condition on cyclics forces every quasi-injective module to be injective. A ring with all its quasi-injective left R-modules injective will be called a *left QI-ring.* According to Boyle [1], a left QI-ring is left noetherian. Note that since a simple module is quasi-injective, a left QI-ring is a left V-ring.

As with Cozzens' example, all the known examples of left QI-rings are left hereditary. Cozzens and Johnson [5] produced examples of two sided noetherian V-rings which Boyle and Goodearl [3] demonstrated to be neither hereditary nor QI. Also, there is no known example of a one sided noetherian V-ring or QI-ring. In this paper, we will consider the problem of determining when a left noetherian left V-ring is left hereditary and when a left V-ring is a right V-ring.

Throughout, all rings will be associative with identity, all R-modules will be unitary left R-modules and maps between modules will be R-homomorphisms. If N is a submodule of a module M, then we will write $N \leqq M$. In case $N \cap K \neq 0$ for all $0 \neq K \leqq M$, then N is called *essential* in M and we will write $N \leqq_e M$. For a module M, $E(M)$, $\operatorname{Soc} M$ and $K \dim M$ will denote the injective hull, socle and Krull dimension of M respectively. It is assumed that the reader is familiar with the notions of singular, nonsingular and uniform modules as presented in [9]. We also use the notions of Krull dimension, critical module and smooth module as given in [11]. Throughout this paper, whenever we use the terms hereditary, noetherian, V-ring or QI-ring unqualified by "left" or "right", this will mean that the term applies to both the left and right.

In § 2, left noetherian left V-rings which satisfy the *restricted left minimum* (RLM) condition are examined. A module M satisfies the RLM condition provided M/K is artinian whenever $K \leqq_e M$. It is shown that the RLM condition is equivalent to $E(R)/R$ semisimple. As a consequence, hereditary is equivalent to $E(R)/R$ semisimple in the two sided case.

The purpose of § 3 is to further investigate the role $E(R)/R$ plays in determining when a noetherian V-ring is hereditary. A necessary condition for hereditary is that R be *critically nice* (all finitely generated uniform modules are critical). In this case, R is hereditary iff $E(R)/R$ is injective and smooth.

In § 4, left-right symmetry is examined. A theorem of Faith's which states that a left QI-domain with the RLM condition is right QI iff it is right Goldie is extended to left noetherian left V-rings.

2. The restricted left minimum condition.

A module M is said to satisfy the *restricted left minimum* condition, denoted RLM, provided M/K is artinian for all $K \leqq_e M$. A ring R is said to satisfy the RLM condition provided the left R-module R satisfies the RLM condition. In this section, we investigate left noetherian left V-rings which satisfy the RLM condition.

LEMMA 2.1. *Let R be a semiprime ring with Krull dimension. Then R satisfies the RLM condition iff $K \dim R \leqq 1$.*

Proof. By Gordon and Robson [11; 6.1], $K \dim R = \sup \{K \dim R/I + 1 \mid I \leqq_e R\}$. If R satisfies the RLM condition, then $K \dim R/I \leqq 0$ for all essential left ideals I. Thus, $K \dim R \leqq 1$. Conversely, if $K \dim R \leqq 1$, then $K \dim R/I \leqq 0$ for all $I \leqq_e R$. Hence, R/I is artinian for all essential left ideals I.

The RLM condition has been shown by Faith [7] to be sufficient for a left QI ring to be left hereditary. Michler and Villamayor [12] have shown that $K \dim R \leqq 1$ is sufficient for a left noetherian left V-ring R to be left hereditary. Therefore, if all cyclic singular left R-modules are semisimple, then by 2.1, $K\dim R \leqq 1$ and R is left hereditary. As the next result shows, $K\dim R \leqq 1$ and the RLM condition on R are equivalent to all singular (cyclic) left R-modules semisimple.

THEOREM 2.2. *If R is a left noetherian left V-ring, then the following are equivalent*:
(1) *R satisfies the RLM condition.*
(2) $K\dim R \leqq 1$.
(3) *All singular left R-modules are semisimple.*
Furthermore, if (1)-(3) *hold, then R is left hereditary.*

Proof. (1) implies (3). Let M be a singular left R-module and let $0 \neq x \in M$. Then $Rx \cong R/I$ where $I \leqq_e R$ and hence, Rx is artinian. Thus, $\operatorname{Soc} Rx \leqq_e Rx$. Since R is a left noetherian left V-ring, $\operatorname{Soc} Rx$ is injective. Therefore, $\operatorname{Soc} Rx$ is a direct summand of Rx. This is impossible unless $\operatorname{Soc} Rx = Rx$. Therefore, every cyclic submodule of M and hence, M is semisimple.

(3) implies (1). Let I be an essential left ideal of R. Since R/I is singular, R/I is finitely generated semisimple. It follows that R/I is a finite direct sum of simple modules. Therefore, R/I is artinian.

The equivalence of (1) and (2) follows from 2.1.

According to 2.2, if a left noetherian left V-ring R satisfies the RLM condition, then $E(R)/R$ is a semisimple left R-module. In this case, $E(R)/R$ semisimple characterizes the RLM condition.

THEOREM 2.3. *Let R be a left noetherian left V-ring. Then R satisfies the RLM condition iff $E(R)/R$ is a semisimple left R-module.*

Proof. Suppose $E(R)/R$ is semisimple. Clearly, it suffices to show that every cyclic singular left R-module is semisimple. Let $I \leqq_e R$. Then there is a regular $c \in I$ and $Rc \leqq_e R$. The map $R \rightarrow Rc$ given by $r \rightarrow rc$ extends to an isomorphism $E(R) \rightarrow E(Rc)$. Since $E(R) = E(Rc)$, passing to the quotient yields an isomorphism $E(R)/R \cong E(R)/Rc$. Thus, $E(R)/Rc$ is semisimple. Now, $R/I \leqq E(R)/I \cong (E(R)/Rc)/(I/Rc)$. Therefore, R/I is semisimple.

The converse follows from 2.2.

For a two sided noetherian V-ring R, Michler and Villamayor [12] have demonstrated that $K\dim R \leqq 1$ and hereditary are equivalent. This result together with 2.3 allows us to characterize hereditary in terms of the left R-module $E(R)/R$. This is in contrast to Boyle and Goodearls result in [3] where $E(R)/R$ is required to be injective on both sides.

COROLLARY 2.4. *A noetherian V-ring R is hereditary iff $E(R)/R$ is a semisimple left R-module.*

Proof. According to Michler and Villamayor [12; 4.4], hereditary is equivalent to $K\dim R \leqq 1$. The result follows from 2.2 and 2.3.

3. **$E(R)/R$ and critically nice rings.** A module U is called *critical* provided $K\dim U/K < K\dim U$ for every $0 \neq K \leqq U$. Boyle [2] has shown that every finitely generated uniform left R-module over a left QI-ring is critical. Following Golan and Papp [8], we will call a ring over which every finitely generated uniform left R-module is *critically nice.* Since a hereditary noetherian V-ring is a QI-ring (Boyle [1; 5]), critically nice is necessary for a noetherian V-ring to be hereditary. Our purpose here will be to examine $E(R)/R$ when R is critically nice and extend some of our previous results.

LEMMA 3.1. *If R is a left noetherian ring, then the following are equivalent:*

(1) *R is critically nice.*

(2) *If $A \neq 0$ is finitely generated, then every finitely generated submodule of $E(A)/A$ has Krull dimension strictly less than the Krull dimension of A.*

Proof. (1) implies (2). Let $A \neq 0$ be finitely generated and let $F \leqq E(A)/A$ be finitely generated. There are $U_1, \cdots, U_n$ uniform submodules of A such that $U_1 \oplus \cdots \oplus U_n \leqq_e A$. Then F is an epimorphic image of a finitely generated $F' \leqq E(U_1)/U_1 \oplus \cdots \oplus E(U_n)/U_n$. Since F' is finitely generated, there are finitely generated $F_i/U_i \leqq E(U_i)/U_i$ such that $F' \leqq F_1/U_1 \oplus \cdots \oplus F_n/U_n$. Therefore, $K\dim F \leqq K\dim F' \leqq K\dim (F_1/U_1 \oplus \cdots \oplus F_n/U_n) = K\dim F_j/U_j$ for some j. Since $U_j \leqq A$, $K\dim U_j \leqq K\dim A$. Also, F_j is critical. Thus, $K\dim F \leqq K\dim F_j/U_j < K\dim F_j = K\dim U_j \leqq K\dim A$.

(2) implies (1). Let $U \neq 0$ be finitely generated and uniform, and let $0 \neq K \leqq U$. Then $K\dim K \leqq K\dim U$. Since $U/K \leqq E(U)/K = E(K)/K$, $K\dim U/K < K\dim K \leqq K\dim U$.

A module M is called *smooth* provided $K\dim F = K\dim H$ for all nonzero finitely generated submodules F, H of M. According to 2.4 and 2.2, R hereditary implies that $E(R)/R$ is smooth and injective. In case R is critically nice, the following result shows that the reverse implication holds.

Note that by [7; 2, 3], we may freely use the hypothesis that our ring is a simple ring.

THEOREM 3.2. *A simple noetherian V-ring R is hereditary iff R is critically nice and $E(R)/R$ is smooth and injective.*

Proof. Sufficiency. According to 2.4, it suffices to show that $E(R)/R$ is semisimple. Consequently, it suffices to show that every cyclic submodule of $E(R)/R$ is injective. Let $0 \neq C \leqq E(R)/R$ be cyclic. Then $C \cong R/I$ where $I \leqq_e R$. As in the proof of 2.3, $E(R)/R \cong E(R)/Rc$ where $c \in I$ is regular. Thus, $E(R)/Rc = E(R/Rc) \oplus E'$. Also, $E(R)/R \cong (E(R)/Rc)/(R/Rc) \cong E(R/Rc)/(R/Rc) \oplus E'$. Thus, if $0 \neq F \leqq E(R/Rc)/(R/Rc)$ is finitely generated, then $K\dim F < K\dim R/Rc$ by 3.1. However, F imbeds in $E(R)/R \cong E(R)/Rc$ and hence, $K\dim F = K\dim R/Rc$ which is a contradiction. It follows that $E(R/Rc) = R/Rc$. Thus, $R/Rc = E(I/Rc) \oplus E''$. Now, $R/I \cong (R/Rc)/(I/Rc) \cong E(I/Rc)/(I/Rc) \oplus E''$. Thus, if $0 \neq K \leqq E(I/Rc)/(I/Rc)$ is finitely generated, then $K\dim K < K\dim I/Rc$ by 3.1. However, since R/I imbeds in $E(R)/R \cong E(R)/Rc$ and K imbeds in R/I, $K\dim R/I = K\dim K = K\dim R/Rc = K\dim I/Rc$ which is a contradiction. It follows that $E(I/Rc) = I/Rc$. Therefore, $R/I \cong E''$ is injective.

Necessity follows from the remark prior to 3.1 and by 2.4.

Since a QI-ring is critically nice by Boyle [2], we immediately obtain the following corollary.

COROLLARY 3.3. *A QI-ring R is hereditary iff $E(R)/R$ is smooth and injective.*

4. Left-right symmetry. In this section, we examine the question of symmetry for left noetherian left V-rings which satisfy the RLM condition. We determine that right Goldie is equivalent to the ring being a right noetherian right V-ring. As a corollary to this result, we obtain a theorem of Faith's.

LEMMA 4.1. *Let R be a simple right Goldie ring. Then R is right noetherian iff R satisfies the ascending chain condition on finitely generated essential right ideals.*

Proof. The forward implication is trivial. For the reverse implication, let $U \neq 0$ be a uniform right ideal of R. By [10; 1.2], there is a $1-1$ map $R \to U^n$ where U^n is a direct sum of n copies of U for some n. Thus, if every submodule of U is finitely generated, then U and hence R is right noetherian. Therefore, it suffices to show that every uniform right ideal of R is finitely generated. If not, then there is a uniform right ideal U with an infinite ascending chain $K_1 < K_2 < \cdots < U$ where each K_i is finitely generated. Since R is right Goldie, there are finitely generated right ideals $U_1, \cdots, U_m$ such that $U \oplus U_1 \oplus \cdots \oplus U_m \leqq_e R$. Thus, if $F_i = K_i \oplus U_1 \oplus \cdots \oplus U_m$, then $F_i \leqq_e R$ for all i and $F_1 < F_2 < \cdots < R$ is an infinite ascending chain which is a contradiction. Therefore, every uniform right ideal of R is finitely generated.

THEOREM 4.2. *Let R be a simple left noetherian left V-ring which satisfies the RLM condition. Then R is a right noetherian right V-ring iff R is right Goldie.*

Proof. Sufficiency. By 2.2, R is left hereditary. If R is right noetherian, then by Small [13], R is right hereditary, and by Boyle and Goodearl [3; 2], R is a right V-ring. Thus, it suffices to show that R is right noetherian.

Let $0 \neq I_1 \leqq I_2 \leqq \cdots \leqq R$ where each I_i is a finitely generated essential right ideal of R. Since R is right Goldie, there is a regular $c \in I_1$ and hence, $I_0 = cR$ is essential in R. For every i, let $I_i^* = \mathrm{Hom}_R(I_i, R)$ and let $g_i: I_i^* \to I_{i-1}^*$ be given by $g_i(f) = f \mid I_{i-1}$. Since each I_i is essential in R and R is nonsingular, g_i is $1-1$ for all i. Define a $1-1$ map $h_i: I_i^* \to R$ for all i by $h_i(f) = f(c)$. Let $J_i = h_i(I_i^*)$ for all i. It is easily verified that each J_i is a left ideal and that since each g_i is $1-1$, $R \geqq J_0 \geqq J_1 \geqq \cdots$. Also, since each I_i contains the inclusion map, $c \in J_i$ for all i. Thus, since R/Rc is artinian, there is an n for which $J_n = J_{n+k}$ for all k. It is well known that this forces $I_n = I_{n+k}$ for all k. By 4.1, R is right noetherian.

Necessity is trivial.

COROLLARY 4.3 [6; 22]. *Let R be a left QI-domain which satisfies the RLM condition. If R is right Goldie, then R is a right QI-ring.*

Proof. By 4.2, R is a right noetherian right V-ring. By Small [13], R is right hereditary. According to Boyle [1; 5], R is a right QI-ring.

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