

ON THE CHARACTERIZATION OF AN INTERESTING PROPERTY OF THE ARCSIN DISTRIBUTION

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The property that product and sum of independent identically distributed random variables are equidistributed is characterized when the original distribution has all moments. This equidistribution property is shown to relate in a natural way to the uniform distribution on a compact group. The result proves a conjecture due to Norton.

Introduction. By the notation $U \sim V$ we will indicate that the random variables U and V are identically distributed. We are concerned with the property

$$(*) \quad Z_1 Z_2 \sim Z_1 + Z_2,$$

where Z, Z_1, Z_2 are independent and identically distributed. All our comments and results assume that all the moments

$$(1) \quad d_i = EZ^i, \quad (i = 0, 1, 2, \dots)$$

exist. We do not know if any distribution (or random variable Z) has property (*) but does not possess all moments.

Clearly, (*) implies that

$$(2) \quad d_n^2 = \sum_{k=0}^n \binom{n}{k} d_k d_{n-k}, \quad (n = 0, 1, 2, \dots).$$

Equivalently,

$$(3) \quad (d_n - 1)^2 = 1 + \sum_{k=1}^{n-1} \binom{n}{k} d_k d_{n-k}, \quad (n = 1, 2, \dots).$$

As was observed by Norton (1978), (2) implies that $-2 \leq Z \leq 2$. After all, since $|d_k|^{1/k} \leq (d_{2n})^{1/2n}$, (2) implies $d_{2n}^2 \leq 2^{2n} d_{2n}$, thus, $E|Z|^{2n} \leq 2^{2n}$ for all n . Since the distribution of a bounded random variable is completely determined by its moment sequence, condition (3) is both necessary and sufficient for (*).

It was shown by Norton (1975) that (*) holds for the Arcsin distribution on $[-2, +2]$, that is, when Z is distributed according to the probability density

$$(4) \quad f(z) = \pi^{-1}(4 - z^2)^{-1/2} \quad \text{for } -2 \leq z \leq +2,$$

($f(z) = 0$, otherwise). The moments of this distribution are

$$(5) \quad d_{2m}^* = \binom{2m}{m}; \quad d_{2m+1}^* = 0, \quad (m = 0, 1, 2, \dots).$$

Norton verified (*) by showing that (2) holds when $d_i = d_i^*$.

Relation (3) may be used to recursively calculate the moments d_n . For $n = 1$, one has $(d_1 - 1)^2 = 1$ so that $d_1 = 0$ or $d_1 = 2$. Since $Z \leq 2$, the latter leads to a degenerate solution of (*), namely,

$$(6) \quad Z = 2.$$

If $d_1 = 0$ then (3) yields $(d_2 - 1)^2 = 1$ so that $d_2 = 0$ or $d_2 = 2$. The case $d_2 = EZ^2 = 0$ leads to the degenerate solution

$$(7) \quad Z = 0$$

of (*). If $d_1 = 0, d_2 = 2$ then (3) yields $(d_3 - 1)^2 = 1$ so that $d_3 = 0$ or $d_3 = 2$. Merely assuming that $d_1 = 0, d_2 = 2$, it follows from (3) that $(d_4 - 1)^2 = 1 + 8d_1d_3 + 6d_2^2 = 25$, hence, $d_4 = 6$. If $d_1 = d_3 = 0$ then $(d_5 - 1)^2 = 1 + 10d_1d_4 + 20d_2d_3 = 1$ so that $d_5 = 0$ or $d_5 = 2$. And so on.

If one insists upon a distribution which is symmetric about zero, then of necessity $d_{2k+1} = 0$ for all k and the above calculations yield that $d_k = d_k^*$ for all k . This shows that the Arcsin law is the only symmetric distribution which satisfies (*).

Setting aside the Arcsin distribution and the degenerate solutions (6) and (7), it follows from the above recursive scheme that to any other solution $\{d_k\}$ of (3) there corresponds a unique positive integer m such that

$$(8) \quad d_k = d_k^* \quad \text{for } k = 0, 1, \dots, 2m, 2m + 2; \quad d_{2m+1} = 2;$$

(in particular, $d_1 = d_3 = \dots = d_{2m-1} = 0$). With the d_j as in (8), Norton (1978) showed that the Hankel determinant

$$A_{m+1} = \det (d_{i+j}; i, j = 0, 1, \dots, m + 1)$$

is equal to zero for $m \leq 14$. Afterwards, Shantaram (1978) established the same for all $m \geq 1$.

Let Z be any random variable whose moments satisfy (8). It follows from $A_{m+1} = 0$ that there exist numbers a_0, a_1, \dots, a_m not all zero such that $E(\sum_{j=0}^{m+1} a_j z^j)^2 = 0$. Equivalently, the support of Z is a finite set consisting of at most $m + 1$ points. It follows from well-known results (see Karlin and Studden (1966), page 42) that the distribution of Z on hand (if it exists) is completely determined by the moments (8).

Note that the Arcsin law is the only nondiscrete solution of (*). As to the discrete solutions, Norton (1978) conjectured the following.

Conjecture. Let m be any positive integer. Then there does exist a random variable $Z(m)$ whose moments satisfy (8). Moreover, the support of $Z(m)$ consists of precisely $m + 1$ points. Finally, this random variable $Z(m)$ has property (*).

The conjecture turns out to be correct. More precisely, we shall prove the following.

THEOREM 1. *Let Z be a random variable having all moments. Then in order that Z satisfies (*), it is necessary and sufficient that one of the following is true.*

(i) Z has the Arcsin distribution (4);

(ii) $Z = 0$;

(iii) *there exists a nonnegative integer m such that Z is distributed as $Z(m)$, where*

$$(9) \quad \begin{aligned} Pr(Z(m) = 2) &= 1/(2m + 1); \\ Pr\left(Z(m) = 2 \cos \frac{2j\pi}{2m + 1}\right) &= 2/(2m + 1) \text{ for } j = 1, 2, \dots, m. \end{aligned}$$

The result due to Shantaram that $\Delta_{m+1} = 0$ is not used in the proof. As will be seen, the solutions (4) and (9) of (*) arise in a very natural way from the properties of a uniformly distributed random variable U taking values in a compact group.

One may ask how far the above considerations carry over to a relation of the form

$$(10) \quad Z_1 Z_2 \cdots Z_r \sim Z_1 + Z_2 + \cdots + Z_s,$$

where the Z_j are independent, each distributed as $Z = Z_1$. Here, $r \geq 2$ and $s \geq 2$ are fixed integers. Provided all moments $d_n = EZ^n$ exist, one easily shows that necessarily $-C \leq Z \leq C$, where $C = s^{1/(r-1)}$. There are always the trivial solutions $Z = 0$ and $Z = C$, and further $Z = -C$ in the case r is odd. We conjecture that no other solutions exist, unless $r = s = 2$.

REMARK. After the present paper was completed, Professor Govind Mudholkar kindly showed us a copy of the paper "On the stochastic equation $X + Y = XY$ ", by R. Shantaram (presented at the International Summer School on Statistical Distributions in Scientific Work, Trieste, Summer 1980). Shantaram's paper is devoted to a proof of exactly the above Theorem 1, but by a very different reasoning.

2. Random variables taking values in a compact group. Let G be a compact abelian group, taking addition as the group operation. By a character g of G we mean a complex-valued function on

G which defines a continuous homomorphism of G into the circle group $\{z: |z| = 1\}$. In particular,

$$g(x + x') = g(x)g(x'); \quad |g(x)| = 1; \quad g(0) = 1.$$

Each character g of G will be labeled by an element y of an index set \hat{G} and written as $g(x) = (x, y)$. The set \hat{G} is made into a (discrete) additive abelian group by defining $(x, y + y') = (x, y)(x, y')$. One has $(x, 0) = 1$ and $(-x, y) = (x, -y)$.

The Haar measure on G is denoted by $m(\cdot)$; $m(G) = 1$. Let U be a random variable taking values in G . It is said to be uniformly distributed on G when $Pr(U \in B) = m(B)$ for each Borel subset B of G . This happens if and only if

$$(11) \quad E(U, y) = 0 \quad \text{for each } y \in \hat{G} \text{ with } y \neq 0.$$

Next, let U_1, U_2 be random variables taking values in G . They are independent, each uniformly distributed on G , if and only if

$$(12) \quad E[(U_1, y_1)(U_2, y_2)] = 0 \quad \text{whenever } (y_1, y_2) \neq (0, 0).$$

Suppose this is true and consider

$$(13) \quad V_1 = U_1 + U_2; \quad V_2 = U_1 - U_2.$$

Then

$$(V_1, y_1)(V_2, y_2) = (U_1, y_1)(U_2, y_1)(U_1, y_2)(U_2, -y_2) = (U_1, y_1 + y_2)(U_2, y_1 - y_2).$$

Using the criterion (12) applied to the pair V_1, V_2 , it follows that V_1, V_2 are independent, each uniformly distributed, if and only if $(y_1, y_2) \neq (0, 0)$ implies that $(y_1 + y_2, y_1 - y_2) \neq (0, 0)$. Equivalently, the character group \hat{G} contains no elements y of order 2, (that is, $y \neq 0$ and $2y = 0$). Equivalently, G does not have any closed subgroup H with $m(H) = 1/2$, (that is, H is of index 2); (if such a subgroup H does exist then $V_1 - V_2 \in H$ showing that V_1, V_2 would be dependent). The above remarks yield the following result.

LEMMA 1. *Let G be a compact abelian group and let U_1, U_2 be independent copies of a random variable U taking values in G . Further consider the random variables*

$$(14) \quad S_1 = h(U_1, U_2); \quad S_2 = h(U_1 + U_2, U_1 - U_2),$$

where h is any measurable function on $G \times G$ taking values in some measurable space.

We assert that S_1 and S_2 are identically distributed as soon as U is uniformly distributed on some closed subgroup G_1 of G with the property that G_1 possesses no closed subgroup H_1 of index 2.

In applying Lemma 1, we shall take G as the additive group of real numbers modulo 2π . It will be convenient to regard G as the additive group of real numbers, identifying numbers x' and x'' such that $x' \equiv x'' \pmod{2\pi}$. A function on G can be represented by a function on R which is periodic with period 2π . Similarly, for a function $h(x_1, x_2)$ on $G \times G$.

THEOREM 2. *Let U be a real random variable, $0 \leq U \leq 2\pi$, and put*

$$(15) \quad Z = 2 \cos U .$$

We assert that Z has property () in each of the following cases.*

(i) *U is uniformly distributed on $[0, 2\pi]$. In this case, Z defined by (15) has the Arcsin distribution (4).*

(ii) *For some nonnegative integer m , the random variable U is uniformly distributed on the finite set $\{2\pi j/(2m+1), j=0, 1, \dots, 2m\}$. In this case,*

$$(16) \quad \begin{aligned} Pr(Z = 2) &= 1/(2m + 1) ; \\ Pr\left(Z = 2 \cos \frac{2\pi j}{2m + 1}\right) &= 2/(2m + 1) \text{ for } j = 1, \dots, m . \end{aligned}$$

Proof. The only proper closed subgroups of the additive group G of real numbers modulo 2π are the cyclic subgroups

$$C_n = \{2\pi j/n, j = 0, 1, \dots, n - 1\} ,$$

($n = 1, 2, \dots$). Hence, the last condition of Lemma 1 is satisfied if and only if either $G_1 = G$ or $G_1 = C_{2m+1}$ ($m = 0, 1, \dots$).

Let G_1 be one of the latter groups and suppose U is uniformly distributed on G_1 . Let U_1, U_2 be independent copies of U . It follows from Lemma 1 that the random variables S_1, S_2 defined by (14) are always identically distributed. Choosing $h(x_1, x_2) = 2 \cos x_1 + 2 \cos x_2$, we find that

$$S_1 = h(U_1, U_2) = 2 \cos U_1 + 2 \cos U_2 = Z_1 + Z_2$$

has the same distribution as

$$S_2 = 2 \cos (U_1 + U_2) + 2 \cos (U_1 - U_2) = 4 \cos U_1 \cos U_2 = Z_1 Z_2 .$$

In other words, $Z = 2 \cos U$ has property (*). Calculating the distribution of Z in each case, one arrives at Theorem 2.

REMARK. Note that the sufficiency part of Theorem 1 is hereby proved. It is easy to verify directly that (16) defines a random variable that has property (*). Put

$$(17) \quad z_j = 2 \cos \left(\frac{2\pi j}{2m + 1} \right), \quad (z_{-j} = z_j).$$

It suffices to verify that, for any function g ,

$$Eg(Z_1 + Z_2) = (2m + 1)^{-2} \sum_{j=-m}^m \sum_{k=-m}^m g(z_j + z_k)$$

is equal to

$$Eg(Z_1 Z_2) = (2m + 1)^{-2} \sum_{j=-m}^m \sum_{k=-m}^m g(z_j z_k).$$

From (17), $z_j + z_k = z_r z_s$ when both $j \equiv r + s \pmod{2m + 1}$ and $k \equiv r - s \pmod{2m + 1}$. Equivalently, when $2r \equiv j + k \pmod{2m + 1}$ and $2s \equiv j - k \pmod{2m + 1}$. This one-one correspondence between pairs (j, k) and (r, s) yields the desired result.

3. Proof of necessity in Theorem 1. We are concerned with random variables Z which have property (*) and further possess all moments $d_k = EZ^k$. We may assume that $Pr(Z \neq 0) > 0$ and that Z does not have the Arcsin distribution. As we have seen, this implies that $-2 \leq Z \leq +2$ and further that $\{d_k\}$ satisfies (8) for some unique nonnegative integer m .

For each integer $m \geq 0$, let K_m denote the class of all distributions (probability measures) on $[-2, +2]$ such that the corresponding moment sequence $\{d_k\}$ satisfies (8). Clearly, K_0 contains only the degenerate distribution at $z = 2$. It remains to prove that the only distributions in $\bigcup_{m=0}^{\infty} K_m$ are the distributions of the random variables $Z(0), Z(1), \dots$ defined by (9).

Let μ_m denote the distribution of $Z(m)$. Its support is of the form

$$(18) \quad A(m) = \{z_{m,j} = 2 \cos(2\pi j/(2m + 1)); j = 0, 1, \dots, m\}.$$

We know by Theorem 2 that $Z(m)$ has property (*), hence, μ_m belongs to some unique class K_M . We claim that $M \leq m$.

For, otherwise, from (8) with m replaced by M , the moments d_k of $Z(m)$ satisfy $d_1 = d_3 = \dots = d_{2m-1} = d_{2m+1} = 0$. Equivalently,

$$\sum_{j=0}^m p_j (z_{m,j})^{2k-1} = 0 \quad \text{for } k = 1, 2, \dots, m + 1,$$

where $p_j = Pr(Z(m) = z_{m,j})$. Note that $z_{m,h} \neq -z_{m,j}$, in fact, the $z_{m,j}^2$ ($j = 0, 1, \dots, m$) are distinct and nonzero. But then the above system would imply that $p_j = 0$ for all j .

We conclude that

$$\{\mu_0, \mu_1, \dots, \mu_m\} \subset \bigcup_{M=0}^m K_M, \quad (m = 0, 1, 2, \dots).$$

This shows that it suffices to prove that each class $K_{\mathcal{Y}}$ contains at most one distribution. Since all the μ_m are different, this would imply that $K_m = \{\mu_m\}$ for all m .

There are many ways of showing that K_m consists of at most a single distribution. As observed in the paragraph following (8), this follows for instance from Shantaram's (1978) result that $\Delta_{m+1} = 0$.

As an independent proof, we even claim that there can be at most one distribution μ on $[-2, +2]$ which satisfies

$$(19) \quad d_{2k-1} = 0 \quad \text{for } k = 1, \dots, m; d_{2m+1} = 2.$$

Here, m is a fixed nonnegative integer (and nothing is assumed about the moments d_{2k}). Namely, consider the polynomial

$$(20) \quad f(z) = 1 + T_{2m+1}(-z/2) = 1 + a_0z + a_1z^3 + \dots + a_mz^{2m+1}.$$

Here, $T_n(x)$ denotes the n th Tchebycheff polynomial. For $x \in [-1, +1]$, it is defined by $T_n(x) = \cos n\theta$ where $\cos \theta = x$. Writing $\cos n\theta$ in complex form, we see that the leading coefficient in $T_n(x)$ equals 2^{n-1} . Therefore, the leading coefficient a_m of $f(z)$ equals

$$a_m = 2^{2m}(-1/2)^{2m+1} = -1/2.$$

Integrating $f(z)$ relative to a measure μ satisfying (19), it follows that

$$\int f(z)\mu(dz) = 1 + a_m d_{2m+1} = 0.$$

Since obviously $f(z) \geq 0$ on $[-2, +2]$, we conclude that μ must be supported by $Z(f) = \{z: f(z) = 0\}$. It is easily seen that $Z(f)$ is precisely the set $A(m)$ defined by (18). Condition (19) then implies that $q_j = \mu(\{z_{m,j}\})$ satisfies the system of $m + 1$ equations

$$\sum_{j=0}^m q_j = 1; \quad \sum_{j=0}^m q_j (z_{m,j})^{2k-1} = 0 \quad \text{for } k = 1, \dots, m.$$

Since the corresponding matrix has rank $m + 1$, there can be at most one solution.

4. Related problems. The referee kindly pointed to the related paper by Arnold and Groeneveld (1980). It contains the following results.

(A) If the random variable X is symmetric (about 0) then

$$(21) \quad X^2 \sim (1 + X)/2$$

if and only if X has the Arcsin distribution on $[-1, +1]$. The corresponding density is

$$f(x) = \pi^{-1}(1 - x^2)^{-1/2} \quad \text{for} \quad -1 < x < 1.$$

(B) Suppose $-1 \leq X \leq +1$ and further that both X and $X^2 - 1/2$ are symmetric. Then

$$(22) \quad 2X(1 - X^2)^{1/2} \sim X$$

if and only if X has the Arcsin distribution on $[-1, +1]$.

(C) Let X, X_1, X_2 be i.i.d. and suppose that both X and $X^2 - 1/2$ are symmetric. Then

$$(23) \quad X_1^2 - X_2^2 \sim X_1 X_2$$

if and only if X has the Arcsin distribution on $[-1, +1]$.

The proofs are easy. Simple counterexamples show that the above symmetry conditions cannot be omitted.

It would be desirable to characterize the properties (21), (22) and (23) without making any symmetry assumption. Without proof, we mention the following results.

(A1) If X satisfies (21) then necessarily $-1 \leq X \leq +1$. Thus, one may write $X = \cos \pi Y$ with $0 \leq Y \leq 1$. Property (21) is equivalent to $TY \sim Y$, where

$$(24) \quad Ty = 1 - 2|y - 1/2|,$$

thus, T defines a continuous map of $[0, 1]$ into itself.

(A2) Suppose X satisfies (21) and, moreover, has an absolutely continuous distribution (relative to Lebesgue measure). Then Y has the uniform distribution on $[0, 1]$, equivalently, X has the Arcsin distribution on $[-1, +1]$.

(A3) We conjecture that (21) admits at least one solution X having a purely singular distribution (with no atoms).

(A4) Suppose the random variable X has a finite support. Then X satisfies (21) if and only if its distribution is a convex linear combination of the uniform distributions on the different (mutually disjoint) finite closed orbits of the continuous transformation $Sx = 2x^2 - 1$ of $[-1, +1]$ into itself.

A number $x \in [-1, +1]$ belongs to such an orbit if and only if $x = \cos \pi y$ with $y = h/(2m + 1)$ as a rational number with odd denominator, ($h \in \{0, 1, \dots, 2m\}$). Besides $y = 1$, the excluded rational values y (such as $1/2$ and $1/6$) are precisely those whose expansion $y = \cdot y_1 y_2 \dots$ to the base 2 is not purely periodic.

(A5) Let m be a nonnegative integer and put $d = (2m + 1)^{-1}$. Then $X = \cos \pi(U + jd/3)$ satisfies (21) as soon as $j \in \{0, 1, 2\}$ while U has a uniform distribution on $\{hd; h = 0, 1, \dots, 2m\}$.

(B1) Suppose X satisfies $-1 \leq X \leq +1$ and (22). Then $W = 1 - 2X^2$ satisfies

$$W^2 = 1 - 4X^2(1 - X^2) \sim 1 - X^2 = (1 + W)/2,$$

and we are back to Problem A.

(B2) Given $-1 \leq X \leq +1$, one may write $X = U \sin \pi Y/2$ where either $U = \pm 1$, $0 < Y \leq 1$ or $U = 0$, $Y = 0$. Condition (22) says precisely that $U \sin \pi Y/2 \sim U \sin \pi Y$. Equivalently, $(U, Y) \sim (U, TY)$ with T as in (24). That is, property (22) holds if and only if each of the measures

$$\mu_\varepsilon(A) = Pr(U = \varepsilon, Y \in A), \quad (\varepsilon = +1 \text{ or } -1)$$

is invariant under T .

(B3) Suppose X satisfies (22) and, moreover, is absolutely continuous. Then each of the measures μ_ε is a multiple of the uniform distribution on $[0, 1]$, in particular, U and Y are independent. The present density of X is of the form

$$\begin{aligned} f(x) &= c_1(1 - x^2)^{-1/2} \quad \text{for } 0 < x < 1; \\ &= c_2(1 - x^2)^{-1/2} \quad \text{for } -1 < x < 0, \end{aligned}$$

with the c_i as nonnegative constants, ($c_1 + c_2 = 2/\pi$).

(C) Let X satisfy (23). One easily sees that $EX^{2k+1} = 0$ ($k = 0, 1, \dots$), in so far these moments exist. Hence, if X is also bounded then it must be symmetric.

Condition (23) is satisfied by $X = c \sin 2\pi Y$, where $c \geq 0$ is constant and Y is uniform on a set $\{h/(2m+1); h = 0, 1, \dots, 2m\}$. In the limit $m \rightarrow \infty$, this gives rise to the Arcsin distribution on $[-c, +c]$. We conjecture that (23) has no other solutions.

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