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SHADOW AND INVERSE-SHADOW INNER PRODUCTS FOR A CLASS OF LINEAR TRANSFORMATIONS

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Suppose $\{H, (\cdot, \cdot)\}$ is a complete inner product space and H_1 is a dense subspace of H. In case T is a linear transformation from H_1 to H_1 (perhaps not bounded), a necessary and sufficient condition is obtained in Theorem 1 for the existence of an inner product $(\cdot, \cdot)_1$ for H_1 such that (i) the identity is continuous from $\{H_1, (\cdot, \cdot)_1\}$ to $\{H, (\cdot, \cdot)\}$ and (ii) T is bounded in $\{H_1, (\cdot, \cdot)_1\}$. When this condition holds, the inverse-shadow inner product is defined on H_1, for sufficiently large positive numbers β, by $(x, y)_{\beta,T} = \sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y)$. An extension of Theorem 1 provides a necessary and sufficient condition for the existence of an inner product $(\cdot, \cdot)_1$ for H_1 such that $\{H_1, (\cdot, \cdot)_1\}$ is complete and (i) and (ii) hold. This latter condition, stated in Theorem 5 in terms of a pair of inverse-shadow inner products, depends on a description of those complete inner product spaces $\{H_1, (\cdot, \cdot)_1\}$, with H_1 dense in H, for which (i) holds. According to this description, given in Theorem 4, each such inner product $(\cdot, \cdot)_1$ is a scalar-multiple of an inverse-shadow inner product $(\cdot, \cdot)_{\delta,C}$, where C is a bounded operator on H mapping H_1 to H_1 and $\delta = 1$.

This pattern was developed in an investigation, other results of which are in [4]. If H_1 is a linear subspace of H, $(\cdot, \cdot)_1$ is an inner product for H_1, and the identity is continuous from $\{H_1, (\cdot, \cdot)_1\}$ to $\{H, (\cdot, \cdot)\}$, $\{H_1, (\cdot, \cdot)_1\}$ is said in [6] to be continuously situated in $\{H, (\cdot, \cdot)\}$. The setting in Theorem 4 of a pair of complete inner product spaces, one continuously situated in the other, is discussed in [1], [2], [6], and [7]. Additional results in Theorems 2 and 3 relate the shadow inner product, the inner product $((1 - T^*T/\beta^2)\cdot, \cdot)'$ in those theorems, and the inverse-shadow inner product $(\cdot, \cdot)_{\beta,T}$. In contrast to Theorem 4, an example at the end of the paper shows that $\{H_1, (\cdot, \cdot)_{\beta,T}\}$ may be complete even when the closure in $H \times H$ of T is not a function.

Here is an example to which Theorem 1 applies (with $H = H_1$). Start with a complete infinite dimensional inner product space $\{H', (\cdot, \cdot)'\}$, a one-to-one (continuous) operator T on H' with range a dense, proper subspace of H', and a closed subspace Z of H' such that $Z \cap T(H')$ is $\{0\}$. Now, with P the orthogonal projection of H' onto $Z^{\perp}$, there is, by the Axiom of Choice, an algebraic complement H_1 of Z in H' of which $T(H')$ is a subspace and, with $(\cdot, \cdot)$ the inner product on H_1 such that $(x, y) = (Px, Py)'$, $\{H_1, (\cdot, \cdot)\}$ is complete and for x in H_1 $(x, x) \leqq (x, x)'$. Yet the restriction of T to H_1 is not continuous in $\{H_1(\cdot, \cdot)\}$. Of course, the above construction uses the Axiom of Choice, as the result of [8] implies it must. However, this use is not in constructing T but in selecting the subspace H_1 of H'.

Throughout the paper, $\{H, (\cdot, \cdot)\}$ is a complete infinite dimensional inner product space and H_1 a dense subspace of H. If some variation of the symbols '$(\cdot, \cdot)$' denotes an inner product for the space S, then the corresponding variation of '$\|\cdot\|$' denotes the corresponding norm for S. For instance, $\|x\|_{\beta,T} = [(x, x)_{\beta,T}]^{1/2}$. An operator on $\{H, (\cdot, \cdot)\}$ is a continuous linear transformation from all of H to (into) H. A closed operator in $\{H, (\cdot, \cdot)\}$ is a linear transformation from a dense subspace of H to H whose graph is closed in $H \times H$. If Z and Z' are two subspaces of H such that $Z \cap Z'$ is $\{0\}$ and H is the linear span of Z and Z', then Z is said to be an algebraic complement in H of Z' and that linear transformation ϕ on H such that ϕ is the identity 1 on Z and 0 on Z' is called the algebraic projection of H onto Z with kernel Z'. If Z is a subset of H, $\bar{Z}$ is the closure of Z in H.

THEOREMS AND EXAMPLES

THEOREM 1. *Suppose that T is a linear transformation from H_1 to H_1. In order that there be a norm $\|\cdot\|_1$ for H_1 such that* (i) *there is a positive number c such that $\|\cdot\| \leqq c\|\cdot\|_1$ on H_1 and* (ii) *T is continuous in $\{H_1, \|\cdot\|_1\}$ it is necessary and sufficient that there be a positive number β such that for x in H_1 $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2$ converges. In case there is such a norm $\|\cdot\|_1$, if β is a number exceeding the operator-norm for T in $\{H_1, \|\cdot\|_1\}$ then for x and y in H_1 the formula $(x, y)_{\beta,T} = \sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y)$ defines an inner product $(x, y)_{\beta,T}$ for H_1 such that*

(1) *there is a positive number d such that for x in H_1 $\|x\| \leqq \|x\|_{\beta,T} \leqq d\|x\|_1$,*

(2) *for x in H_1 $\lim_{p\to\infty} \|(T/\beta)^p x\|_{\beta,T} = 0$, and*

(3) *for x and y in H_1 $(Tx, Ty)_{\beta,T} = \beta^2[(x, y)_{\beta,T} - (x, y)]$.*

Proof. In case there is a positive number β for which $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2$ converges on H_1, we have for x and y in H_1 and n a positive integer,

$$\sum_{p=0}^{n} |((T/\beta)^p x, (T/\beta)^p y)|$$

$$\leqq \sum_{p=0}^{n} \|(T/\beta)^p x\| \, \|(T/\beta)^p y\|$$

$$\leqq \left(\sum_{p=0}^{n} \|(T/\beta)^p x\|^2\right)^{1/2}\left(\sum_{p=0}^{n} \|(T/\beta)^p y\|^2\right)^{1/2},$$

so that $\sum_{p=0}^{\infty}((T/\beta)^p x, (T/\beta)^p y)$ converges absolutely. Moreover, the formula $(x, y)_{\beta,T} = \sum_{p=0}^{\infty}((T/\beta)^p x, (T/\beta)^p y)$ defines as inner product for H_1.

Suppose that there is a norm $\|\cdot\|_1$ for H_1 for which (i) and (ii) hold. Suppose n is a positive integer, β is a positive number, and r is a number greater than 1 such that for x in H_1 $r\|Tx\|_1 \leqq \beta\|x\|_1$. Then for x and y in H_1

$$\begin{aligned} &\sum_{p=0}^{n} |((T/\beta)^p x, (T/\beta)^p y)| \\ &\qquad \leqq \sum_{p=0}^{n} \|(T/\beta)^p x\| \, \|(T/\beta)^p y\| \\ \text{(A)} \qquad &\qquad \leqq c^2 \sum_{p=0}^{n} \|(T/\beta)^p x\|_1 \|(T/\beta)^p y\|_1 \\ &\qquad \leqq c^2 \sum_{p=0}^{n} \|x\|_1 \|y\|_1 (1/r^{2p}) \\ &\qquad = c^2 \|x\|_1 \|y\|_1 r^2/(r^2-1) . \end{aligned}$$

Thus, for x and y in H_1 the series $\sum_{p=0}^{\infty}((T/\beta)^p x, (T/\beta)^p y)$ converges absolutely and, replacing y by x in (A), we have

$$\text{(B)} \qquad \sum_{p=0}^{n} \|(T/\beta)^p x\|^2 \leqq c^2(\|x\|_1)^2 r^2/(r^2-1) .$$

Note that (1) follows from (B) with $d = cr/(r^2-1)^{1/2}$. To establish (2), observe that for x in H_1

$$(\|(T/\beta)^p x\|_{\beta,T})^2 = \sum_{q=0}^{\infty} \|(T/\beta)^{p+q} x\|^2 \longrightarrow 0 \quad \text{as} \quad p \longrightarrow \infty ,$$

since $\sum_{q=0}^{\infty} \|(T/\beta)^q x\|^2$ converges. The equality (3) is established by noting that

$$\begin{aligned} &(Tx, Ty)_{\beta,T} \\ &\qquad = \sum_{p=0}^{\infty}((T/\beta)^p Tx, (T/\beta)^p Ty) \\ &\qquad = \beta^2 \sum_{p=1}^{\infty}((T/\beta)^p x, (T/\beta)^p y) \\ &\qquad = \beta^2\left[\sum_{p=0}^{\infty}((T/\beta)^p x, (T/\beta)^p y) - (x, y)\right] \\ &\qquad = \beta^2[(x, y)_{\beta,T} - (x, y)] . \end{aligned}$$

The following example is offered in connection with Lemma 1. This lemma is useful in the proof of Theorems 3 and 4.

EXAMPLE 1. Suppose that S is the subspace of $L^2[0, 1]$ of all absolutely continuous f on $[0, 1]$ such that f' is in $L^2[0, 1]$ and for such f $Tf = f'$, so that T is a closed operator in $L^2[0, 1]$. Suppose H_1 is the set of all f in S such that for $p \geqq 0$ $T^p f$ is in S and $\sum_{p=0}^{\infty} \int_0^1 |T^p f|^2$ converges. Then H_1 is a dense subspace of $L^2[0, 1]$ and, with $\beta = 1$ and $(f, g)_{\beta,T} = \sum_{p=0}^{\infty} \int_0^1 [T^p f][T^p g]^*$ on H_1, $\{H_1, (\cdot, \cdot)_{\beta,T}\}$ is complete.

LEMMA 1. *Suppose that T is a closed operator in $\{H, (\cdot, \cdot)\}$ and $\beta > 0$. Then the set H_2 of all x in H such that for $p > 0$ x is in the domain of T^p and $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2$ converges is a linear space such that $T(H_2)$ lies in H_2. Also, if $(\cdot, \cdot)_{\beta,T}$ is the inner product for H_2 given, as in Theorem 1, by $(x, y)_{\beta,T} = \sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y)$ then $\{H_2, (\cdot, \cdot)_{\beta,T}\}$ is complete. In case T is self-adjoint in $\{H, (\cdot, \cdot)\}$, then the restriction of T to H_2 is self-adjoint in $\{H_2, (\cdot, \cdot)_{\beta,T}\}$.*

The following argument is offered. In general (when T is only closed and not defined everywhere), H_2 need not be dense in H. Suppose x is in H_2. Then $\sum_{p=0}^{\infty} \|(T/\beta)^p Tx\|^2 = \beta^2 \sum_{p=1}^{\infty} \|(T/\beta)^p x\|^2$, so that Tx is in H_2. To show that H_2 is a linear space, suppose S_1 is the linear space of all H-valued sequences, S_2 is the subspace of S_1 to which z belongs only in case $\sum_{p=0}^{\infty} \|z_p\|^2$ converges, and for z and w in S_2 $\langle z, w\rangle = \sum_{p=0}^{\infty} (z_p, w_p)$, so that $\{S_2, \langle\cdot, \cdot\rangle\}$ is a complete inner product space. Suppose D is the set of all x in H such that for $p > 0$ x is in the domain of T^p and $\tilde{T}$ the linear transformation from D to S_1 such that for $p \geqq 0$ $(\tilde{T}x)_p = (T/\beta)^p x$. Note that $H_2 = \tilde{T}^{-1}(S_2)$, a linear space, and that $\tilde{T}$, restricted to H_2, is a linear isometry from $\{H_2, (\cdot, \cdot)_{\beta,T}\}$ onto a subspace of S_2. Suppose y is a convergent sequence in $\{H_2, (\cdot, \cdot)_{\beta,T}\}$. Then $\tilde{T}y$ is convergent in S_2, with limit z in S_2. Since, for $p \geqq 0$ the sequence $\{(T/\beta)^p y, (T/\beta)^{p+1} y\}$ has values in the closed transformation T/β and limit $\{z_p, z_{p+1}\}$ in $H \times H$, $z_{p+1} = (T/\beta) z_p$. Thus, for $p \geqq 0$ $z_p = (T/\beta)^p z_0$, so that $z = \tilde{T} z_0$. Since $\tilde{T}$ is an isometry, y has limit z_0 in $\{H_2, (\cdot, \cdot)_{\beta,T}\}$. Suppose T is self-adjoint in $\{H, (\cdot, \cdot)\}$. Then for x and y in H_2

$$\begin{aligned}(Tx, y)_{\beta,T} &= \sum_{p=0}^{\infty} ((T/\beta)^p Tx, (T/\beta)^p y) \\ &= \sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p Ty) = (x, Ty)_{\beta,T} ,\end{aligned}$$

so that T is self-adjoint on the complete space $\{H_2, (\cdot, \cdot)_{\beta,T}\}$.

EXAMPLE 2. This example shows that in case $\{H, (\cdot, \cdot)\}$ is separable the set of linear transformations T with domain H and range lying in H for which there is a positive number β such that $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2$ converges on H is not a linear space.

Suppose y is in H, $\|y\| = 1$, and Y is the linear span of $\{y\}$. Suppose $\{e_m\}_1^{\infty}$ is a complete orthonormal sequence in $H \ominus Y$. Suppose for $m > 0$ $u_m = e_m + (m!)y$. The linear span U of $\{u_m\}_1^{\infty}$ is dense in H. One sees this by noting that $y = \lim_{m\to\infty}(u_m/m!)$. Hence, for $p > 0$ $e_p = u_p - (p!)y$ is in $\bar{U}$. Thus, the linear space $\bar{U}$ includes both Y and $H \ominus Y$. Suppose that Z is an algebraic complement of Y in H of which U is a subspace. Suppose ϕ is the algebraic projection of H onto Z with kernel Y and that C is the operator on H such that $Cy = 0$ and for m a positive integer $Ce_m = e_{m+1}$. Since the operator-norm of C is 1, $\sum_{p=0}^{\infty} \|(C/2)^p x\|^2$ converges on H. Since for $p > 0 (\phi - 1)^p = (-1)^{p+1}(\phi - 1)$, $\sum_{p=0}^{\infty} \|[(\phi - 1)/2]^p x\|^2$ converges on H.

Suppose T is $C + (\phi - 1)$ and m is the number-sequence such that $m_1 = 1$ and for $n > 0$ $m_{n+1} = (n+1)! - m_n$. Then for $n > 0$ $T^n(e_1) = e_{n+1} + m_n y$ and $\|T^n e_1\|^2 = 1 + m_n^2$. Note that for $n \geqq 1$ $n! - (n-1)! \leqq m_n \leqq n!$, so that $m_{n+1} \geqq n!$. Thus, for $\beta > 0$ $\sum_{p=0}^{\infty} \|(T/\beta)^p e_1\|^2$ diverges.

THEOREM 2. *Suppose that $\{H', (\cdot, \cdot)'\}$ is a complete inner product space, T is an operator on $\{H', (\cdot, \cdot)'\}$, and H_1 is a dense subspace of H' such that $T(H_1)$ lies in H_1. Suppose, moreover, that there is a positive number β such that for each of x and y in H_1 $(x, y)' = \sum_{p=0}^{\infty}((T/\beta)^p x, (T/\beta)^p y)$. Then* (i) *$\beta$ is not less than the operator-norm for T in $\{H', (\cdot, \cdot)'\}$,* (ii) *with T^* the adjoint of T in $\{H', (\cdot, \cdot)'\}$ and x and y in H_1 $(x, y) = ((1 - T^*T/\beta^2)x, y)'$, and* (iii) *in case $H' \neq H_1$ and $\{H_1, (\cdot, \cdot)\}$ is complete, so that $H = H_1$, then β is the operator-norm for T in $\{H', (\cdot, \cdot)'\}$ and for T on H_1 in $\{H_1, (\cdot, \cdot)'\}$.*

Proof. Since H_1 is dense in H' and T continuous on H', the operator-norm for T in $\{H', (\cdot, \cdot)'\}$ is the operator-norm for T on H_1 in $\{H_1, (\cdot, \cdot)'\}$. Suppose that for x and y in H_1 $(x, y)' = \sum_{p=0}^{\infty}((T/\beta)^p x, (T/\beta)^p y)$. Then for x in H_1

$$(\|Tx\|')^2 = \beta^2[(\|x\|')^2 - \|x\|^2] \leqq \beta^2(\|x\|')^2 .$$

Thus, β is not less than the operator-norm for T in $\{H', (\cdot, \cdot)'\}$. Also, on H_1

$$\begin{aligned}(x, y) &= (x, y)' - ((T/\beta)x, (T/\beta)y)' \\ &= ((1 - T^*T/\beta^2)x, y)' ,\end{aligned}$$

so that (ii) is established.

To prove (iii), note that, since $H' \neq H_1$, H_1 is not closed in H'.

Also, the identity function from $\{H_1, (\cdot, \cdot)'\}$ to $\{H_1, (\cdot, \cdot)\}$ is continuous. Since $\{H_1, (\cdot, \cdot)\}$ is complete, the identity function from $\{H_1, (\cdot, \cdot)\}$ to $\{H_1, (\cdot, \cdot)'\}$ is not continuous. By the Closed Graph theorem, the set Z of all $\|\cdot\|'$-limits in H' of H_1-sequences having $\|\cdot\|$-limit 0 is nondegenerate. Since Z is the kernel of $(1 - T^*T/\beta^2)^{1/2}$, there is a nonzero point x of H' such that $x = (T^*T/\beta^2)x$. Thus, $(\|Tx\|')^2 = \beta^2(\|x\|')^2$. In view of (i), (iii) is established.

REMARK. Here I will describe why I call an inner product, $((1 - T^*T/\beta^2)\cdot, \cdot)'$, a shadow inner product. The point of view taken by the author is that one starts with $\{H, (\cdot, \cdot)\}$, a linear transformation T from H to H, not continuous in $\{H, (\cdot, \cdot)\}$, and a positive number β such that $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2$ converges on H. (T might be the transformation $\phi - 1$ of Example 2 with $\beta = 2$). One builds the space $\{H, (\cdot, \cdot)_{\beta,T}\}$ with a completion $\{H'(\cdot, \cdot)'\}$ so that H is a proper subspace of H', dense in H'. Now T has continuous linear extension to H', also denoted by T, with adjoint T^* in $\{H', (\cdot, \cdot)'\}$. Then by Theorem 2, $(x, y) = ((1 - T^*T/\beta^2)x, y)'$ on H. The identity function from $\{H, (\cdot, \cdot)'\}$ to $\{H, (\cdot, \cdot)\}$ is continuous. If $\{H, (\cdot, \cdot)\}$ is complete, by Note 5 of [4], the set Z of all $\|\cdot\|'$-limits in H' of sequences in H with $\|\cdot\|$-limit 0 is closed in H' and also an algebraic complement of H in H', and if P is the orthogonal projection of H' onto $Z^\perp$ then $(\cdot, \cdot)$ is equivalent on H to $(P\cdot, P\cdot)'$. That is, the inner product $((1 - T^*T/\beta^2)x, y)'$ on H is equivalent to the inner product $(Px, Py)'$ on H, the inner product in H' of the shadow of x in $Z^\perp$ with the shadow in $Z^\perp$ of y. Another point of view, starting with a complete space $\{H', (\cdot, \cdot)'\}$, an operator T on $\{H', (\cdot, \cdot)'\}$, and a dense, proper subspace H_1 of H', and yielding a shadow inner product $((1 - T^*T)\cdot, \cdot)'$ for H_1 such that $\{H_1, ((1 - T^*T)\cdot, \cdot)'\}$ is complete, will be pursued in Example 3.

THEOREM 3. *Suppose, as in Theorem 2, that $\{H', (\cdot, \cdot)'\}$ is a complete inner product space, that H_1 is a dense subspace of H', and that T is an operator on $\{H', (\cdot, \cdot)'\}$ such that $T(H_1)$ lies in H_1. Suppose that β is a positive number and that, with T^* the adjoint of T in $\{H', (\cdot, \cdot)'\}$,* (i) *β is not less than the operator-norm for T in $\{H', (\cdot, \cdot)'\}$ and* (ii) *$1 - T^*T/\beta^2$ is a one-to-one transformation on H_1. Then for x and y in H_1 the formula $(x, y)'' = ((1 - T^*T/\beta^2)x, y)'$ defines an inner product $(\cdot, \cdot)''$ for H_1 such that if $(\cdot, \cdot)$ denotes $(\cdot, \cdot)''$ on H_1 then for x in H_1 $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2$ converges, with limit not exceeding $(\|x\|')^2$. In case $\lim_{p\to\infty} (\|(T/\beta)^p x\|') = 0$ on H_1, then on H_1 $(x, y)' = (x, y)_{\beta,T}$ and if, in addition, $\{H_1, (\cdot, \cdot)\}$ is complete, so that $(1 - T^*T/\beta^2)^{1/2}(H_1)$ is closed in H', and $H' \neq H_1$ then the restriction of T to H_1 is not continuous in $\{H_1, (\cdot, \cdot)\}$. (Despite the conven-*

tion of the introduction, here $(\cdot, \cdot)$ *is not given beforehand).*

Proof. Note that, since $1 - T^*T/\beta^2$ is a one-to-one function when restricted to H_1, $\{H_1, (\cdot, \cdot)''\}$ is isometrically isomorphic to the subspace $(1 - T^*T/\beta^2)^{1/2}(H_1)$ of $\{H', (\cdot, \cdot)'\}$. Thus, writing $(\cdot, \cdot)$ in place of $(\cdot, \cdot)''$, $\{H_1, (\cdot, \cdot)\}$ is complete if and only if $(1-T^*T/\beta^2)^{1/2}(H_1)$ is closed in H'. Suppose n is a positive integer and each of x and y is in H_1. We have

$$\text{(C)}\qquad \begin{aligned}\sum_{p=0}^{n}&((T/\beta)^p x, (T/\beta)^p y)\\ &= \sum_{p=0}^{n}((T/\beta)^p x, (T/\beta)^p y)' \\ &\quad - \sum_{p=0}^{n}((T/\beta)^{p+1} x, (T/\beta)^{p+1} y)' \\ &= (x, y)' - ((T/\beta)^{n+1}x, (T/\beta)^{n+1}y)' .\end{aligned}$$

Hence, in case $\lim_{p\to\infty} \|(T/\beta)^p x\|' = 0$ on H_1 then on H_1 $(x, y)' = (x, y)_{\beta,T}$. Now for x in H_1 the number-sequence $\{\|(T/\beta)^p x\|'\}_{p=0}^{\infty}$ is non-increasing with limit α_x. By (C), for x in H_1

$$\begin{aligned}\sum_{p=0}^{\infty}&\|(T/\beta)^p x\|^2\\ &= (\|x\|')^2 - (\alpha_x)^2 \leqq (\|x\|')^2 .\end{aligned}$$

Suppose $H' \neq H_1$, $(x, y)' = (x, y)_{\beta,T}$ on H_1, and $\{H_1, (\cdot, \cdot)\}$ is complete. Then, by Lemma 1, in case T on H_1 is continuous in $\{H_1, (\cdot, \cdot)\}$, $\{H_1, (\cdot, \cdot)'\}$ is complete, so that H_1 is closed in H'. Since H_1 is dense in H' and $H_1 \neq H'$, H_1 is not closed in H'. Hence, T on H_1 is not continuous in $\{H_1, (\cdot, \cdot)\}$.

EXAMPLE 3. Suppose that on l^2 $\langle f, g\rangle = \sum_{p=0}^{\infty} f_p g_p^*$ and that y is the point of l^2 such that $y_0 = 1$ and for $p > 0$ $y_p = 0$. Suppose Y is the linear span of $\{y\}$, P the orthogonal projection of l^2 onto $Y^\perp$, and T the operator on l^2 such that $T(c)$ is the sequence d, with $d_0 = \sum_{p=1}^{\infty} c_p/2^{p+1}$, $d_1 = c_0$, and for $p > 1$ $d_p = c_{p-1}/2^{2p-1}$. Now $T^*(c)$ is the sequence e such that $e_0 = c_1$ and for $p > 0$ $e_p = c_0/2^{p+1} + c_{p+1}/2^{2p+1}$ and $T^*T(c)$ the sequence f such that $f_0 = c_0$ and for $p > 0$ $f_p = [\sum_{p=1}^{\infty} c_q/2^{q+1}]/2^{p+1} + c_p/2^{4p+2}$. Hence,

$$\begin{aligned}\langle(1 - T^*T)c, c\rangle&\\ &= \sum_{p=1}^{\infty}[1 - 1/2^{4p+2}]|c_p|^2 - \sum_{p=1}^{\infty}\left\{\left[\sum_{q=1}^{\infty} c_q/2^{q+1}\right]c_p^*/2^{p+1}\right\}\\ &= \sum_{p=1}^{\infty}[1 - 1/2^{4p+2}]|c_p|^2 - \left|\sum_{p=1}^{\infty} c_p/2^{p+1}\right|^2\\ &\geqq (63/64)\sum_{p=1}^{\infty}|c_p|^2 - \left[\sum_{p=1}^{\infty}|c_p|^2\right]\left[\sum_{p=1}^{\infty} 1/2^{2p+2}\right]\end{aligned}$$

$$\geqq (1/2) \sum_{p=1}^{\infty} |c_p|^2 .$$

By the above inequality,

$$\text{(D)} \qquad \langle Pc, Pc\rangle \geqq \langle (1 - T^*T)c, c\rangle \geqq (1/2)\langle Pc, Pc\rangle .$$

Since $\langle c, c\rangle - \langle Tc, Tc\rangle \geqq 0$ on l^2, the operator-norm for T does not exceed 1. However, $T^2(c) = g$, where $g_0 = c_0/4 + \sum_{p=2}^{\infty} (c_{p-1})/2^{3p}$, $g_1 = \sum_{p=1}^{\infty} c_p/2^{p+1}$, $g_2 = c_0/8$, and for $p > 2$ $g_p = (c_{p-2})/2^{4p-4}$. Computation reveals that the operator-norm for T^2 does not exceed 1/2. Hence, $\lim_{p\to\infty} \langle T^p c, T^p c\rangle$ is 0 on l^2. Note that $T(l^2) \cap Y$ is $\{0\}$. Also, with z the l^2-sequence such that for $p \geqq 0$ z_p is the sequence w with $w_q = 2^{p+1}$ or 0 accordingly as $q = p$ or not, Tz has limit y in l^2. Hence, y is in $\overline{T(l^2)}$. Since $\overline{PT(l^2)}$ is $Y^\perp$, we conclude that $T(l^2)$ is dense in l^2.

Suppose H_1 is an algebraic complement of Y in l^2 and $T(l^2)$ is a subspace of H_1. Then the formula $(x, y)'' = \langle Px, Py\rangle$ defines an inner product for H_1 such that $\{H_1, (\cdot, \cdot)''\}$ is complete. By (D), the formula $(x, y) = \langle (1 - T^*T)x, y\rangle$ defines an inner product for H_1 equivalent to $(\cdot, \cdot)''$. Of course, with $\beta = 1$, by Theorem 3 $\langle \cdot, \cdot\rangle = (\cdot, \cdot)_{\beta,T}$ on H_1. It is of interest to note that $[(x, y)'']_{\beta,T}$ $(=\sum_{p=0}^{\infty} \langle PT^p x, PT^p y\rangle)$ is equivalent to $\langle \cdot, \cdot\rangle$ on H_1. For

$$(1/2)[\|x\|'']^2 \leqq \|x\|^2 \leqq [\|x''\|^2]$$

implies

$$(1/2)[(x, x)'']_{\beta,T} \leqq (x, x)_{\beta,T} \leqq [(x, x)'']_{\beta,T}$$

on H_1.

Note 1. An argument for most of the following, known to the author through work of MacNerney [6], may be found in [1] (Lemma, p. 316), in which it is partly attributed to Friedrichs [3]. No argument will be offered here.

Suppose $\{H_1, (\cdot, \cdot)'\}$ is complete and continuously situated in $\{H, (\cdot, \cdot)\}$, in the sense that H_1 lies in H and there is a positive number c such that $\|\cdot\| \leqq c\|\cdot\|'$ on H_1, that H_1 is dense in H, and that B is the adjoint of the identity function from $\{H_1, (\cdot, \cdot)'\}$ to $\{H, (\cdot, \cdot)\}$, so that B is that linear transformation from H to H_1 such that for x in H_1 and y in H $(x, y) = (x, By)'$. Suppose C is an operator on $\{H, (\cdot, \cdot)\}$. Then

(1) B is positive definite in $\{H, (\cdot, \cdot)\}$ and the operator-norm for B in $\{H, (\cdot, \cdot)\}$ does not exceed c;

(2) with $B^{1/2}$ the positive definite square-root of B in $\{H, (\cdot, \cdot)\}$

and $B^{-1/2} = (B^{1/2})^{-1}$, $H_1 = B^{1/2}(H)$ and $(\cdot, \cdot)' = (B^{-1/2}\cdot, B^{-1/2}\cdot)$ on H_1;

(3) if $C(H)$ lies in H_1 then C is continuous from $\{H, (\cdot, \cdot)\}$ to $\{H_1, (\cdot, \cdot)'\}$;

(4) if $CB = BC$, then $CB^{1/2} = B^{1/2}C$ so that $C(H_1)$ lies in H_1 and for x and y in H, with $x \neq 0$, $\|CB^{1/2}x\|'/\|B^{1/2}x\|' = \|Cx\|/\|x\|$ and $(CB^{1/2}x, B^{1/2}y)' = (Cx, y)$; hence, the operator-norm in $\{H_1, (\cdot, \cdot)'\}$ for the restriction C_1 of C to H_1 is the operator-norm for C in $\{H, (\cdot, \cdot)\}$ and if C is nonnegative in $\{H, (\cdot, \cdot)\}$ C_1 is nonnegative in $\{H_1, (\cdot, \cdot)'\}$; and (5) if $C(H)$ is dense in H and C is one-to-one the formula $(x, y)'' = (C^{-1}x, C^{-1}y)$ defines an inner product for $C(H)$ such that $\{C(H), (\cdot, \cdot)''\}$ is complete and continuously situated in $\{H, (\cdot, \cdot)\}$ and the adjoint of the identity function from $\{C(H), (\cdot, \cdot)''\}$ to $\{H, (\cdot, \cdot)\}$ is CC^* on H, where C^* is the adjoint of C as an operator of H into itself. Moreover, for the adjoint $C^+: C(H) \to H$ of $C: H \to C(H)$ we have $CC^* = C^+C$ (or $C^+ = CC^*C^{-1}$).

THEOREM 4. *Suppose that H_1 is a dense subspace of H. Then in order that $(\cdot, \cdot)_1$ be such an inner product for H_1 that $\{H_1, (\cdot, \cdot)_1\}$ is complete and continuously situated in $\{H, (\cdot, \cdot)\}$ it is necessary and sufficient that for some operator C on $\{H, (\cdot, \cdot)\}$ and positive number d H_1 is the set of all x in H such that $\sum_{p=0}^{\infty} \|C^p x\|^2$ converges and, if each of x and y is in H_1, $(x, y)_1 = d \sum_{p=0}^{\infty} (C^p x, C^p y)$.*

Proof. The sufficiency of the condition follows from Lemma 1. To argue necessity, let e be a number such that for x in H_1 $\|x\|^2 \leqq e(\|x\|_1)^2$ and $(\cdot, \cdot)'$ be $e(\cdot, \cdot)_1$ on H_1. Then the complete inner product space $\{H_1, (\cdot, \cdot)'\}$ is continuously situated in $\{H, (\cdot, \cdot)\}$ and the operator-norm for the identity function from $\{H_1, (\cdot, \cdot)'\}$ to $\{H, (\cdot, \cdot)\}$ does not exceed 1. Hence, with B as in Note 1, the operator-norm for B in $\{H, (\cdot, \cdot)\}$ does not exceed 1. Suppose that C is $(1 - B)^{1/2}$ on H, so that $B = 1 - C^2$. Since $BC = CB$, by Note 1 $C(H_1)$ lies in H_1, the restriction of C to H_1 is nonnegative in $\{H_1, (\cdot, \cdot)'\}$, and the operator-norm for this restriction in $\{H_1, (\cdot, \cdot)'\}$, does not exceed 1. By Theorem 3, $\sum_{p=0}^{\infty} \|C^p x\|^2$ converges on H_1. (Note that $\{H', (\cdot, \cdot)'\}$ in Theorem 3 is replaced by $\{H_1, (\cdot, \cdot)'\}$ here and that $T = C$, $1 - T^*T = B$, $((1 - C^2)x, y)' = (Bx, y)' = (x, y)$.) Suppose that $\{H'', (\cdot, \cdot)''\}$ is the complete inner product space of all x in H for which $\sum_{p=0}^{\infty} \|C^p x\|^2$ converges with $(x, y)'' = \sum_{p=0}^{\infty} (C^p x, C^p y)$. Note that, since H_1 lies in H'', H'' is dense in H and $(1 - C^2)(H)$ lies in H''. Also, by Lemma 1, $C(H'')$ lies in H'' and the restriction of C to H'' is self-adjoint in H''. By Note 1, $1 - C^2$ is continuous from $\{H, (\cdot, \cdot)\}$ to $\{H'', (\cdot, \cdot)''\}$. Suppose each of x and y is in H''. Then, by Theorem 2, $(x, y) = (x, (1 - C^2)y)''$. (The $\{H', (\cdot, \cdot)'\}$ of Theorem 2 is $\{H'', (\cdot, \cdot)''\}$ now, $\beta = 1$ and $T = C$; the H_1 of Theorem 2 is H'' now.)

Suppose z is in H, x is in H'', and y is a sequence in H'' with limit z in H. Then

$$(x, z) = \lim (x, y) = \lim (x, (1 - C^2)y)'' = (x, (1 - C^2)z)'' ,$$

so that $1 - C^2$ is the adjoint of the identity function from $\{H'', (\cdot, \cdot)''\}$ to $\{H, (\cdot, \cdot)\}$. Hence, $H'' = (1 - C^2)^{1/2}(H) = H_1$ and for x and y in H_1, by Note 1,

$$\begin{aligned}(x, y)_1 &= (1/e)(x, y)' \\ &= (1/e)((1 - C^2)^{-1/2}x, (1 - C^2)^{-1/2}y) \\ &= (1/e)(x, y)'' \\ &= (1/e) \sum_{p=0}^{\infty} (C^p x, C^p y) .\end{aligned}$$

The theorem is established, taking d as $1/e$.

It may be noted that an argument for Theorem 4 could be based on a theorem, Theorem 2 of [5], of the author and Note 1. The argument given above is more closely related to the other theorems of this paper.

THEOREM 5. *Suppose that H_1 is a dense subspace of H and T is a linear transformation from H_1 to H_1. Then in order that there be an inner product $(\cdot, \cdot)_1$ for H_1 such that $\{H_1, (\cdot, \cdot)_1\}$ is complete and continuously situated in $\{H, (\cdot, \cdot)\}$ and T is continuous in $\{H_1, (\cdot, \cdot)_1\}$ it is necessary and sufficient that for some pair, β and γ, of positive numbers and some operator C on $\{H, (\cdot, \cdot)\}$ H_1 is the set of all x in H for which $\sum_{p=0}^{\infty} \|C^p x\|^2$ converges and for x in H_1 $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2 \leqq \gamma \sum_{p=0}^{\infty} \|C^p x\|^2$.*

Proof. To argue necessity, suppose b is the operator-norm for T in $\{H_1, (\cdot, \cdot)_1\}$ and $\beta = 2b$. By Theorem 4, there is an operator C in $\{H, (\cdot, \cdot)\}$ and a positive number d such that H_1 is the set of all x in H for which $\sum_{p=0}^{\infty} \|C^p x\|^2$ converges, with limit $(1/d)(\|x\|_1)^2$. Now, with $e = (1/d)^{1/2}$, $\|x\| \leqq e\|x\|_1$ and

$$\begin{aligned}\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2 &\leqq e^2 \sum_{p=0}^{\infty} (\|(T/\beta)^p x\|_1)^2 \\ &\leqq e^2(4/3)(\|x\|_1)^2 = (4/3) \sum_{p=0}^{\infty} \|C^p x\|^2 ,\end{aligned}$$

on H_1, so that the condition follows with $\gamma = 4/3$.

To argue the sufficiency of the condition, suppose $(x, y)_1 = \sum_{p=0}^{\infty} (C^p x, C^p y)$ on H_1, so that $\{H_1, (\cdot, \cdot)_1\}$ is complete and continuously situated in $\{H, (\cdot, \cdot)\}$, and set $(x, y)_2 = \sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y)$ on

H_1. Now T on H_1 is continuous in $\{H_1, (\cdot, \cdot)_1\}$ and $\|x\|_2 \leqq \gamma^{1/2}\|x\|_1$ on H_1. Suppose T is not continuous in $\{H_1, (\cdot, \cdot)_1\}$. Then, by the Closed Graph theorem, there is an H_1-sequence x with limit 0 in $\{H_1, (\cdot, \cdot)_1\}$ such that Tx has limit $y \neq 0$ in $\{H_1, (\cdot, \cdot)_1\}$. Since $\|z\|_2 \leqq \gamma^{1/2}\|z\|_1$ on H_1, x has limit 0, and Tx limit y, in $\{H_1, (\cdot, \cdot)_2\}$. But Tx has limit 0 in $\{H_1, (\cdot, \cdot)_2\}$. Thus, $y = 0$. This is a contradiction.

EXAMPLE. There is a dense subspace H_1 of H and a linear transformation T on H_1 such that $T(H_1)$ lies in H_1, the formula $(x, y)_1 = \sum_{p=0}^{\infty}(T^p x, T^p y)$ defines on H_1 an inner product such that $\{H_1, (\cdot, \cdot)_1\}$ is complete, and yet T is not a closed operator in $\{H, (\cdot, \cdot)\}$.

Suppose C is an operator on H such that the set H_2 of all x in H for which $\sum_{p=0}^{\infty}\|C^p x\|^2$ converges is a dense proper subspace of H. Suppose y is not in H_2, H_1 is the linear span of $\{y\}$ and H_2, and ϕ is the algebraic projection of H_1 onto H_2 with kernel the linear span Y of $\{y\}$. Suppose T is $C\phi + 1/2(1 - \phi)$ on H_1. Since $C(H_2)$ lies in H_2, T^p is C^p on H_2. Since the set of all x for which $\sum_{p=0}^{\infty}\|T^p x\|^2$ converges is a linear space including both Y and H_2, this set is H_1. Define $(x, y)_1$ to be $\sum_{p=0}^{\infty}(T^p x, T^p y)$ on H_1. Then H_2 is a complete subspace of $\{H_1, (\cdot, \cdot)_1\}$. Since Y is one-dimensional, $\{H_1, (\cdot, \cdot)_1\}$ is complete. Now, since y is not in H_2, $Cy \neq (1/2)y$ so that T does not lie in C. Yet the closure of T in $H \times H$ includes C. Hence, the closure of T in $H \times H$ is not a function.

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