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ON LOCAL ISOMETRIES OF FINITELY COMPACT METRIC SPACES

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By local isometries we mean mappings which locally preserve distances. Local isometries which do not increase distances are called nonexpansive local isometries. A few of the main results are:

1. Let f be a local isometry (nonexpansive local isometry) of a finitely compact metric space (M, ρ) into itself. If for each (some) $z \in M$ the sequence $\{f^n(z)\}$ is bounded, then there exists a unique decomposition of M into disjoint open sets, $M = M_0^f \cup M_1^f \cup \cdots$, such that (i) f maps M_0^f injectively into itself, and (ii) $f(M_{i+1}^f) \subset M_i^f$ for each $i = 0, 1, \cdots$. Moreover, f maps M_0^f homeomorphically (isometrically) onto itself.

2. Let f be a nonexpansive local isometry (local isometry) of a connected (convex) finitely compact metric space (M, ρ) into itself. If for some $z \in M$ the sequence $\{f^n(z)\}$ is bounded, then f is an isometry onto.

1. Introduction. Let f be a mapping of a metric space (M, ρ) into a metric space (N, σ). We will call f a *local isometry* if for each $z \in M$ there is a neighborhood U_z of z such that $\sigma(f(x), f(y)) = \rho(x, y)$ for all $x, y \in U_z$. If f is a local isometry and also a nonexpansive mapping (i.e., $\sigma(f(x), f(y)) \leqq \rho(x, y)$ for all $x, y \in M$), we will say that f is a *nonexpansive local isometry*.

A metric space (M, ρ) is said to be *finitely compact* [2] if each bounded and closed subset of M is compact.

The purpose of this paper is to extend the results of the author's paper [4] to those local isometries f of a finitely compact metric space (M, ρ) into itself which have the property that for each $z \in M$ the sequence $\{f^n(z)\}$ is bounded. In § 2 we give some more notation and preliminary lemmas. Section 3 contains the main results. Roughly speaking, the main theorem is: Let f be a local isometry (nonexpansive local isometry) of a finitely compact metric space (M, ρ) into itself. If for each (for some) $z \in M$ the sequence $\{f^n(z)\}$ is bounded, then there exists a unique decomposition of M into disjoint open sets, $M = M_0^f \cup M_1^f \cup \cdots$, such that (i) f maps M_0^f injectively into itself, (ii) $f(M_i^f) \subset M_{i-1}^f$ for each $i \geqq 1$. Moreover, f maps M_0^f homeomorphically (isometrically) onto itself.

It should be noted that open surjective local isometries were studied by Busemann [2], [3], Kirk [5], [6], [7] and Szenthe [8], [9], [10], in the special case where (M, ρ) is a G-space (Busemann [2] called them "locally isometric mappings"). In [5] Kirk proved that if an open local isometry f of a G-space (M, ρ) onto itself has a fixed point, then f is an isometry (from which it follows that if the isometries of (M, ρ) onto itself form a transitive group, then each open surjective local isometry is an isometry). Later Kirk [6] proved that if an open local isometry f of a G-space (M, ρ) onto itself has the property that for some $z \in M$ the sequence $\{f^n(z)\}$ is bounded, then f is an isometry.

In § 4 and § 5 of the present paper, by using the results of § 3, we extend the above results of Kirk to the case of general local isometries of finitely compact metric spaces.

2. Preliminaries.

(2.1) DEFINITION. Let ρ_i, $i = 0, 1$, be metrics on a set M. We shall say that ρ_1 *is locally identical* with ρ_0 if the identity mapping, id_M, of M is a local isometry of (M, ρ_0) into (M, ρ_1). We shall say that ρ_1 and ρ_0 *are locally identical* if ρ_i is locally identical with ρ_j, for all $i, j = 0, 1$.

(2.2) DEFINITION. Let f be a mapping of a metric space (M, ρ) into itself. Then the function ρ_f defined by

$$\rho_f(x, y) = \sup_{n \geqq 0} \rho(f^n(x), f^n(y)) \quad \text{for all} \quad x, y \in M ,$$

(where $f^0 = \mathrm{id}_M$, $f^{n+1} = f \circ f^n$) is called the *induced metric* on M.

(2.3) REMARKS. (i) Let ρ_i, $i = 0, 1$, be metrics on a set M such that ρ_1 and ρ_0 are locally identical. Then ρ_1 and ρ_0 are topologically equivalent. If (M, ρ_0) is finitely compact and $\rho_1 \geqq \rho_0$, then (M, ρ_1) is also finitely compact. If f is a local isometry of (M, ρ_0) into itself, then f is also a local isometry of (M, ρ_1) into itself.

(ii) Let f be a mapping of a metric space (M, ρ) into itself such that for each $z \in M$ the sequence $\{f^n(z)\}$ is bounded. Then for each $x, y \in M$, $\rho_f(x, y) < \infty$, and hence the induced metric, ρ_f, is a metric on the set M such that

(1) $\rho_f \geqq \rho$,

(2) f is a nonexpansive mapping of the metric space (M, ρ_f) into itself , and

(3) $\rho_f = \rho$ if and only if f is a nonexpansive mapping of (M, ρ) into itself .

In [4] we proved the following theorem ((4.3) of [4]).

(2.4) THEOREM. *Let f be a local isometry of a compact metric* *space (M, ρ) into itself. Then there exists a unique decomposition of M into disjoint open sets,*

$$M = M_0^f \cup \cdots \cup M_n^f \quad (0 \leqq n) ,$$

such that (i) $f(M_0^f) = M_0^f$, (ii) $f(M_i^f) \subset M_{i-1}^f$ *and* $M_i^f \neq \emptyset$ *for each* i, $1 \leqq i \leqq n$. *Moreover, the induced metric* ρ_f *is a metric on* M *such that* ρ_f *and* ρ *are locally identical and* f *is a nonexpansive local isometry of* (M, ρ_f) *into itself which maps* M_0^f *isometrically onto itself.*

From this theorem we have

(2.5) COROLLARY. *Let f be a one-to-one local isometry of a compact metric space (M, ρ) into itself. Then $f(M) = M$.*

Proof. If f is one-to-one, then by (2.4), $M = M_0^f$ and hence $f(M) = M$.

REMARK. If f is a local isometry of a compact metric space (M, ρ) into itself and if N is a compact subset of M such that $f(N) \subset N$, then the restriction of f to N, f/N, is also a local isometry. For convenience, $N = N_0^f \cup \cdots \cup N_{n(N)}^f$ will denote the decomposition of N defined by (2.4) for f/N.

(2.6) PROPOSITION. *Let f be a local isometry of a compact metric space (M, ρ) into itself. If N is a compact subset of M such that $f(N) \subset N$, then*

$$N_i^f = N \cap M_i^f \quad \textit{for each} \quad i = 0, \cdots, n(N) ,$$

where $n(N) = \max\{i \geqq 0: N \cap M_i^f \neq \emptyset\}$.

Proof. By (2.4), it is sufficient only to show that $f(N \cap M_0^f) = N \cap M_0^f$. However, it follows from (2.4) that f maps $N \cap M_0^f$ isometrically into itself. Hence, by (2.5), $f(N \cap M_0^f) = N \cap M_0^f$ as desired.

We will need the following.

(2.7) LEMMA. *Let f be a local isometry of a metric space (N, ρ) into itself. If N is a compact subset of M, then there exists a number $\delta > 0$ such that for each $z \in N$,*

$$(4) \qquad \rho(f(x), f(y)) = \rho(x, y) ,$$

for all $x, y \in S_\rho(z, \delta) = \{p \in M: \rho(z, p) < \delta\}$.

The straightforward verification of (2.7) is omitted.

The convexity in this paper is to be understood in the sense of Menger (cf. [1, p. 40]). A subset N of a metric space (M, ρ) is, accordingly, convex if for each two distinct points $x, y \in N$, there exists a point $z \in N$, $z \neq x, y$, such that $\rho(x, y) = \rho(x, z) + \rho(z, y)$.

Also, we will use

(2.8) LEMMA. *If f is a local isometry of a convex and complete metric space (M, ρ) into itself, then f is a nonexpansive local isometry.*

Proof. Let x and y be given points of M such that $x \neq y$. Since M is convex and complete, by a theorem of Menger (cf. [1, p. 41]) there exists a metric segment $L \subset M$ whose extremities are x and y; that is, a subset isometric to an interval of length $\rho(x, y)$. Since L is compact, it follows that there exists a finite sequence $z_0, z_1, \cdots, z_k$ of points of L such that $z_0 = x$, $z_k = y$ and

$$\rho(f(z_i), f(z_{i+1})) = \rho(z_i, z_{i+1}) \quad \text{for each} \quad i = 0, \cdots, k-1$$

and

$$\rho(x, y) = \sum_{i=0}^{k-1} \rho(z_i, z_{i+1}) .$$

Thus,

$$\rho(f(x), f(y)) \leqq \sum_{i=0}^{k-1} \rho(f(z_i), f(z_{i+1})) = \sum_{i=0}^{k-1} \rho(z_i, z_{i+1}) = \rho(x, y) .$$

This proves that f is a nonexpansive mapping, and hence a non-expansive local isometry.

3. Local isometries and decomposition theorems. We shall now prove the following extension of (2.4).

(3.1) THEOREM. *Let f be a local isometry of a finitely compact metric space (M, ρ) into itself. If for each $z \in M$ the sequence $\{f^n(z)\}$ is bounded, then there exists a unique decomposition of M into disjoint open sets,*

$$M = M_0^f \cup M_1^f \cup \cdots , \tag{5}$$

such that

$$f \text{ maps } M_0^f \text{ injectively into itself} , \tag{6}$$

$$f(M_i^f) \subset M_{i-1}^f \quad \text{for each} \quad i = 1, 2, \cdots . \tag{7}$$

Moreover, the induced metric, ρ_f, is a metric on M such that ρ_f and ρ are locally identical, (M, ρ_f) is a finitely compact metric space and f is a nonexpansive local isometry of (M, ρ_f) into itself which maps M_0^f isometrically onto itself.

Proof. In the proof, for each $A \subset M$ and $\delta > 0$, $S_\rho(A, \delta)$ is the δ-ball in M about A and $\operatorname{cl} A$ ($\operatorname{Int} A$) is the closure (interior) of A. For each $z \in M$ we denote: $c(z) = \operatorname{cl}\{f^n(z): n \geqq 0\}$.

We first define a sequence A_n, $n = 0, 1, \cdots$, of compact subsets of M such that

$$(8) \qquad f(A_n) \subset A_n \quad \text{for each} \quad n = 0, 1, \cdots,$$

$$(9) \qquad A_n \subset \operatorname{Int} A_{n+1} \quad \text{for each} \quad n = 0, 1, \cdots,$$

$$(10) \qquad \bigcup_{n=0}^{\infty} A_n = M .$$

For each $z \in M$, let $\delta_z > 0$ be a number defined by (2.7) for the compact set $c(z)$ and let $V_z = S_\rho(c(z), \delta_z)$. Thus, for each $z \in M$, V_z is an open and bounded subset of M and using (4) and the fact that $f(c(z)) \subset c(z)$, we have $f(V_z) \subset V_z$. Since (M, ρ) has a countable base of neighborhoods, there exists a sequence z_n, $n = 0, 1, \cdots$, of points of M such that $\bigcup_{n=0}^{\infty} V_{z_n} = M$. Define the sets A_n, $n = 0, 1, \cdots$, inductively, as follows: $A_0 = \operatorname{cl} V_{z_0}$ and $A_{n+1} = \bigcup_{i=0}^{k(n)} \operatorname{cl} V_{z_i}$, where $k(n)$ is an integer such that $k(n) > n$ and $A_n \subset \bigcup_{i=0}^{k(n)} V_{z_i}$. Clearly, the sets A_n, $n = 0, 1, \cdots$, satisfy conditions (8), (9) and (10), and are compact.

It follows now from (2.4), that for each $n \geqq 0$, there exists a sequence $(A_n)_i^f$, $i = 0, 1, \cdots$, of disjoint subsets of A_n such that

$$(11) \qquad (A_n)_i^f \cap \operatorname{Int} A_n \text{ is open}, \quad \text{for each} \quad i = 0, 1, \cdots,$$

$$(12) \qquad \bigcup_{i=0}^{\infty} (A_n)_i^f = A_n ,$$

$$(13) \qquad f \text{ maps } (A_n)_0^f \text{ injectively into itself},$$

$$(14) \qquad f((A_n)_i^f) \subset (A_n)_{i-1}^f , \quad \text{for each} \quad i = 1, 2, \cdots .$$

By (2.6), we have

$$(15) \qquad (A_n)_i^f = A_n \cap (A_{n+1})_i^f , \quad \text{for all} \quad n, i = 0, 1, \cdots .$$

Now, for each $i = 0, 1, \cdots$, we define the set M_i^f as follows:

$$M_i^f = \bigcup_{n=0}^{\infty} (A_n)_i^f .$$

Then, by (15) and the fact that $(A_n)_i^f$, $i \geqq 0$, are disjoint, the sets M_i^f, $i \geqq 0$, are disjoint. By (9) and (15),

$$(A_n)_i^f \subset (A_{n+1})_i^f \cap \operatorname{Int} A_{n+1} \subset (A_{n+1})_i^f ,$$

hence,

$$M_i^f = \bigcup_{n=0}^{\infty} ((A_{n+1})_i^f \cap \operatorname{Int} A_{n+1}) , \quad \text{for each} \quad i = 0, 1, \cdots ,$$

and therefore, by (11), the sets M_i^f, $i \geqq 0$, are open. By (10) and (12),

$$\bigcup_{i=0}^{\infty} M_i^f = \bigcup_{i,n=0}^{\infty} (A_n)_i^f = \bigcup_{n=0}^{\infty} A_n = M ,$$

and it follows from (13), (14) and (15) that the sets M_i^f, $i \geqq 0$, satisfy conditions (6) and (7). This proves the existence of the desired decomposition of M.

In order to prove the uniqueness, it is sufficient only to show that for each decomposition of M into disjoint open sets, $M = \bigcup_{i=0}^{\infty} M_i$, conditions (6) and (7) imply

$$M_0 = \{z \in M: f(c(z)) = c(z)\} . \tag{16}$$

Let us assume, $M = \bigcup_{i=0}^{\infty} M_i$ is a decomposition of M into disjoint open sets, satisfying conditions (6) and (7). If $z \in M_0$, then (6) implies that the restriction of f to $c(z)$ is a one-to-one local isometry of $c(z)$ into itself. Since $c(z)$ is compact, it follows from (2.5) that $f(c(z)) = c(z)$. Conversely, if $z \notin M_0$, then $z \in M_n$ for some $n \geqq 1$. Using (7) and the fact that M_i, $i \geqq 0$, are disjoint and open, we obtain

$$f(c(z)) \subset c(f(z)) \subset M_0 \cup \cdots \cup M_{n-1} ,$$

hence $z \in c(z) \backslash c(f(z))$, i.e., $c(z) \neq c(f(z))$. Therefore (16) follows as desired.

Finally, by (ii) of (2.3), the induced metric, ρ_f, is a metric on M and it follows from (8), (9), (10) and (2.4) that ρ_f and ρ are locally identical (cf. also (1)). Hence, by (1) and (i) of (2.3), the metric space (M, ρ_f) is finitely compact and, by (2), f is a nonexpansive local isometry of (M, ρ_f) into itself. It follows from (2.4) and (15) and the definition of M_0^f that f maps M_0^f isometrically onto itself with respect to the metric ρ_f. This completes the proof.

(3.2) Remark. Let f be a nonexpansive mapping of a metric space (M, ρ) into itself. If for some $z \in M$ the sequence $\{f^n(z)\}$ is bounded, then for each $x \in M$ the sequence $\{f^n(x)\}$ is bounded.

Indeed, since f is nonexpansive, then for all $x, z \in M$ and each $i = 0, 1, \cdots$, we have

$$\rho(f^i(x), \{f^n(z)\}) \leqq \rho(f^i(x), f^i(z)) \leqq \rho(x, z) ,$$

hence, if $\{f^n(z)\}$ is bounded, then also $\{f^n(x)\}$ is bounded.

The following theorem is an immediate consequence of (3.1), (3.2) and (3).

(3.3) THEOREM. *Let f be a nonexpansive local isometry of a finitely compact metric space (M, ρ) into itself. If for some $z \in M$ the sequence $\{f^n(z)\}$ is bounded, then there exists a unique decomposition of M into disjoint open sets,*

$$M = M_0^f \cup M_1^f \cup \cdots ,$$

such that (i) *f maps M_0^f injectively into itself,* (ii) *$f(M_i^f) \subset M_{i-1}^f$ for each $i = 1, 2, \cdots$. Moreover, f maps M_0^f isometrically onto itself.*

We have the following corollaries

(3.4) COROLLARY. *Let f be a local isometry of a finitely compact metric space (M, ρ) into itself. If for each $z \in M$ the sequence $\{f^n(z)\}$ is bounded, then the following are equivalent:*

(i) *f is one-to-one,*

(ii) *f is a homeomorphism of M onto itself,*

(iii) *f is an isometry with respect to the induced metric ρ_f.*

Proof. The proof follows from (3.1), since each of (i)-(iii) is equivalent to $M_0^f = M$.

(3.5) COROLLARY. *Let f be a nonexpansive local isometry of a finitely compact metric space (M, ρ) into itself. If for some $z \in M$ the sequence $\{f^n(z)\}$ is bounded, then the following are equivalent:*

(i) *f is one-to-one,*

(ii) *f is a homeomorphism of M onto itself,*

(iii) *f is an isometry onto.*

Proof. This follows from (3.3) (or from (3.4) and (3)).

4. Some consequences. As an immediate consequence of (3.1), we get

(4.1) THEOREM. *Let f be a local isometry of a connected finitely compact metric space (M, ρ) into itself. If for each $z \in M$ the sequence $\{f^n(z)\}$ is bounded, then the induced metric, ρ_f, is a metric on M such that ρ_f and ρ are locally identical, (M, ρ_f) is a finitely compact metric space and f is an isometry of (M, ρ_f) onto itself. In particular, f is a homeomorphism of M onto itself.*

As an immediate consequence of (3.3), we get

(4.2) THEOREM. *Let f be a nonexpansive local isometry of a connected finitely compact metric space (M, ρ) into itself. If for some $z \in M$ the sequence $\{f^n(z)\}$ is bounded, then f is an isometry onto.*

The corresponding statement concerning local isometries of convex finitely compact metric spaces is stated next.

(4.3) THEOREM. *Let f be a local isometry of a convex finitely compact metric space (M, ρ) into itself. If for some $z \in M$ the sequence $\{f^n(z)\}$ is bounded, then f is an isometry onto.*

Proof. Since (M, ρ) is convex and complete, by (2.8), f is a nonexpansive local isometry. Hence, our assertion follows from (4.2).

Finally, we note the following special cases of (4.2) and (4.3).

(4.4) COROLLARY. *Let f be a nonexpansive local isometry of a connected finitely compact metric space (M, ρ) into itself. If f has a fixed (periodic) point, then f is an isometry onto.*

(4.5) COROLLARY. *Let f be a local isometry of a convex finitely compact metric space (M, ρ) into itself. If f has a fixed (periodic) point, then f is an isometry onto.*

REMARK. Theorems (4.2) and (4.3) extend the result of [6]; Corollaries (4.4) and (4.5) extend Theorem 1 of [5] to the case of general local isometries of finitely compact metric spaces.

5. A condition on (M, ρ) under which local isometries are isometries. In this section, by using (3.3), we extend Theorem 3 of [5]. First, we shall prove

(5.1) PROPOSITION. *Let f be a nonexpansive local isometry of a finitely compact metric space (M, ρ) into itself. If (M, ρ) has a transitive group of isometries, then there exists a sequence N_n, $n = 0, 1, \cdots$, of open and closed subsets of M such that $M = \bigcup_{n=0}^{\infty} N_n$ and for each $n \geqq 0$, f maps N_n isometrically onto an open closed subset of M.*

Proof. Let $z \in M$. Then, by assumption, there exists an isometry g_z of (M, ρ) onto itself such that $g_z(f(z)) = z$. Since $g_z \circ f$ is a nonexpansive local isometry, it follows from (3.3) that there is an open and closed set N_z such that $z \in N_z$ and $g_z \circ f$ maps N_z isometrically onto itself. Hence $g_z^{-1}(N_z)$ is open and closed, and f maps N_z isometrically onto $g_z^{-1}(N_z)$. Since (M, ρ) is separable, our assertion follows.

The next two results follow immediately from (5.1) and (2.8) (or, in a direct fashion, from (4.4) and (4.5)).

(5.2) THEOREM. *If a connected finitely compact metric space (M, ρ) has a transitive group of isometries, then each nonexpansive local isometry of (M, ρ) into itself is an isometry onto.*

(5.3) THEOREM. *If a convex finitely compact metric space (M, ρ) has a transitive group of isometries, then each local isometry of (M, ρ) into itself is an isometry onto.*

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