PACIFIC JOURNAL OF MATHEMATICS
Vol. 103, No. 2, 1982

THE TWO-OBSTACLE PROBLEM FOR THE BIHARMONIC OPERATOR

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In this work we consider a two-obstacle problem for the plate, namely, the problem of finding a minimizer u of

$$\int_{\Omega} |\Delta v|^2 dx\ ,\ \textbf{subject to}\ (v-h) \in H_0^2(\Omega)\ ,\qquad \phi \leqq v \leqq \psi$$

where Ω is a bounded domain in R^n; $n = 2, 3$. We prove that $u \in C^{1,1}$ and that, in general, $u \notin C^2$.

1. The main results. Let Ω be a bounded domain in R^n ($n = 2, 3$) with $C^{2+\alpha}$ boundary $\partial\Omega$, where $0 < \alpha < 1$. Let $h(x)$ be a function in $C^{2+\alpha}(\bar{\Omega})$, and let $\phi(x)$, $\psi(x)$ be functions in $C^4(\bar{\Omega})$ satisfying

$$\begin{aligned} &\phi \leqq \psi \qquad \text{in} \quad \Omega\ , \\ &\phi < h < \psi \quad \text{on} \quad \partial\Omega\ . \end{aligned} \tag{1.1}$$

Then the set

$$K = \{v; (v-h) \in H_0^2(\Omega), \phi \leqq v \leqq \psi \text{ a.e.}\}$$

is nonempty.

Consider the variational inequality: find u such that

$$\min_{v \in K} \int_{\Omega} |\Delta v|^2 dx = \int_{\Omega} |\Delta u|^2 dx\ , \qquad u \in K\ . \tag{1.2}$$

By standard results [4] [5] this problem has a unique solution. We shall prove:

THEOREM 1.1. *u belongs to $C^{1,1}(\Omega)$.*

That means that $\nabla^2 u \in L^\infty(\Omega)$.

We shall also show that, in general,

$$u \notin C^2 \quad \text{locally}\ . \tag{1.3}$$

For the corresponding variational inequality (for Δ^2) with one obstacle only (i.e., $u \geqq \phi$ instead of $\phi \leqq u < \psi$) it was proved by Caffarelli and Friedman [1] that, for $n \geqq 2$, $u \in C^{1,1}$ locally and, for $n = 2$, $u \in C^2$ locally.

Notice that if in Theorem 1.1 $\phi < \psi$ in a subdomain Ω_0 of Ω, then the coincidence sets $\{u = \phi\}$, $\{u = \psi\}$ are disjoint in Ω_0 (since u

is continuous). Thus (1.3) can only hold (at least for $n = 2$) in a neighborhood of a point x^0 for which $\phi(x^0) = \psi(x^0)$.

In §2 we shall prove that $\Delta u \in L^\infty(\Omega)$ and in §3 we shall complete the proof of Theorem 1.1. An example for which (1.3) holds is given in §4.

2. Δu is bounded. Set

$$\phi_\varepsilon = \phi - \varepsilon\,, \qquad \varepsilon > 0\,,$$
$$K_\varepsilon = \text{the set } K \text{ with } \phi \text{ replaced by } \phi_\varepsilon\,.$$

Denote by u_ε the solution of the variational inequality (1.2) with K replaced by K_ε. Clearly,

$$\int_\Omega |\Delta u_\varepsilon|^2 dx \leqq C\,, \qquad C \text{ independent of } \varepsilon\,.$$

Since $n \leqq 3$ we can apply Sobolev's inequality to deduce that

(2.1) u_ε is uniformly continuous in x, with modulus of continuity independent of ε.

It follows that the coincidence sets

$$I_\varepsilon^+ = \{u_\varepsilon = \psi\}\,, \qquad I_\varepsilon^- = \{u_\varepsilon = \phi\}\,,$$

are closed disjoint sets. Furthermore, by (1.1), (2.1),

$$d(I_\varepsilon^\pm, \partial\Omega) \geqq \delta > 0\,, \qquad \delta \text{ independent of } \varepsilon\,, \tag{2.2}$$

where

$$d(A, B) = \text{dist.}\,(A, B)\,.$$

We now claim that

$$u_\varepsilon \longrightarrow u \text{ uniformly in } \Omega\,, \qquad \text{as } \varepsilon \longrightarrow 0\,. \tag{2.3}$$

Indeed for any sequence $\varepsilon_m \to 0$ there is a subsequence $\varepsilon_{m'} \to 0$ such that

$$u_{\varepsilon_{m'}} \longrightarrow \bar{u} \qquad \text{weakly in } H^2(\Omega)\,.$$

The variational inequality for $u_{\varepsilon_{m'}}$ can be written in the form (Minty's lemma)

$$\int_\Omega \Delta v \cdot \Delta(v - u_{\varepsilon_{m'}}) \geqq 0 \qquad \text{for every } v \in K_{\varepsilon_{m'}}\,.$$

Taking $m' \to \infty$ we get

$$\int_\Omega \Delta v \cdot \Delta(v-u) \geqq 0 \qquad \text{for every} \quad v \in K\,,$$

so that u is the solution u of (1.2); this completes the proof of (2.3).

Since I_ε^+, I_ε^- are disjoint closed sets, there is a version of Δu which is subharmonic and upper semicontinuous in $\Omega\backslash I_\varepsilon^+$ and superharmonic and lower semicontinuous in $\Omega\backslash I_\varepsilon^-$; this is proved exactly as in [1].

Set

$$\Omega_r = \{x \in \Omega;\, d(x, \partial\Omega) > r\}\,, \qquad r > 0\,.$$

Let ζ be a $C_0^\infty(\Omega)$ function such that

$$\zeta = 1 \quad \text{in} \quad \Omega_{\delta/2}\,, \qquad \zeta = 0 \quad \text{in} \quad \Omega\backslash\Omega_{\delta/4}\,,$$

$$0 \leqq \zeta \leqq 1 \quad \text{elsewhere}\,; \qquad \delta \text{ as in (2.2)}\,.$$

We can represent Δu_ε as in [1; (3.8)] in the form

$$\Delta u_\varepsilon(x) = -\int_{\Omega_\delta} V(x, y)d\mu(y) + \gamma(x) \tag{2.4}$$

where $|\gamma(x)|$ is a bounded function in $\Omega_{\delta/2}$, with an upper bound independent of ε, $d\mu = \Delta^2 u_\varepsilon$ and V is Green's function for $-\Delta$, for a ball containing $\bar\Omega$; here we have used the fact (which follows from (2.2)) that $\Delta^2 u_\varepsilon = 0$ in $\Omega\backslash\Omega_\delta$ and, consequently, the first two derivatives of u_ε are bounded in $\Omega_{\delta/2}$ by a constant independent of ε.

Notice that μ is a signed measure; it can be written as a difference $\mu_1 - \mu_2$ of two positive measures, where μ_1 is $\Delta^2 u_\varepsilon$ supported on I_ε^- and μ_2 is $\Delta^2 u_\varepsilon$-supported on I_ε^+.

Introduce the notation:

$$B(y, \rho) = \{x;\, |x-y| < \rho\}\,, \qquad B(\rho) = B(0, \rho)\,,$$

$$S_\rho(y) = \partial B(y, \rho)\,, \qquad S_\rho = \partial B(\rho)\,,$$

$$|S_\rho| = \text{surface area of } S_\rho\,.$$

We reason as in [1]. Let $x_0 \in I_\varepsilon^-$. Then

$$u_\varepsilon(x_0) = \frac{1}{|S_\delta|}\int_{S_\delta(x_0)} u_\varepsilon - \int_{B_\delta(x_0)} G\Delta u_\varepsilon\,,$$

$$\phi_\varepsilon(x_0) = \frac{1}{|S_\delta|}\int_{S_\delta(x_0)} \phi_\varepsilon - \int_{B_\delta(x_0)} G\Delta \phi_\varepsilon\,.$$

Here G denotes

$$C\Big(\frac{1}{r} - \frac{1}{\delta}\Big) \qquad \text{in} \quad R^3\,,$$

$$C \log \frac{r}{\delta} \qquad \text{in} \quad R^2$$

for some constant $C > 0$. Since

$$u_\varepsilon(x_0) = \phi_\varepsilon(x_0)$$

$$\int_{S_\delta(x_0)} u_\varepsilon \geqq \int_{S_\delta(x_0)} \phi_\varepsilon$$

and

$$\frac{1}{|S_\delta|} \int_{S_\delta(x_0)} \Delta u_\varepsilon$$

is a monotone function of δ, for $\delta \to 0$, we get

$$\Delta u_\varepsilon(x_0) \geqq \Delta\phi_\varepsilon(x_0) \qquad \text{if} \quad x_0 \in \operatorname{supp} \mu_1 . \tag{2.5}$$

Similarly

$$\Delta u_\varepsilon \leqq \Delta\psi_\varepsilon \qquad \text{on} \quad \operatorname{supp} \mu_2 . \tag{2.6}$$

The function

$$\hat{V}(x) = \int_{\Omega_\delta} V(x, y) d\mu(y) \tag{2.7}$$

satisfies, by (2.4)-(2.6),

$$\begin{aligned} \hat{V}(x) &\leqq C \qquad \text{on} \quad \operatorname{supp} \mu_1 , \\ \hat{V}(x) &\geqq -C \qquad \text{on} \quad \operatorname{supp} \mu_2 \end{aligned}$$

where C is a constant independent of ε. As in the proofs of Theorems 1.6, 1.10 of [3], we then have

$$\limsup_{d(x, \operatorname{supp} \mu_1) \to 0} \hat{V}(x) \leqq C , \qquad \limsup_{d(x, \operatorname{supp} \mu_2) \to 0} \hat{V}(x) \geqq -C .$$

Hence, by the maximum principle,

$$|\hat{V}(x)| \leqq C \qquad \text{in} \quad \Omega_\delta$$

and (2.4) gives

$$|\Delta u_\varepsilon| \leqq C \qquad \text{in} \quad \Omega_{\delta/2}$$

with another C. Taking $\varepsilon \to 0$ and recalling (2.3), we obtain:

LEMMA 2.1. *Δu is in $L^\infty(\Omega)$.*

3. $u \in C^{1,1}$. Let

$$w \in H^2(\Omega) , \qquad \Delta w \in L^\infty(\Omega) , \qquad w \geqq 0 ,$$

and set

$$J = \{x \in \Omega; w(x) = 0\}, \qquad \|\Delta w\|_{L^\infty(\Omega)} \leqq M_0 .$$

LEMMA 3.1. *There exists a constant M depending only on M_0 such that if $x_0 \in J$ then*

$$(3.1) \quad |w(x)| \leqq M|x - x_0|^2, \qquad |\nabla w(x)| \leqq M|x - x_0| \qquad \text{if } x \in B(x_0, \rho/2)$$

where $\rho = d(x_0, \partial\Omega)$.

Proof. Take for simplicity $x_0 = 0$ and consider the function

$$w_\rho(x) = \frac{1}{\rho^2} w(\rho x) \qquad \text{in} \quad B(1) .$$

Then

$$w_\rho(0) = 0, \qquad |\Delta w_\rho(x)| = |(\Delta w)(\rho x)| \leqq M_0 .$$

Consider the function

$$\lambda(x) = -\int_{B(1)} V(x-y)\Delta w_\rho(y)dy \qquad \text{in} \quad B(1)$$

when V is Green's function for $-\Delta$ in $B(1)$. Then

$$\Delta\lambda = \Delta w_\rho$$

and

$$(3.2) \qquad \|\lambda\|_{L^\infty(B(1))} \leqq C_1, \qquad |\nabla\lambda|_{L^\infty(B(1))} \leqq C_1$$

where the C_i will be used to denote constants depending only on M_0.

The function

$$(3.3) \qquad z = w_\rho - \lambda$$

is harmonic in $B(1)$ and

$$|z(0)| = |\lambda(0)| \leqq C_1, \qquad z \geqq -C_1 .$$

By Harnack's inequality we obtain

$$|z(x)| \leqq C_2 \qquad \text{in} \quad B(3/4) ;$$

therefore

$$|\nabla z(x)| \leqq C_3 \qquad \text{in} \quad B(1/2) .$$

Recalling (3.2), (3.3) are get

$$|w_\rho(x)| \leqq M, \qquad |\nabla w_\rho(x)| \leqq M \qquad \text{in} \quad B(1/2)$$

and (3.1) follows.

Set

$$\begin{aligned} I^- &= \{x \in \Omega;\, u(x) = \phi(x)\}\ , \\ I^+ &= \{x \in \Omega;\, u(x) = \psi(x)\}\ , \\ I &= I^- \cup I^+\ . \end{aligned}$$

Since $u \in C(\bar{\Omega})$,

$$d(I, \partial\Omega) > 0\ . \tag{3.4}$$

In view of Lemma 2.1 we can apply Lemma 3.1 to $u - \phi$ and conclude, upon using also (3.4), that

$$\begin{aligned} |(u - \phi)(x)| &\leqq M(d(x, I^-))^2\ , \\ |\nabla(u - \phi)(x)| &\leqq M d(x, I^-)\ . \end{aligned} \tag{3.5}$$

Similar estimates hold for $u - \psi$.

LEMMA 3.2. *There exists a positive constant N such that*

$$|D^2 u(x)| \leqq N \qquad in \quad \Omega\backslash I\ . \tag{3.6}$$

Proof. Let $x^0 \in \Omega_\delta\backslash I$, $d(x^0, I) < d(I, \partial\Omega)$. Suppose for definiteness that

$$d(x^0, I) = d(x^0, I^-)\ .$$

Consider the function

$$w_d(x) = \frac{1}{d^2}(u - \phi)(d(x - x^0)) \qquad (d = d(x^0, I))$$

and take for simplicity $x^0 = 0$. Then, by (3.5),

$$\left.\begin{aligned} |w_d(x)| &\leqq M \\ |\nabla w_d(x)| &\leqq M \end{aligned}\right\} \quad \text{in} \quad B(1)\ .$$

Also

$$\Delta^2 w_d(x) = \Delta^2 \phi(dx)\ . \tag{3.7}$$

By elliptic estimates it then follows that

$$|D^2 w_d(x)| \leqq C \qquad \text{in} \quad B(1/2)\ . \tag{3.8}$$

Thus

$$|D^2(u - \phi)(x)| \leqq C \qquad \text{in} \quad B\left(x^0, \frac{1}{2}d\right)$$

and consequently,

$$|D^2 u(x)| \leqq C \qquad \text{if} \quad |x - x^0| < \frac{1}{2} d(x^0, I)$$

provided $d(x^0, I) < d(I, \partial\Omega)$. Recalling (3.4), the assertion (3.6) follows.

We can now complete the proof of Theorem 1.1. Let e_1 be the unit vector in the direction of the positive x_1-axis and $h = h_1 e_1$, h_1 real. Consider the finite difference

$$D_h^2 u(x) = \frac{u(x+h) + u(x-h) - 2u(x)}{2h_1^2}$$

for $x \in \Omega$ and $|h_1|$ small enough.

If $d(x, I) < 4|h_1|$ then we choose a point $x_0 \in I$ with $|x - x_0| = d(x, I)$ and suppose, for definiteness, that $x_0 \in I^-$. Using (3.5) we get

$$\begin{aligned} |D_h^2(u-\phi)(x)| &\leqq \frac{1}{h_1^2}\{|u(x+h) - \phi(x+h)| + |u(x-h) - \phi(x-h)| \\ &\quad + 2|u(x) - \phi(x)|\} \\ &\leqq \frac{1}{h_1^2} C h_1^2 , \end{aligned}$$

so that

$$|D_h^2 u(x)| \leqq C + |D_h^2 \phi(x)| .$$

If $d(x, I) > 4|h_1|$ then

$$|D_h^2 u(x)| = |D_{x_1 x_1} u(\bar{x})|$$

for some $\bar{x}$ in $\Omega \backslash I$, and $d(\bar{x}, I) < 2d(x, I)$. Using Lemma 3.2 we obtain

$$|D_h^2 u(x)| \leqq M .$$

We have thus proved that for any $x \in \Omega$

$$|D_h^2 u(x)| \leqq C \qquad \text{if} \quad |h_1| \quad \text{is small enough} ,$$

where C is a constant independent of x, h_1. This implies that

$$\frac{\partial^2 u}{\partial x_1^2} \in L^\infty(\Omega) .$$

Similarly one can show that each second derivative of u belongs to $L^\infty(\Omega)$.

REMARK 1. The assumption $\phi, \psi \in C^4(\bar{\Omega})$ was used in order to deduce (3.8) from (3.7). One can actually justify this derivation assuming merely that $\phi, \psi \in C^{2+\alpha}(\bar{\Omega})$.

REMARK 2. The assumption $n = 2, 3$ made in Theorem 1.1 is used only at one point, namely, in deducing (2.1). The remaining arguments are all valid for any $n \geqq 2$.

REMARK 3. Theorem 1.1 extends, with obvious modifications in the proof, to the case $n = 1$.

4. Counterexample. We shall show by a counterexample that, in general, u is not in C^2, locally.

Take Ω the unit ball in R^n, $n \geqq 2$, and

$$\phi(x) = -|x|^2 - |x|^4 ,$$
$$\psi(x) = |x|^2 + |x|^4 .$$

For K we take

$$K = \left\{v \in H^2(\Omega);\ \phi \leqq v \leqq \psi;\ v = A, \frac{\partial v}{\partial \nu} = B \text{ on } \partial\Omega\right\}$$

where A, B are constants satisfying

$$|A| < 2 \tag{4.1}$$

and

$$2A \neq B , \quad \text{or} \quad |A| > 1 , \quad \text{or} \quad |B| > 2 . \tag{4.2}$$

Notice that

$$\phi = -2 < A < 2 = \psi \quad \text{on} \quad \partial\Omega$$

and that K is nonempty.

THEOREM 4.1. *If* (4.1), (4.2) *hold then the solution u is not in C^2, locally in Ω.*

Proof. Notice that

$$I^+ \cap I^- = \{0\} . \tag{4.3}$$

It is clear, by symmetrization, that the solution u must be a function of $\rho = |x|$. We shall write

$$u = u(\rho) , \quad \phi = \phi(\rho) , \quad \psi = \psi(\rho) .$$

Since $u(\rho)$ is in H^2, it is continuously differentiable for $0 < \rho < 1$. In view of (4.3), u then has the same regularity properties in $\Omega\backslash\{0\}$ as the solution of the one obstacle problem; i.e., by [2] [6],

$$u(\rho) \in C^2(0, 1) . \tag{4.4}$$

We claim that

$$\operatorname{int} I^+ = \emptyset \ . \tag{4.5}$$

Indeed (cf. [1]) in $\operatorname{int} I^+$ we have $\Delta^2 u = \Delta^2 \psi > 0$ and also (since $u > \phi$ in a neighborhood of $(\operatorname{int} I^+)\backslash\{0\}$) $\Delta^2 u \leqq 0$; thus (4.5) follows.

Similarly one shows that $\operatorname{int} I^- = \emptyset$.

LEMMA 4.2. *There holds*:

$$0 \in \overline{I\backslash\{0\}} \qquad \textit{where} \quad I = I^+ \cup I^- \ . \tag{4.6}$$

Proof. If the assertion is not true then

$$\Delta^2 u(\rho) = 0 \qquad \text{if} \quad 0 < \rho < \delta \ , \qquad \text{for some} \quad \delta > 0 \ .$$

Thus

$$\left(\frac{d^2}{d\rho^2} + \frac{n-1}{\rho}\frac{d}{d\rho}\right)^2 u(\rho) = 0 \ .$$

One can now either use a general theorem on removable singularities for solution of $\Delta^2 w = 0$ or else write u explicitly (i.e.,

$$u = c_1 + c_2\rho^2 + c_3 \log \rho + c_4 \rho^2 \log \rho \qquad \text{if} \quad n = 2 \ , \quad \text{etc.})$$

in order to deduce (after making use of the fact that $\phi \leqq u \leqq \psi$) that $u(\rho) = c\rho^2$ if $0 < \rho < \delta$ and $|c| < 1$.

By analytic continuation we then get $u = c\rho^2$ if $0 < \rho < 1$. Hence $B = 2A$ and $|A| < 1$. Since, by (4.1), $|A| < 2$, we now get a contradiction to (4.2).

LEMMA 4.3. *Suppose*

$$\alpha, \beta \in I^+ \ , \qquad 0 < \alpha < \beta < 1 \ , \qquad (\alpha, \beta) \subset (0, 1)\backslash I \ .$$

Then there exists a $\bar\rho \in [\alpha, \beta]$ *such that*

$$\Delta u(\bar\rho) = \Delta\psi(\bar\rho) \ .$$

Proof. Since $\psi - u$ takes minimum at α, β, we have (using (4.4))

$$\Delta(\psi - u)(\alpha) \geqq 0 \ , \qquad \Delta(\psi - u)(\beta) \geqq 0 \ .$$

Hence if the assertion is not true then

$$\Delta(\psi - u)(\rho) > 0 \qquad \text{for all} \quad \rho \in [\alpha, \beta] \ .$$

Recalling that $(\psi - u)(\alpha) = (\psi - u)(\beta) = 0$, and applying the maximum principle, we get $\psi < u$ in (α, β), which is impossible.

LEMMA 4.4. *There holds*:

$$0 \in \overline{I^- \backslash \{0\}}\,, \qquad 0 \in \overline{I^+ \backslash \{0\}}\,. \tag{4.7}$$

Proof. It is enough to prove the first assertion. If this assertion is not true then

$$(0, \delta) \cap I^- = \varnothing \qquad \text{for some} \quad \delta > 0\,. \tag{4.8}$$

By Lemma 4.2 we then have

$$0 \in \overline{I^+ \backslash \{0\}}\,.$$

Recalling (4.5) we deduce that there exist

$$\alpha_i \in I^+\,, \qquad \beta_i \in I^+ \qquad (i = 1, 2)$$

such that

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \delta$$

and

$$(\alpha_i, \beta_i) \subset (0, 1) \backslash I\,.$$

From Lemma 4.3 it follows that there exist $\rho_i \in [\alpha_i, \beta_i]$ such that

$$\Delta(\psi - u)(\rho_i) = 0\,. \tag{4.9}$$

Since u does not touch the lower obstacle in $0 < \rho < \delta$, we have

$$\Delta^2 u \leqq 0 \qquad \text{in} \quad 0 < \rho < \delta$$

and consequently,

$$\Delta^2(\psi - u) > 0 \qquad \text{in} \quad (\rho_1, \rho_2)\,.$$

We can therefore apply the maximum principle to conclude that

$$\Delta(\psi - u)(\rho) < 0 \qquad \text{in} \quad (\rho_1, \rho_2)\,.$$

But this contradicts the fact that $\Delta(\psi - u)(\alpha_2) \geqq 0$.

From Lemma 4.4 it follows that there exist sequences $\rho_m \to 0$, $\tilde{\rho}_m \to 0$ such that

$$u(\rho) = \rho^2 + \rho^4 \qquad \text{if} \quad \rho = \rho_m\,,$$
$$u(\rho) = -\rho^2 - \rho^4 \qquad \text{if} \quad \rho = \tilde{\rho}_m\,.$$

This implies that $u \notin C^2$ in any neighborhood of $\rho = 0$.

REMARK. In the above example u touches both the upper obstacle and the lower obstacle (by Lemma 4.4).

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Received September 29, 1980. This paper is partially supported by the National Science Foundation Grants 7406375 A01 and MCS 791 5171 and by C.N.R. of Italy through L.A.N. of Pavia.

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