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POINTWISE ERGODIC THEOREMS IN l.c.a. GROUPS

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Let G be a l.c.a. group and $\{T_g\}$ be a representation of G such that each T_g is a measure-preserving transformation on some probability space $(\Omega, \mathscr{F}, P)$. Let $\{\mu_n\}$ be a sequence of probability measures on G. We are interested in the a.e. convergence or summability of $\int_G f(T_g w) d\mu_n(g)$, for $f \in L_1(\Omega)$. Some examples and counterexamples are given, and some partial results are obtained.

1. Let G be a locally compact abelian group (l.c.a.), and let $\hat{G}$ be its dual group. $\hat{G}$ consists of all continuous homomorphisms of G of absolute value one. $\hat{G}$ is again l.c.a. Denote by $\hat{G}_d$ the l.c.a. group obtained from $\hat{G}$ by endowing it with the discrete topology, and by $\bar{G}$ the dual of $\hat{G}_d$. $\bar{G}$ is a compact group known as the Bohr compactification of G, and G is a dense subset of $\bar{G}$. If m is normalized Haar measure on $\bar{G}$, then $m(G) = 0$. Note that G and $\bar{G}$ have the same characters, namely the elements of $\hat{G}$. Now if μ is a finite measure on the Borel sets of G, we may without loss of generality consider it to be a measure on $\bar{G}$, for if B is a Borel subset of $\bar{G}$ we can define $\mu(B) = \mu(B \cap G)$. If $\{\mu_n, n = 1, 2, \cdots\}$ is a sequence of probability measures on G, we shall call it an ergodic sequence if μ_n, considered as a sequence of measures on $\bar{G}$, converges weakly to m, the Haar measure on $\bar{G}$. The reason for this terminology is that it was shown in Blum and Eisenberg, [2], that if $U = \{U_g\}$ is a strongly continuous unitary representation of G on some Hilbert space H, and if we consider the sequence $\int_G U_g f d\mu_n(g)$, which is defined weakly for each $f \in H$, then if $\{\mu_n\}$ is an ergodic sequence we have a strong limit $\int_G U_g f d\mu_n(g) = Pf$ for every $f \in H$, where P is the orthogonal projection on the closed linear subspace of H consisting of those elements of H invariant under each U_g. Moreover if this version of the mean ergodic theorem is to hold for every strongly continuous unitary representation of G, then it is necessary that $\{\mu_n\}$ be ergodic.

In this paper we shall be concerned with pointwise ergodic theorems.

Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\{T_g\}$ be a group of measure-preserving transformations of Ω into itself such that the corresponding unitary operators U_g on $L_2(\Omega)$ are a strongly continuous representation of G. We show by a simple example that the pointwise ergodic theorem does not hold for every ergodic sequence

$\{\mu_n\}$ on G. We then show that for certain ergodic sequences the pointwise ergodic theorem does hold for a set which is dense in $L_1(\Omega)$, but not necessarily for all of $L_1(\Omega)$. Finally we exhibit certain ergodic sequences for which the pointwise ergodic theorem does hold.

2. Let $G = Z$ and for each positive integer n define μ_n by putting mass $1/[\sqrt{n}]$ on the integers $n+1, \cdots, n+[\sqrt{n}]$, where $[x]$ is the longest integer not exceeding x. Now let $\hat{\mu}_n(\alpha)$ be the Fourier transform of μ_n for $0 \leqq \alpha < 2\pi$. Then $\hat{\mu}_n(\alpha) = (1/[\sqrt{n}]) \sum_{j=n+i}^{n+[\sqrt{n}]} e^{ij\alpha}$ and $\hat{\mu}_n(0) \to 1$ while $\hat{\mu}_n(\alpha) \to 0$ for $0 < \alpha < 2\pi$. But if m is Haar measure on Z then $\hat{m}(\alpha) = \begin{cases} 1, & \alpha = 0 \\ 0, & 0 < \alpha < 2\pi \end{cases}$. Thus $\{\mu_n\}$ is an ergodic sequence. Now let $(\Omega, \mathscr{F}, P)$ be a probability space, and let T be an invertible ergodic measure-preserving transformation of Ω onto itself. It was shown by Akcoglu and Del Junco, [1], that there exists a set $A \in \mathscr{F}$ with $0 < P(A) < 1/2$. and a set $B \in \mathscr{F}$ with $P(B) > 1/2$ such that for $w \in B$ we have

$$\frac{1}{[\sqrt{n}]} \sum_{j=n+1}^{n+[\sqrt{n}]} \chi_A(T^{-j}w) = 1$$

infinitely often, where χ_A is the indicator of A. In fact by a slight modification of their argument and by taking lim sup one can make $P(A)$ arbitrary small and $P(B) = 1$. In any case it is clear that the individual ergodic theorem does not hold for this ergodic sequence $\{\mu_n\}$.

3. Now suppose $\{\mu_n\}$ is a sequence of probability measures on G, each of which is absolutely continuous with respect to the Haar measure on G. Denote by φ_n its density with respect to Haar measure, i.e., $\mu_n(A) = \int_A \varphi_n(g)dg$, for each Borel subset A of G, where dg is Haar measure on G. For $\gamma \in \hat{G}$ we shall write $\hat{\varphi}_n(\gamma)$ for the Fourier transform of μ_n, i.e., $\hat{\varphi}_n(\gamma) = \int_G \langle g, \gamma\rangle \varphi_n(g)dg$. Here $\langle g, \gamma\rangle$ is the usual notation for the character γ evaluated at g.

Then we have

THEOREM 1. *Suppose for each compact subset K of $\hat{G}$ with $0 \notin K$, and every sequence $\{\gamma_n\}$ with $\gamma_n \in K$ for all n we have $\sum_{n=1}^{\infty} |\hat{\varphi}_n(\gamma_n)|^2 < \infty$. Then there exists a set D, dense in $L_2(\Omega)$, and hence in $L_1(\Omega)$, such* that *for $f \in D$ we have* $\lim_{n\to\infty} \int_G f(T_g w) d\mu_n(g) = Pf$ a.e.

Here P is as above the orthogonal projection of f onto the closed subspace of $L_2(\Omega)$ of elements invariant under each U_g. Before proving the theorem we exhibit a class of examples for which the hypotheses of the theorem are satisfied. Let μ be a probability measure on G which is absolutely continuous with respect to Haar measure on G, and suppose $|\hat{\mu}(\gamma)| < 1$ for $\gamma \neq 0$. Then if we let μ_n be the n-fold convolution of μ with itself it is easily verified that the hypotheses of the theorem are satisfied. For example, if $G = Z$ and μ puts mass p on 0 and mass $1 - p$ on 1, where $0 < p < 1$, then $\hat{\mu}(\gamma) = p + (1 - p)e^{i\gamma}$ for $0 < \gamma < 2\pi$ so that $|\hat{\mu}(\gamma)| < 1$. In this case μ_n is the binomial distribution with parameters n and p.

Proof of the theorem. Let $E(\cdot)$ be the spectral measure associated with $\{U_g\}$. Note that $E(\{0\})f = Pf$ for all $f \in L_2(\Omega)$. Denote by $\mathscr{E}$ the closed subspace of $L_2(\Omega)$ spanned by the eigenfunctions of $\{U_g\}$.

Let $f \in \mathscr{E}^{\perp}$. Then $(E(d\gamma)f, f)$ is a continuous, regular Borel measure on $\hat{G}$ and for given $\varepsilon > 0$ we can find a compact set $K \subset \hat{G}$ with $0 \notin K$ such that

$$\| E(K)f - f \|_2 < \varepsilon .$$

Since $PE(K)f = 0$ we shall show that $\lim_{n\to\infty} \int_G E(K)f(T_g w)d\mu_n(g) = 0$ a.e. Henceforth we shall write this integral as $\int_G E(K)f(T_g w)\varphi_n(g)dg$. Let $\delta > 0$ and define $A_{n,\delta} = \left\{w \,\middle|\, \left| \int_G E(K)f(T_g w)\varphi_n(g)dg \right| > \delta\right\}$. Then

$$\begin{aligned}
P\{A_{n,\delta}\} &\leqq \frac{1}{\delta^2}\left\| \int_G E(K)f(T_g w)\varphi_n(g)dg \right\|_2^2 \\
&= \frac{1}{\delta^2} \int_G \int_{\hat{G}} \langle g, \gamma\rangle Ed(\gamma)(E(K)f)\varphi_n(g)dg \Big\|_2^2 \\
&= \frac{1}{\delta^2} \int_{\hat{G}} |\hat{\varphi}_n(\gamma)|^2 (Ed(\gamma)E(K)f, E(K)f) \\
&= \frac{1}{\delta^2} \int_K |\hat{\varphi}_n(\gamma)|^2 (E(d\gamma)f, f) \leqq \frac{1}{\delta^2} \max_{\gamma \in K} |\hat{\varphi}_n(\gamma)|^2 \| f \|_2^2 .
\end{aligned}$$

But $\hat{\varphi}_n(\gamma)$ is continuous and therefore there exists $\gamma_n \in K$ such that $|\hat{\varphi}_n(\gamma_n)|^2 = \max_{\gamma \in K} |\hat{\varphi}_n(\gamma)|^2$. Hence by the hypotheses we have $\sum_{n=1}^{\infty} P(A_n, \delta) < \infty$. It follows from the Borel-Cantelli lemma that $P(A_\delta) = 1$ where $A_\delta = \{w \mid w$ is in at most finitely many $A_{n,\delta}\}$ and similarly if $A = \bigcap_{k=1}^{\infty} A_{1/k}$, then $P\{A\} = 1$. But for $w \in A$ we have $\lim_{n\to\infty} \int_G E(K)f(T_g w)\varphi_n(g)dg = 0 = PE(K)f$. Thus we approximate each $f \in \mathscr{E}^{\perp}$ by a function for which the theorem holds.

If $f \in \mathscr{E}$ and $\varepsilon > 0$ there exist finitely many eigenfunctions, say $f_{\gamma_1}, \cdots, f_{\gamma_M}$ with $\gamma_j \neq 0$, $j = 1, \cdots, M$ such that

$$\left\| E(\{0\})f + \sum_{j=1}^{M} f_{\gamma_j} - f \right\|_2 < \varepsilon .$$

Then

$$\lim_{n\to\infty}\int_G\Big[E(\{0\})f+\sum_{j=1}^{M}f_{\gamma_j}(T_gw)\Big]\varphi_n(g)dg$$
$$=Pf+\sum_{j=1}^{M}\lim_{n\to\infty}\int_G\langle g,\gamma_j\rangle f_{\gamma_j}(w)\varphi_n(g)dg$$
$$=Pf+\sum_{j=1}^{M}f_{\gamma_j}(w)\lim_{n\to\infty}\hat{\varphi}_n(\gamma_j)=P(f)$$

since clearly $\lim_{n\to\infty}|\hat{\varphi}_n(\gamma_j)|=0,\ j=1,\cdots,M$.

This concludes the proof of the theorem.

4. As was mentioned earlier the theorem does not hold in general for all of $L_1(\Omega)$, or indeed for all bounded functions. As an example, let $G=Z$ and let μ_n be $B(n,p)$, the binomial distribution on $0,1,\cdots,n$ with $0<p<1$. Let $(c_j,\ j=0,1,2,\cdots)$ be a sequence of 0's and 1's. Then it was shown by Diaconis and Stein, [4], that $\lim_{n\to\infty}\sum_{j=0}^{n}c_j\binom{n}{j}p^j(1-p)^{n-j}=L$ if and only if for every $\varepsilon>0$ we have $\lim_{n\to\infty}1/\varepsilon\sqrt{n}\sum_{j=n+1}^{n+[\varepsilon\sqrt{n}]}c_j=L$. Now if T is invertible and ergodic we saw that we can choose a set A with $0<P(A)<1/2$ and a set B with $P(B)=1$ such that for $w\in B$ we have

$$\limsup_n\frac{1}{[\sqrt{n}]}\sum_{j=n+1}^{n+[\sqrt{n}]}\chi_A(T^{-j}w)=1\,.$$

By choosing $c_j=\chi_A(T^{-j}w)$ it follows from the Diaconis-Stein result with $\varepsilon=1$ that the individual ergodic theorem fails to hold for this sequence $\{\mu_n\}$. As was mentioned earlier, this sequence does satisfy the hypotheses of the theorem.

It is of some interest to point out that in the case when μ_n is the n-fold convolution of a measure μ on G, we do have a version of the individual ergodic theorem as follows:

THEOREM 2. *Let μ be a probability measure on G, and let $\{T_g\}$ be a measure-preserving representation of G on some probability space $(\Omega,\mathscr{F},P)$. For each n let μ_n be the n-fold convolution of μ with itself. Let $f\in L_1(\Omega)$. Then* $\lim_{N\to\infty}1/N\sum_{j=1}^{N}\int_G f(T_gw)d\mu_n(g)$ *exists* a.e.

The proof follows at once from the Dunford-Schwartz ergodic theorem. (See e.g., Garsia, [5].) Define the operator S on $L_1(\Omega)$ by $(Sf)(w)=\int_G f(T_gw)d\mu(g)$. Then clearly $\|S\|_1\leqq 1$ and $\|S\|_\infty\leqq 1$. It is easily verified that $(S^nf)(w)=\int_G f(T_gw)d\mu_n(g)$, and the theorem follows from the Dunford-Schwartz theorem.

The limit in Theorem 2 clearly depends on the nature of μ. If μ is absolutely continuous with respect to Haar measure on G, and if its density φ has a Fourier transform $\hat{\varphi}(\gamma)$ such that $\hat{\varphi}(\gamma) \neq 1$ for $\gamma \neq 0$, then it is not difficult to show that for $f \in L_2(\Omega)$ the limit is again Pf.

5. In the case when $G = Z$ or $G = R$, it is of interest to ask what summability methods other than Cesaro averaging yield the individual ergodic theorem. Some results along this line may be obtained from a paper by Davydov and Ibragimov, [3]. We shall give one of their theorems for the real line R. Let μ be a probability measure on R and for each n let μ_n be the n-fold convolution of μ with itself. Let f be a measurable, real-valued, bounded function on R. Then we have the

THEOREM (*Davydov-Ibragimov*). *Suppose μ belongs to the domain of attraction of a symmetric stable law and suppose for some n the distribution μ_n has an absolutely continuous component. Then*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) d\mu_n(x) = L \quad \textit{if and only if}$$
$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x) dx = L .$$

A similar result holds when $G = Z$, i.e., when μ is a lattice distribution.

The way we can apply this is as follows. Let $(\Omega, \mathscr{F}, P)$ be a probability space, and let $\{T_t, -\infty < t < \infty\}$ be a measurable, measure-preserving flow on Ω which is a representation of R, i.e., $T_0 = I$ and $T_t T_s = T_{t+s}$. Let $g \in L_1(\Omega)$. Then it follows from the individual ergodic theorem that $\lim_{T\to\infty} 1/2T \int_{-T}^{T} g(T_t w) dt$ exists a.e. and equals Pg when $g \in L_2(\Omega)$. Now let w be in this set of probability one. Without loss of generality we may assume that $g(T_t w)$ is bounded in t. Define $f(t) = g(T_t w)$. Then it follows from the theorem above that $\lim_{n\to\infty} \int_{-\infty}^{\infty} g(T_t w) d\mu_n(t)$ exists and equals $\lim_{T\to\infty} 1/2T \int_{-T}^{T} g(T_t w) dt$. For example, if μ is the normal distribution with mean zero and variance $\sigma^2 > 0$, then μ_n is the normal distribution with mean zero and variance $n\sigma^2$, and we have that

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi}\sqrt{n\sigma^2}} \int_{-\infty}^{\infty} g(T_t w) e^{-(1/2n\sigma^2)t^2} dt$$

exists a.e. for $g \in L_1(\Omega)$. In the case when $G = Z$, the Davydov-Ibragimov theorem only requires that μ belong to the domain of attraction of a symmetric stable law. For example, if μ is any distribution on the integers with mean zero and positive second moment the theorem applies.

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